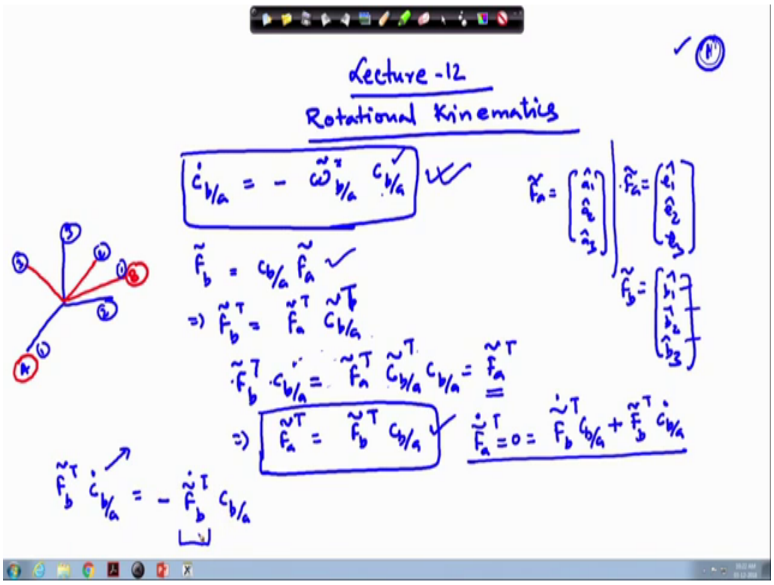
Satellite Attitude Dynamics and Control
Prof. Manoranjan Sinha
Department of Aerospace Engineering
Indian Institute of Technology, Kharagpur

Lecture – 12
Rotational Kinematics (Contd.)

Welcome to the 12th lecture. So, we have been discussing about the Rotational Kinematics, so let us continue with that.

(Refer Slide Time: 00:25)



So, if you remember that in the last lecture we proved using the vector method that c dot b slash a equal to minus omega cross b slash a time c b slash a. So, this is the transformation matrix from a frame to b frame. Similarly this is the angular velocity of the b frame with respect to the a frame and this is the rate of change of the matrix c.

Now, the same thing can be proved also using the vectrix method and this I am doing because the vector method we assume that the vector to be fixed and it may create confusion. So, its a better look from the vectrix method.

So, we having in the vectrix method. So, this is your frame A and frame B is oriented with respect to this somewhere this is your frame B this is frame a. So F b, where F a if you remember that F a have written as let us say let us say a1 cap, a 2 cap, a 3 cap or either we can also write in terms of F a tilde equal to e1 cap, e 2 cap, e 3 cap. So, these are the unit vectors along the direction1, direction 2 and direction 3 similarly, in the B frame you will have F b tilde this will be writing as b1 cap, b 2 cap and b 3 cap where b1 b 2 b 3 these are the unit vectors which are all unit vectors and this called the vectrix as earlier we have denoted it.

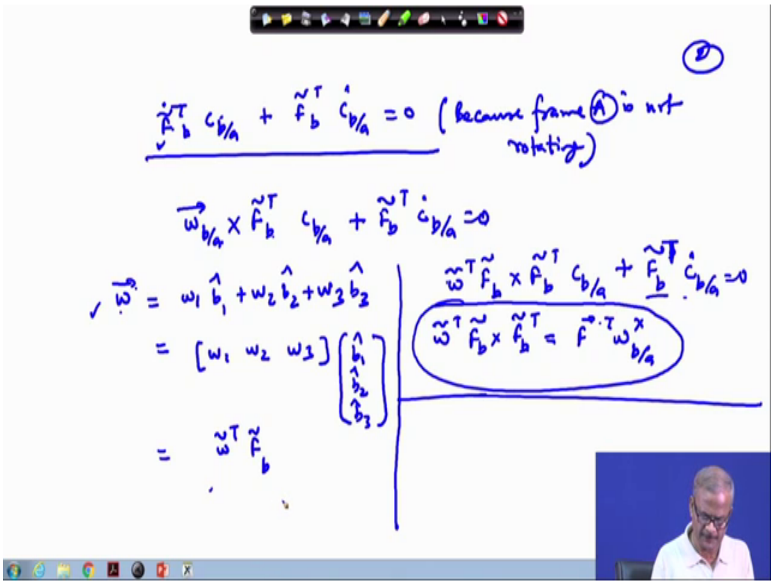
So, this is the 1st direction of this, this is the 2nd direction, and this is the 3rd direction of this of the B frame. So, using this relationship this is transformation of this trigger Fa to triggered b where that trigger b is rotating with respect to triggered a. If we take the transpose of this so, we get F b transpose here and Fa transpose comes here cba transpose goes on this side.

