Introduction to Aerodynamics

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Module No. # 01
Lecture No. # 04
Fluids and Forces in Fluids (Contd.)

Now, we will look to that nature of the surface force. For that, let us consider a small fluid element, small fluid element in the shape of a tetrahedron. In the shape of a tetrahedron, such that three of its faces lies on the three coordinate plane. A tetrahedron that is a figure with four faces but this tetrahedron is such that, it has three faces lying on the three coordinate plane and the fourth face is an inclined face.

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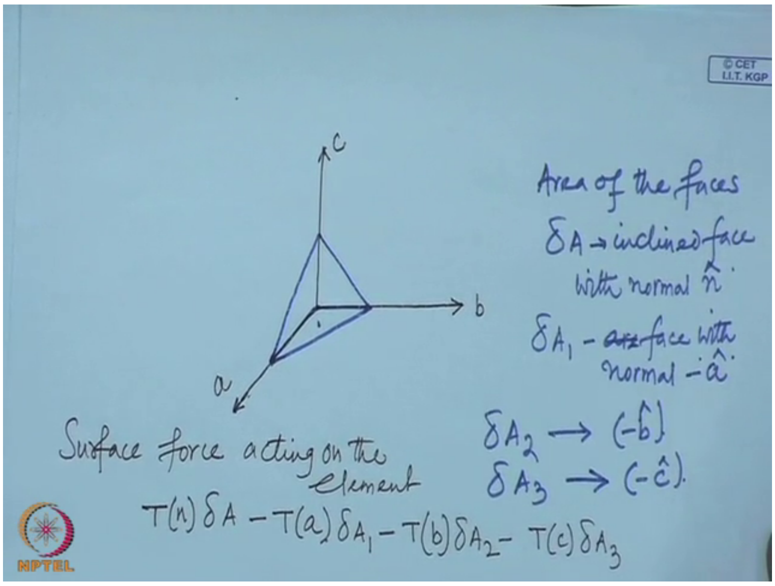

Let us take it this way. And these three mutual orthogonal direction, let us take them to be a, b, c. That is, the unit vector along this direction is a, unit vector along this direction is b and the unit vector along this direction is c. And the tetrahedron, so, this makes one face lying on this coordinate plane, a c coordinate plane. This face lies on coordinate plane bc. And this third face lies on the coordinate plane ab, this face. And the inclined face, now this becomes, now, let us say, what is the force acting on this element? The surface forces that are acting on this element. We did not or other, for sake of simplicity, we will not write that x and t, thus. Because, you are thinking of a small element and in our discussion that the position is remaining unchanged and the time is also not changing. So, that position vector and time in that expression, we will not write. We will simply write T as function of n. Yes. Let us say these elements are, area are: delta a 1 is the area to which a is normal; that is, the face that is lying in the bc plane has area delta a 1. Similarly, the face which is lying on ac has area delta a 2 and this third face which is laying on this plane ab has area delta a 3. And the area of the inclined faces, face inclined face is let say delta a. Area of the faces are like this.

Inclined face, with normal n, this normal we are denoting, let say n. Delta a 1, sorry. Face with normal, actually, it is minus a. We are taking only the normal which is pointing out of the volume. So, the normal is actually minus a. similarly, delta a 2 and delta a 3. Delta a 2 corresponds to minus b and delta a 3. So, what is the total surface force acting on this element? Total surface force acting on this element.

Now, can we express this delta a 1, delta a 2, delta a 3 in terms of delta a? You see, this delta a 1 delta a 2 delta a 3 are the projection of the area delta a in three different, in three coordinate planes. So, what is the relationship? How can you write delta a 1? What is delta a one? Search an example.

