Introduction to Aerodynamics

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Lecture No. # 33

Conformal Transformation

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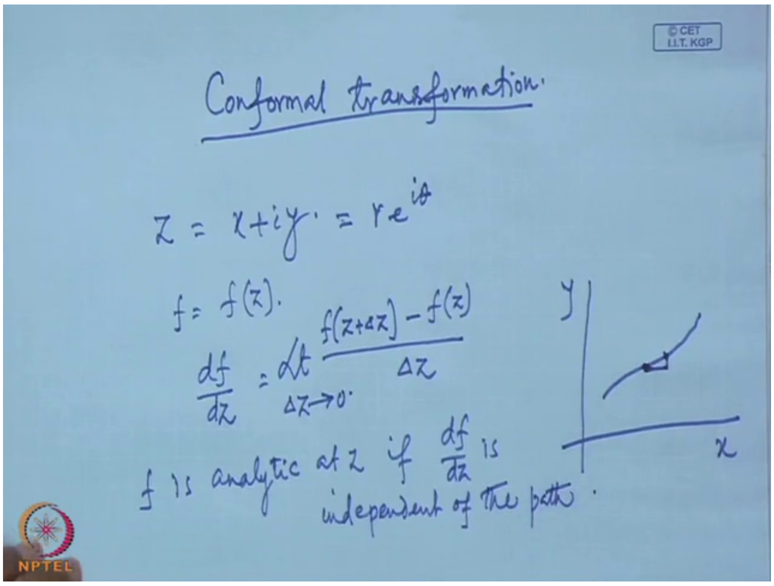

So, we will continue our discussion on conformal transformation. However, before we go to conformal transformation proper, let us recall one or two important results from complex analysis. A complex number is denoted by a real and imaginary part usually written in this fashion, this i stands for imaginary square root of minus one, sometime j is also used for it, we will be using i anyway. This can also be written as r e to the power i theta, where r is square root of x square plus y square, that is distance of the point or the magnitude of the number, if we say just a complex number then it is the magnitude of the number and that theta is its argument.

Since it is a pair or triple, it also represents a point in a complex plane, in that case that point x, y. So, its position vector is r or distance from the origin is r, and theta it is is the angle that makes with the x axis, the argument. A function of a complex number just like any other real function can be differentiated as well as integrated, about the differentiation we have made say if we have a function f of f z, and this by definition as you know. And since this increment delta z, if we look to a say a function represented in the x, y plane anything.

Let us say this is that function then this delta z the increment can follow any path, it can follow this type of path only along x, only along y or on along the curve, any of these path can be chosen and a function is called analytic, if this differentiation or derivative is independent of the path. If we consider this delta z to be simply delta x or simply i delta y, and the derivative is same then the function is called analytic.

So, call f is analytic at z and this result to, what is the condition? This function f is analytic in z, it satisfies what is called as Kosirina condition, which comes this independence, path independent derivative, that means if we evaluate this derivative just by taking delta x or by taking i delta y, in both cases the derivative are same. If that is true then the function is analytic, and this say df, dx equal to df i dy this leads to what we call the Kosirina condition.

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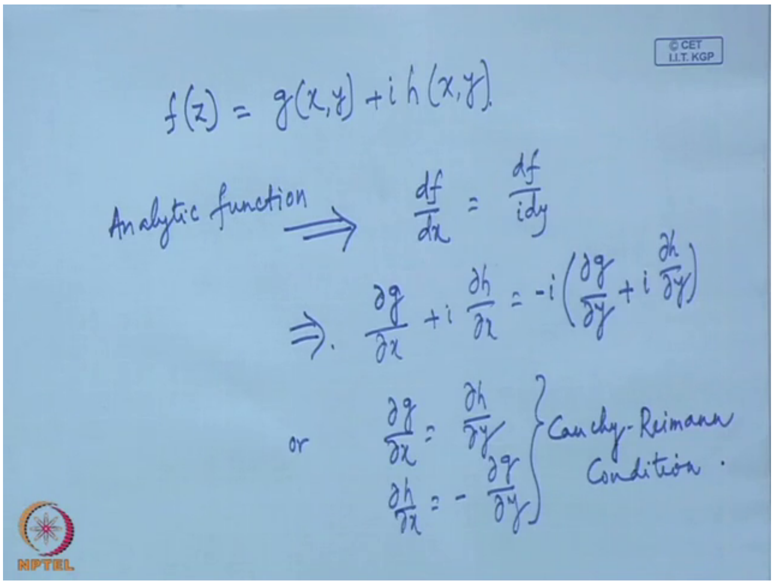

What is that condition for f? See f z like z has two part real and imaginary similarly, this function also as two part a real function and the imaginary part. Let us say this is g x, y plus i h x, y, then that analytic function that implies your and what does it mean? Yes, this gives what, what is df dx? dg, d x plus i dh dx, and what is that 1 by i df di, or should we write in terms of actually, we should have written partial derivative here? Yes 1 by i is minus i, yes then.