Now, this can be rewritten what we can do that we multiply on both side by c b slash a ok. So, if we multiply so, that this quantity is nothing this is the rotation matrix. So, its a determinant if its a product will be I identity matrix and therefore, we have written here as Fa transient F a transpose. So, this is Fa transpose here.

So, Fa transpose this becomes equal to Fb tilde transpose times cba so, this is written here in this place. Now as we know that this frame is 6 transient b is rotating. So, if we differentiate this. So, this becomes F a tilde dot this will be equal to 0 in the earlier case we have taken as the vector. So, it may that might have created some confusion in that if that vector is rotating then what will happen. So, for removing that confusion we are just dealing with the vectrix. So, both the methods used just compare and have a deep inside so how the things are working. So, F dot a transpose this equal to 0, on the left hand side and right hand side we will have F b tilde dot transpose times c b slash a times Fb tilde transpose times c b slash a dot ok.

So, what we get from this place that F tilde b transpose c b slash a equal to c b slash a and here F tilde b transpose this equal to minus with minus sign and this is the quantity we are looking for. So, if we remove this quantity so, our job will be done.

(Refer Slide Time: 04:43)



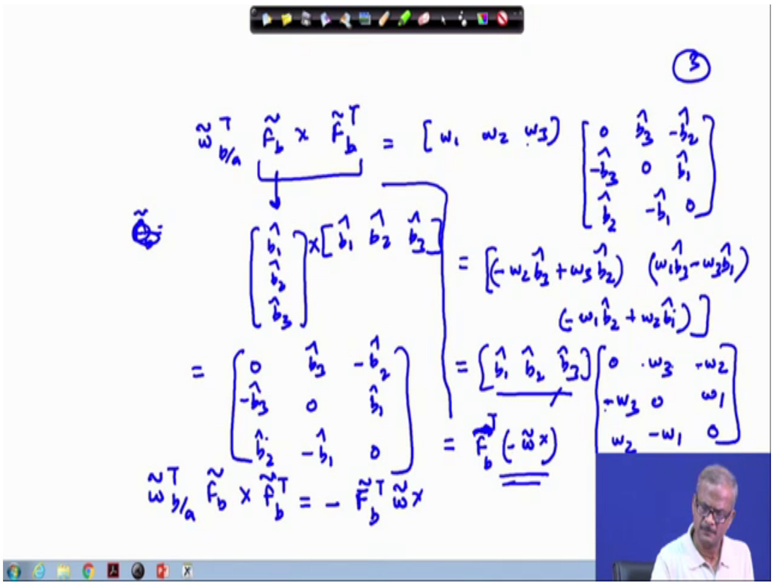
So, now we have F transpose c b slash a plus F tilde transpose b times c dot b slash a this equal to 0 ok. So, now we shall simplify it because, frame a is not rotating. This quantity this is the rate of change of a vector. So, this is the rate of change of vector b here in this is the vectrix. So, this we can write as omega tilde b slash a cross F tilde trans F tilde this is the vector F transpose. So, here it will come as a transpose F transpose b, and then this is being operated by c b slash a plus Fb tilde transpose. Little bit changes its required here and our job will be done so, this omega tilde instead of writing omega tilde let us write this as omega here.

So, this omega as you know this will be omega1 times this is the rotation rate of vector omega sorry this is the rotation rate of the frame B with respect to frame A. So, this we can describe in terms of the vector b itself ok. So, the components of vector b omega 2 b 2 cap plus omega 3 b 3 cap we can write it like this and this can be written as omega1 omega 2 omega 3 in the form of vectrix b1 cap b 2 cap b 3 cap ok. So, this gets reduced to we write it like this omega tilde transpose.

So, here we make the difference like this the matrix rotation and this is just the vector notation we are using. And here we have the vectrix, so this is F tilde b ok. So, this is the vectrix b. Now if we insert this here in this place so, what we get omega tilde transpose F tilde b cross F tilde transpose b c b slash a plus F tilde transpose b times c dot b slash a this equal to 0.