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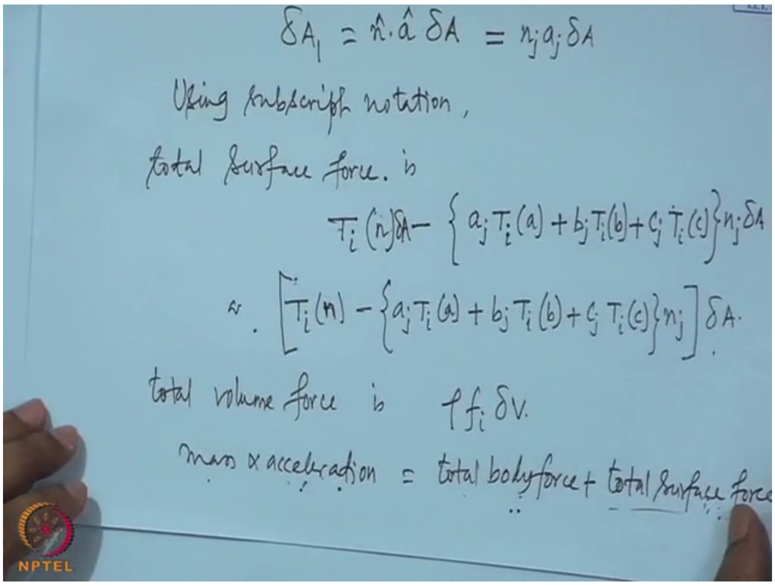

What is the projection of n along a? You know the unit vector. Say, what it is? What is delta a 1, then? N dot a into delta a. And in our notation, we can write this as nj aj delta a and others also. Similarly, delta a 2 delta a 3. Then, if we write it in the earlier equation using the vector notation, this t which is force per unit area is a vector. So, the T will denote as y Ti. Using the subscript notation, that vector T will be written as T i.

Sorry, the first term there is delta a or we can take only one.

And mass into acceleration equal to total body force plus total surface force. This left hand side is proportional to delta v. The first term on the right hand side is proportional to delta v. The second term on the right hand side is proportional to delta a. Now, think that this element is being made smaller and smaller, in such a way that its shape and orientation, it remains same. In such a situation, you can see that the left hand side term and the first term on the right hand side will approach to zero at a much faster rate than the second term. Agreed, because the first term, that is left hand side term and the body force term and this acceleration term or the inertia force term is proportional to delta v. Which is cube of the characteristic size, while this total surface force is proportional to square of the characteristic size. Now, if I am making the element smaller and smaller is what? That is characteristic size is becoming smaller. So, obviously, these will decrease at third sorry at cubic order, while this will decrease at second order. So, thus decrease in these two are much faster than the decrease in the second, the last one, surface force term.

However, this condition, if this fluid is at rest, if the if it is equilibrium, this condition will valid always. See, when in the sense, when this characteristic size or this volume approaches to zero, at that limit also this equation holds or this element in equilibrium. How is it possible? One, two terms are decreasing in, as delta x cube say or delta l cube while another decreasing as delta l square. How come they balance? That is possible, if this is entirely zero. This is possible only, if this total surface force is entirely zero. Otherwise, this will not be balanced by these two, because these are much smaller than this. It cannot balance. Only possibility is that this has to be zero. Only then, it can be balanced. Followed this argument?

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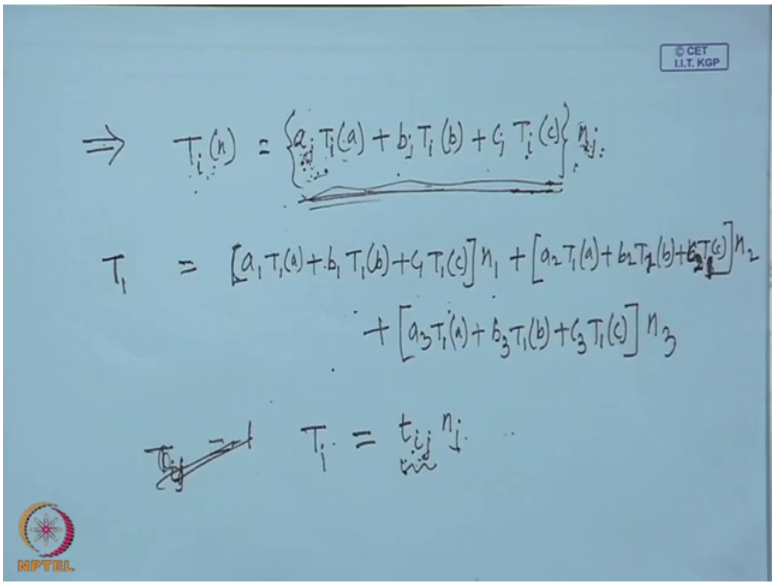

So, this gives such that… So, that will be the first component of this surface force on this. Anyway, this the term in the bracket is called the stress tensor. This is what the stress tensor. And of course, denoted by two subscript and now, we will write that simply as notations. (( )) Say T sorry T ij, sorry Ti equal to, this will write as a small t, t ij nj. This t ij is called the stress tensor which is a second order tensor. And as we have seen here, that irrespective of the choice of the axis system. Independent of the choice of axis system. Like the vector, normal vector and force vector, this stress tensor is also independent of choices, choice of axis system. And this t ij, which you can look to this, you can understand, what is the meaning of this t ij. t ij is the, of course, force per unit area in the direction I, on a plane to which j is normal. t ij is force per unit area, we are not mentioning that at position x, at time t. That is always there. These are all function of position and time, as we have mentioned in the beginning. t ij is force per unit area in the direction i, acting on a surface which has a normal, which has the normal in j direction. Here, the first subscript i refers to the direction of this force and the second subscript j refers to the direction of normal to the plane, on which it is acting.

And again, remember that in the right hand side, which is now written as product of two quantity. One tensor and one vector, one second order tensor and one first order tensor, it is summed over the index j. The index that represents the plane. So, in general, this stress we will have nine elements. Along three direction, and in each direction, again three component because of three direction of the normal. However, we will now see that, as such this is not so, that all these nine components are not independent. And this, we can now again see, using the rotational equilibrium. Using the translational equilibrium, we have found that the surface force is basically of, has the stress nature, the nature of the surface force. And further, if you consider the angular or rotational equilibrium, we can see that all the components are not different. For that, we can consider an element of any shape, you can consider an element of any shape and think about a point, a fixed point within that volume. And then, take moment of all the forces that are acting and equated to the angular rotational inertia force. Moment of inertia into angular acceleration.

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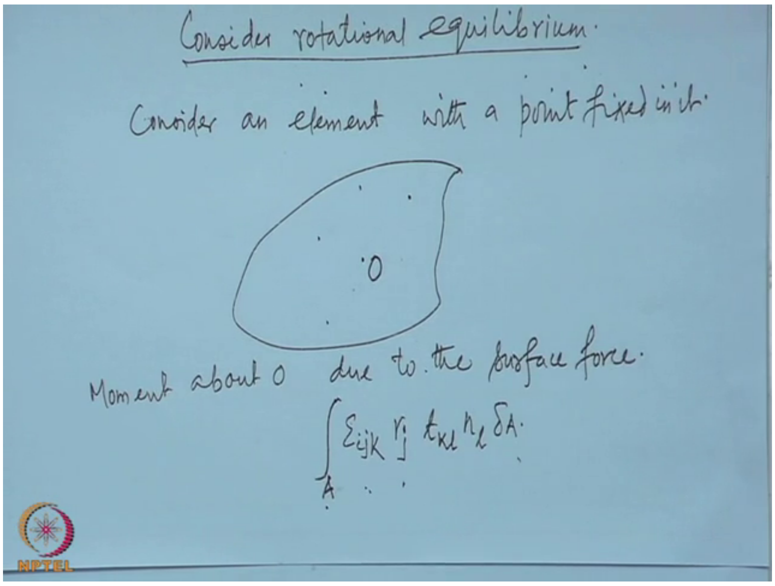

Now, consider rotational equilibrium. Again, let us start with, consider an element, we have just fixing up a point within the control volume, just to sake take moment about that point. Let us say that, this is that point, fixed point called O and some element like this. at any position, surface forces are acting given by that T ij nj. So, what is the moment of the surface surface forces? The surface force is, the surface force acting at any point which position vector is x, is t ij nj which is function of x and t. We are not writing that all the time explicitly. So, t ij nj into of course, delta a, which t ij nj is force per unit area. So, the force is t ij nj delta a. So, what will be the moment due to this force? Cross product of that force with the distance from the point O. Write it, using this index notation.

The distance you can write r. So, it is r cross f. You can start from there, r cross f and r cross f will be denoted by epsilon ijk rj fk. Now, that fk will come from that t ij nj. So, fk, you have to write in such a way the result is, so, t ij nj, you have to write such a way that the result is fk. See, when you write t ij nj, the result is t i. The way we wrote t ij nj, of course, the j is being summed t ij nj, that small t tensor t ij nj. The second index j is summed. So, in the product j will not be there. Only, i will be there. So, if you want to write k, yes, t k. No, but see, that should not be remain there at all. t k if you write t k, and then, what is the second index?