And from here; obviously, we need to reduce it little bit and this quantity we can separate out and write c dot b a. So, what now we have to use here that we use this result F tilde b omega tilde transpose times F tilde b cross F tilde transpose b, this quantity we can rewrite this as F transpose omega b slash a cross and this we can show. Let us do this on the next page may be that will be appropriate or either will go to the next page.

(Refer Slide Time: 08:55)



So, we have to get this quantity b slash a transpose Fb cross F transpose b ok; if you write this quantity this quantity already we have written omega1 omega 2 omega 3 and this is Fb cross. So, this quantity together if you multiply this will get reduce to the format say the here Fb is your Fb tilde just put it here in this place Fb tilde is a vector. So, the b1 cap b 2 cap b 3 cap and this transpose b1 cap b 2 cap b 3 cap and there is a cross sign also.

So, now if we take the this cross product so, b1 times b1 that will be 0 b1 times b 2 is b 3 cap b1 times b 3 is b1 times b 3 this is minus b 2 cap ok. So, if you keep multiplying so, this what you are going to get this screw symmetric matrix and so, the 3rd term 2nd row 3rd term this will b 2 times b 3 equal to b1 cap ok. So, here this will be minus b1 cap and this place b 2. So, remember in this skew symmetric matrix in any column or any row only1 minus sign will be there and the other will be plus1 and obviously, there is1 0 all the way it will look.

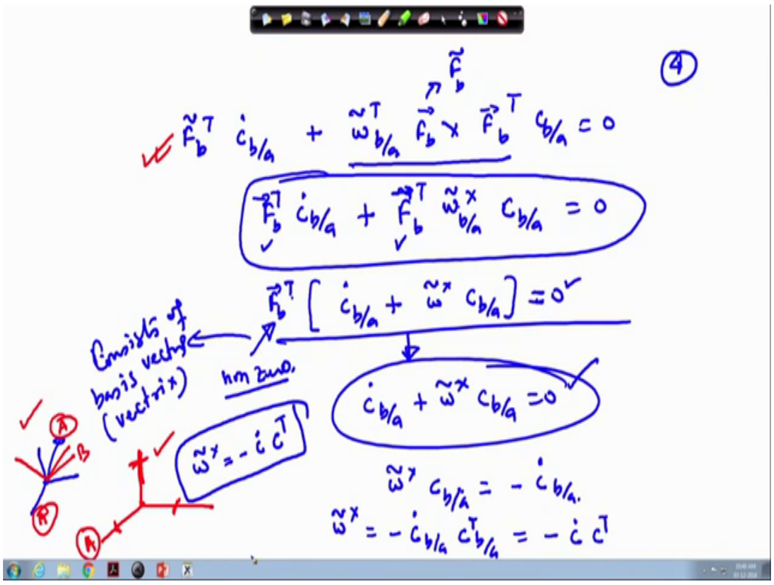
So, we can just insert here in this place 0 b 3 cap minus b 2 cap minus b 3 cap 0 b 2 cap and 0 and then multiply it and solve it. So, after solving you get it is in this format. So, omega1 a after we multiply it so, minus omega 2 times b 3 cap this is the term getting multiplied with this we are multiplying this plus omega 3 times b 2 cap this is the first row ok. So, this will be a single row and there are 3 columns in this.

The other will be from here to here if we look into this. So, omega1 b 3 cap then omega 2 this is 0 omega 3 it minus omega 3 b1 cap and the third one will be omega1 b 2 cap with minus sign and this one omega 2 and so, b1 cap, and this is the last1 will be 0. So, you get this 3 terms. Now, this 3 terms if you rewrite it in terms of if you change the order if you write it like this.

On this side then you will have here 0 omega 3 minus omega 2 0 0 and here omega1 minus omega1 omega 2 and minus omega 3. So, thus the things have got reverse. So, what this quantity is this quantity is nothing but b this is we have written as Fb higher in the form of tilde or transpose whatever you write ok. So, Fb tilde transpose this is the quantity and this quantity is nothing but minus this quantity is minus omega tilde cross because, in the omega tilde cross minus sign comes here this minus sign. So, the things have got reverted, if you put take out the minus sign you will see that here it will be minus and plus and similarly the other places. So, this is the quantity that we are getting.

So, this can be reduced to this format. So, what we are getting omega tilde transpose b slash a Fb tilde cross F tilde transpose b this equal to minus F tilde transpose b times omega tilde cross. And this can be utilized in the expression that we have got.

(Refer Slide Time: 13:43)



So, inserting this in the equation that we have written so, our equation was F tilde b transpose times c dot b slash a plus omega tilde transpose b slash a Fb cross Fb transpose either you can write both way in terms of tilde also it does not matter here. And the quantity from here to here right, now we have derived this, and this we have written as let us check the sign here perhaps sign reversal has taken place. So, this part is and if we multiply this comes here will check the sign here whether the things we have done correctly or not.

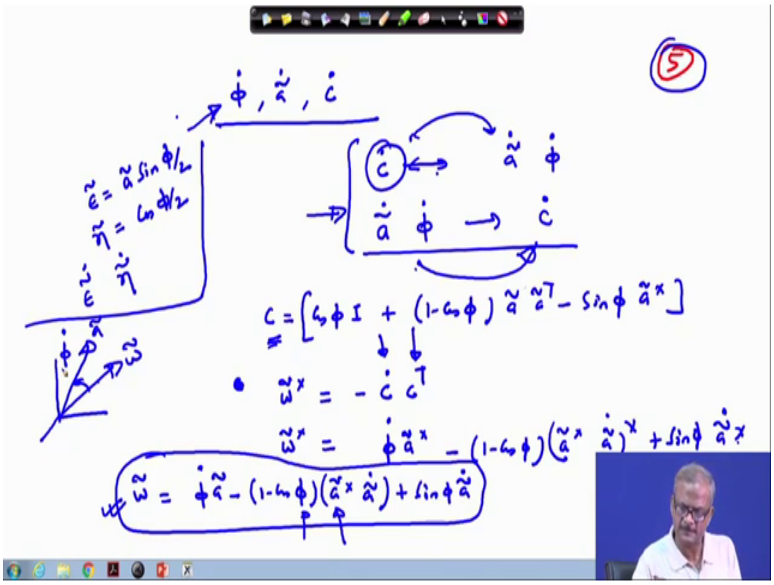
So, b1 cap this is 0 b 2 times minus omega 3, here it comes as b 2 comes with plus sign. So, here we will have plus here minus plus here the sign is got reversed this is minus and here minus. So, this is this is plus. So, what we get here this quantity should be plus not minus. So, we have done this correction now using this we can replace this full part. So, we pick up this particular part F tilde transpose b times omega cross F tilde transpose this is F tilde transpose b times omega b slash a tilde cross, then this c b slash a right hand side this is 0 and Fb tilde transpose c dot b slash a. Now what we see that this quantity this quantity is common to both this terms so, we can take it outside and right hand side c dot b slash a plus omega tilde cross c b slash a this equal to 0.

So, if you look here in this quantity the right hand side this is 0 and this is non 0 quantity because this is the basis vector this consist of consist of basis vector that is this is the vectrix. So, this is non zero quantity. And therefore, this quantity must be 0 c b slash a omega tilde cross c b slash a this equal to 0 and this what we proved using the vector method. So, either using this method or that method there we assumed that in the frame a there is a vector which is fixed, and then the frame b which is oriented with respect to this frame this is frame b and this is frame a. So, b is rotating with respect to this and the vector was fixed.

However, here removing this vector we have just assume the frame and then derive the whole thing equally in the case of the in this case we could have assumed that this is the frame a this vector A is there say. So, this vector A will have certain components along the 3 directions is not it and if the frame B is rotating with respect to frame A. So obviously, if we already fixed the components here along this directions. So, in this frame these components are not going to change. So, this method and the vector method they do not differ they are exactly the same only thing its a matter of representation how we are going to work out.

One may be convenient for some people another may be convenient for some people. So, this vectrix notation you have to get acquainted with then only it will look convenient otherwise for many people this appears to be little complicated. So, if you are not comfortable with this you can just follow the vector method that we have discussed in the lecture 11 ok. So, once we have done this so, the next step will be to our next step will be to get the c matrix in terms of already we have the c matrix of what we have require that we need inter convention say we have the.