If you take i, then in the epsilon ijk, there is i. So, that i should not, because then that will again mean a sum. So, we should not take i but, you see that second index will be the same as the index of n and since it is a dummy index, you can write anything. does not matter. We can write l, write l. So, this term, we will be writing, epsilon ijk rj and to make that force tkl nl into delta a. And of course, this has to be integrated over the bounding surface. So, since, it is integrated over a surface, we will always denote by an integration with notation like this. Similarly, an integration with, over a volume will also denote by an integration with integration subscript v, not by triple integration or double integration. We will usually not write double integration or triple integration, they will be denoted by just single integral sign, with this notation a. So, this is the moment due to the surface forces.

Now, before you proceed further, just think about it. The other moments that will be coming is due to volume forces and the inertia. Angular momentum, rate of change of angular momentum, which will be that moment of inertia into angular acceleration, i into alpha. Now, this moment of inertia and this volume force both are function of volume. They depend on the volume of the element. And those integration, then will also be, in that case also you have to integrate. And those integration will be integration over volume. The moment due to the body forces and the moment inertial moment, they are integration over volume. But, this is integration over area. Since, they will be written in the same equation, it is necessary that all should represent similar type of term. That is, if those two are integration over volume, this also must be integration over volume. Alternately, those two volume integration can be expressed in terms of surface integration. But, it is convenient to express this surface integral over volume integral. So, we now want to express this either volume integration. Here, we have done that already. We have changed integration. So, what it is? Say, just I think about of very simple, you have, this is a vector, here the integral is a vector. So, integration of a vector over it, over a surface is represented by how? How it is transformed to volume integral? Divergence.

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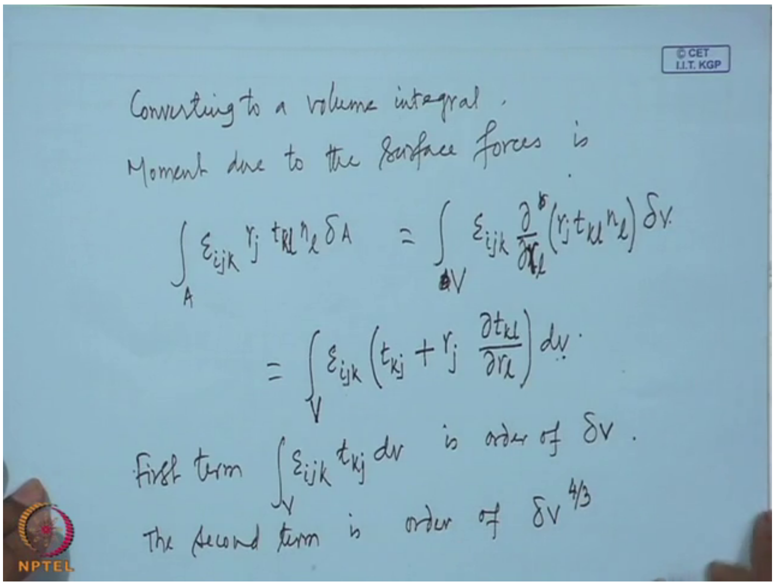

So, we will convert this to a volume integral using this divergence. How much? Epsilon ijk of course, will not be differentiated; sorry epsilon ijk will not be differentiated. It is just a constant plus or minus one or zero. I am sorry not x r. Now, here you see that, this integral has two term. We do not have to evaluate this integral. Now, look to this. This integral has two term. The first term is simply integrated over dv but the second again content or derivative. So, in terms of proportionality with delta v, what will be the result of the first and second term? Yes, the first term will be proportional to delta v or of the order of delta v, we call it. While, the second term will be, second term will be order of area. Another length rj. So, what is it? Of the order of v to the power 4 by 3. So, the first term is of the order of v, the second term delta v, the second term is of the order of delta v to the power 4 by 3 no, no, how can be at this stage T, it is not t kl t kj. (( )) No, no, t kl.

But, after this, there is a differentiation no. See this difference is because of the result of drj drl. You can check, what that different derivative will be? drj dxl thinking of r1 r2 r3 and rl also r1 r2 r3. So, the derivative will be non-zero, when j and al are same, otherwise not. And in that way, that l will become j; that is, this derivative drj drl, the way we have got this first term is simply by differentiating rj with respect to rl and rj. And rl rj when differentiated with respect to rl is 1, only when j and al are same. For j and l different this becomes zero. So, they vanish. So, only t kj part only remains.

nl, how can nl be there? nl is not possible. That normal, the normal with the derivative. rj, this nl with rl. Only, that l part will, only the when al is same as j, only then it remains. And for that this t kl, then becomes t kj. The end result is t kj. For little practice or understanding, you write the terms explicitly, taking and if you do it for once or twice, see these things will become, I mean you will become more confident with these notation.