(Refer Slide Time: 18:53)



Now, we have dealing with the kinematics of rotation. So, will be dealing with phi dot a tilde dot c tilde c dot these are the things we will be handling here ok. And quiet often it may be required that we convert we go from one notation to another notation that is sometimes we may be dealing with c and from this c we need to go to the a tilde dot and the phi dot and vice versa. Sometimes it may be that we are dealing with a dot phi dot and from there we need to go to the c dot.

And here, as we will see what we are doing its not just without purpose, we are going to we have already discussed about the Euler's parameter. So, Euler's parameter of epsilon tilde we have written as a tilde sin phi by 2 and vector tilde equal to cos phi by 2.

So, our objective ultimately is to get this quantities ok. So, once we differentiate this quantities you can see that a dot will come and phi dot will come ok. And a dot now its a described in terms of the c dot if you have the c dot and the vice versa whatever we want to do. So, the thing is that we need to go from this place to this place or either this place to or from this place to this place and the same thing can also be done in terms of the (Refer Time: 20:30). So, let us proceed further.

So, if you remember that using this c equal to cross phi i plus 1 minus cos phi times a tilde transpose a tilde times a tilde transpose minus sin phi times a tilde cos. This is the c matrix and this we have also converted into the in terms of the Euler parameters ok.

So, still what we have do here that we have omega from the previous slide from here what we get we carry it to the next space omega tilde cross this quantity is omega tilde cross times c b slash a this will be equal to minus c dot b slash a and multiplying on both sides by cba transpose. So, this will get reduce to identity matrix on the left hand side and this will be c b slash a times c b slash a transpose.

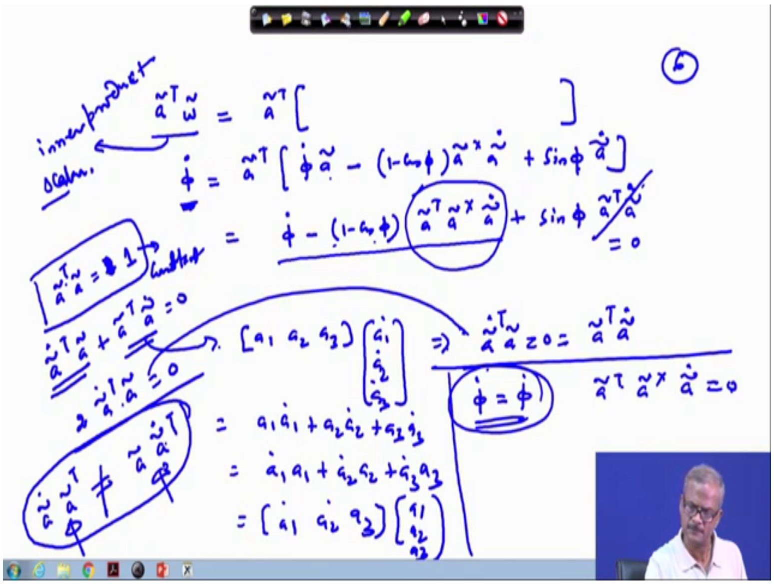
And if we drop these notations so, simply we can write this as c dot times c transpose. So, omega tilde cross omega tilde cross this get reduce to minus c dot times c dot transpose, and this is one important conclusion this will be frequently used ok. So, using this particular expression omega tilde cross this equal to minus c dot times c transpose. So, if we insert this c needs to be inserted here and derivative of c after taking derivative of this c this needs to be inserted here.

This will do as a part of the tutorial. So, it can be reduced to the format omega tilde cross I have already last time perhaps we have written it a tilde cross minus 1 minus cos phi. And if the cross notation is taken from both sides then this get reduced to phi dot a tilde minus 1 minus cos phi a tilde cross times a tilde dot plus sin phi times a tilde dot. So, if you know a dot ok; obviously, you are rotating about the a vector.

So, this needs to be available this should be available. So, a tilde dot phi dot these are the two things which are main. So, if this two things are available and by how much angle you are going to rotate. So, this will give you the omega and vice versa if is the omega so, also we can go and find out a tilde dot and the phi dot. Now if we are looking for rotation about this is say if as we shown last time at if this is my frame and here this is your omega vector and this is a vector ok.

So, component here in this direction this will give you the phi dot the component of omega along a vector this is nothing but our phi dot. So, if we take the component of omega along this direction from this place so, this must result.