Now, let us come to this integration, then that epsilon ijk t kj, this integration of this is zero; that is what, the this term is zero. That is what, we obtained from the equilibrium. Now, look to this particular integration, now this integration can be zero but, this function which is being integrated or the integrant is continuous. And for a continuous, continuous integrant the result can be zero. For all possible choice of v, this choice of v is arbitrary. The volume that elemental volume that we choose that is arbitrary. we can choose any volume. for any arbitrary chosen volume, this result will be true only, if this integral itself is zero. So, it may become zero for some particular choice of v. even, if it non-zero itself but, because of some particular choice of the elemental volume, sometime it may become zero. But, in general for all possible choice of the elemental volume, if this is zero, that is possible only, if the integrant itself is zero. Since, choice of…

And what does it mean? Can you say epsilon ijk t kj equal to zero? Just think about i equal to 1, that case. Only i equal to 1, then j and k can be either 2 or 3. See, if j is 2, k is 3, then it becomes epsilon 1, 2, 3; that is plus 1, t 3 2. And when you make j 3 k 2, this epsilon ijk become minus 1. So, t 3 2 minus t 2 3 equal zero. For i equal to 1, we get this equation. For i equal to 1, this results. We get t 3 2 minus t 2 3 equal to 0 or t 2 3 equal to t 3 2 or whatever way you write.

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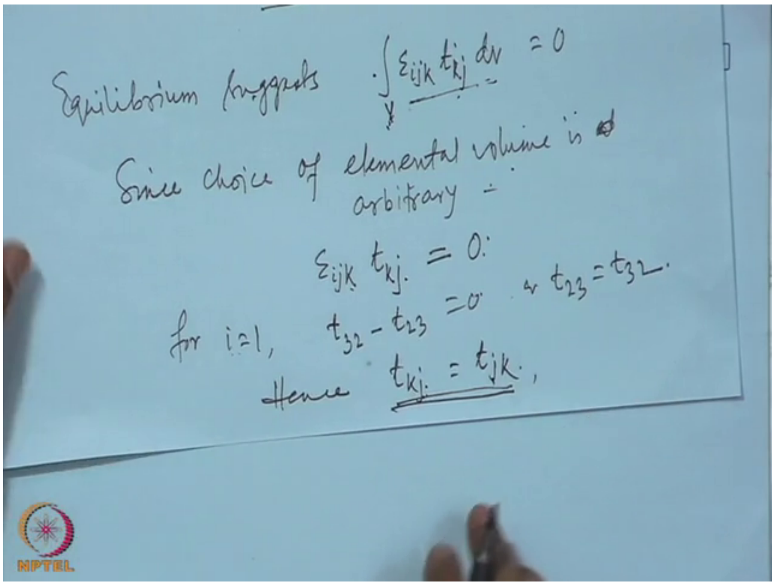

Similarly, for any choice or in one way, this t kj epsilon ijk t kj equal to 0 means, t kj equal to t jk. You can try; this is i equal to 2 or 3, that way. Practice it for a few days. Unless, you will not feel confident with these notations, unless you practice for some cases. So, do that practice and you will see that, there is nothing very terrifying or nothing very special about this notation. It is only convenient and simple. Anyway, so, what does it mean, t k j equal t j k? That is, the stress tensor is symmetric. The stress tensor is symmetric. You know, this is what, the way symmetric matrix are written, when the matrix is symmetric a ij equal to a ji. So, this is the transpose is same as the original, the matrix is same as its transport. So, stress tensor is symmetric.

So, what we have found? Based on our discussion today, we have seen that the nature of surface forces. That surface forces are basically stress and also that this stress is symmetric tensor. And now, we will see certain properties of, I think some tensor, some matrix, you might be familiar with. And you know, what are say tensor in variant? In terms of moment of inertia, you have, that is also another tensor you have you have familiar with? Did you not come across something called invariant? This is we are familiar with, that if we rotate the axis system, then the elements are changed. Which element will change? We will discuss this, in the next class.