(Refer Slide Time: 25:11)



And therefore, if we take the omega tilde this is available to us and we take the component of this along the a directions means, a tilde transpose we have to write like this; so this matrix, then we have to multiply it by a tilde transpose and if we do this. So, we get on the left hand side is the phi dot this is obviously scalar quantity because, this is the inner product. So, here we will have a tilde transpose times phi dot a tilde 1 minus cross phi if we take it inside the brackets.

So, phi tilde dot this quantity becomes1 because a tilde is affinity magnitude then 1 minus cos phi and this get reduced here we have to write it like this and plus sign phi times a tilde transpose times a tilde dot. Now a tilde times a tilde a tilde transpose times a tilde this quantity is I this quantity is1 ok, this is scalar inner product. So, the right hand side this is a constant.

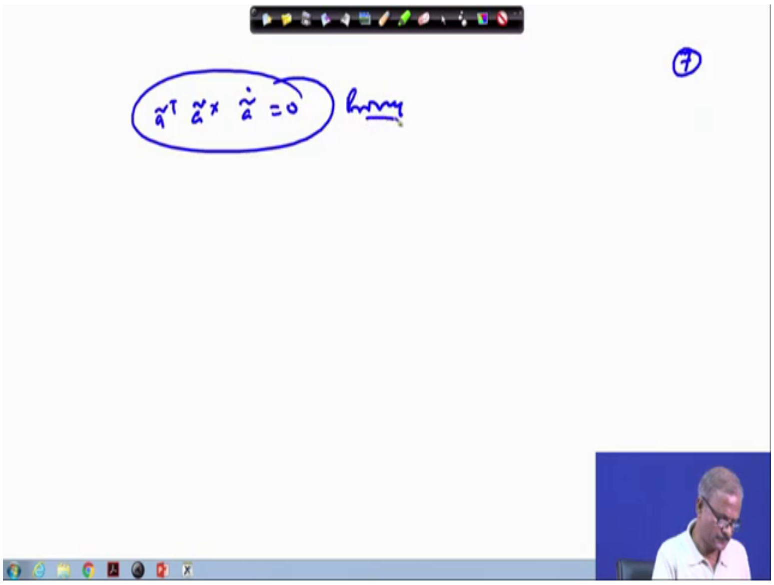
If we differentiate this will be 0 and this is the inner product. So, what it implies this can be written as if head it been the outer product say this quantity if we write it like this. So, this is not equal to a tilde dot a tilde times this quantity is not equal to this quantity see the difference this is the outer product a tilde dot coming here and front here its a coming at the end.

But here in this case because its a inner product if you this will basically give you the term like a1 a 2 a 3 and here you have a1 dot a 2 dot a 3 dot. So, if you take this or a 2 dot plus a 3 times a 3 dot and you know that I can put it in the front. So, because its a scalar quantity so, it does not matter and this can be written as a1 dot a 2 dot a 3 dot times a1 a 2 a 3. So, this two quantities are equal and there for we can write like this and therefore, this implies this implies here a tilde dot times a tilde this equal to 0 or equally we can say this equal to a tilde transpose times a tilde dot.

So, this simple conclusion it can be used here. So, this quantity gets reduced to 0 here we have that dot. So, this quantity gets reduced to 0 and we just have to look into this quantity which is phi dot minus1 minus cos phi and this quantity again we have to resolve it. So, phi dot1 minus cos phi phi dot equal to (Refer Time: 29:41) ultimately what should happen that this quantity should go to 0 because, here only then what we get here phi dot equal to phi dot ok, this phi dot; obviously, on the right hand side we are getting here phi dot because of this.

Now, here if we prove this that this quantity is 0 this part then only our job will be done and the left hand side will be equal to the right hand side otherwise this will be extra term and then things will be in error. So, what it says that the component of omega as we have discussed earlier in the direction of a tilde is nothing, but this phi dot and this if you can prove your job is done. So, what here needs to be proved that a tilde transpose a tilde cross times a tilde dot this quantity is 0. So, we go to the next page now.

(Refer Slide Time: 31:09)



So, we need to prove that a tilde transpose a tilde cross times a tilde dot a tilde transpose times a tilde cross times a tilde dot this equal to 0 this we need to prove. So, we will continue in the next lecture.

Thank you.