So, we have that is all right? These are and we have already seen that for the flow the potential function and stream function satisfies this condition, by our definition of potential function and stream function that satisfy this condition. Now, let us come to the integration of this analytic function, let us say a function f z is defined over a certain domain, and in its domain of definition at each and every point the function is analytic, the analytic it is a point property. A function can be analytic at a point may not be analytic at another point, the point or the function ceases to be analytic is called the singular point. If there is any point in the domain of definition of that function where the function is not analytic, then it is singular at that point as an example, this simple function say f z equal to one by z. This infinite z plan is its domain of definition, and the function is analytic everywhere except at z equal to zero, at z equal to zero it is not define, and call that it is a singularity of the function, the function as its singularity at z equal to zero. So, function can be analytic at some points, may not be analytic at some other points. So now, think about a function which is analytic everywhere, at each and every point within the domain.

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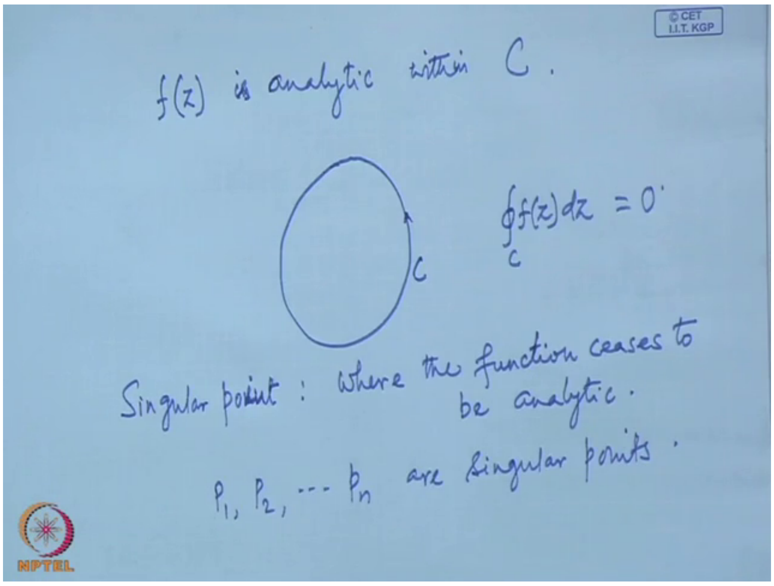

Say within say we call a C, that is we have some domain C and within this domain the function is analytic. Then this integration of this function is also path independent. If we take any path from one point to the other, integration across any path is independent of that choice of path. So, if we have a closed path the integration of is this is zero, so let us call this itself is C then Now, let us say we have a function which is analytic everywhere except at certain points, at few points it is singular, then what will be this integration? First of all remember this singular point, singular point where the function ceases to be analytic, let us say we have few singular points p 1, p 2 say p n are singular points, that is once again

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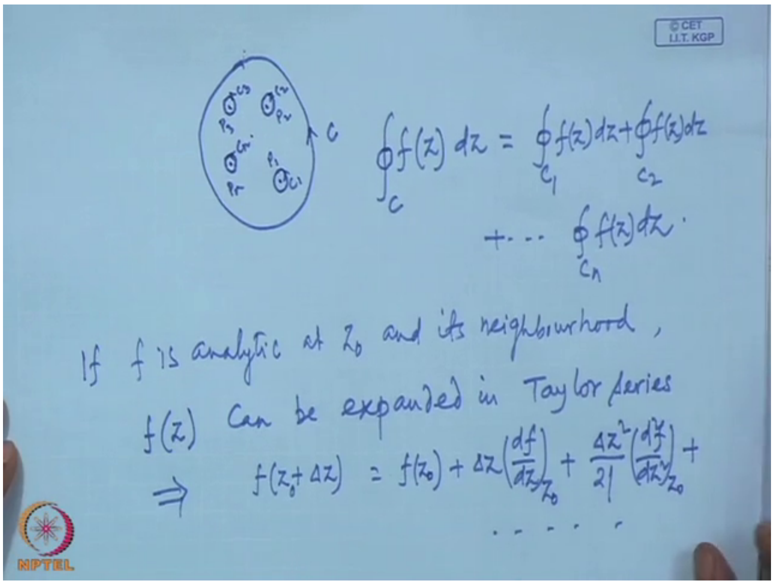

Think about say p 1, these are all singular points, and then what will be this integration? Yes, is it zero again? That is correct it is not zero, but what it is? What we do is we exclude these singular points, how? We consider a small may be circle, need not be circle any closed curve that includes that singular point, we can think about a circle of very small radius about that singular point, so that the singular point is within that circle or we can make it as a general closed curve.

So, let us say we have now each of these singular point, we exclude by drawing a circle. And, like say these points are p 1, p 3 so on p 1, p 2, p 3, p n, we call these curves also say c 1, c 2, c 3, c n. So, you see that this f z dz this can be written as some of the integration for this c 1, c 2, c 3 to c n plus whatever is excluded, now whatever is excluded that is in that part is the function is analytic. So, that integration is zero once again.

So, what it remains is this is if a function is analytic at z zero and its neighborhood, then we can expand the function into let us say just like a real function using Taylor series. If the function is analytic at z zero and its neighborhood then we can expand the function in Taylor series just like a real function equal to… Then something very important that is a Laurent series expansion for a complex function, if this function f z is analytic within an annulus, then with that annulus the function can be expanded using Laurent series, that is if z is annulus.

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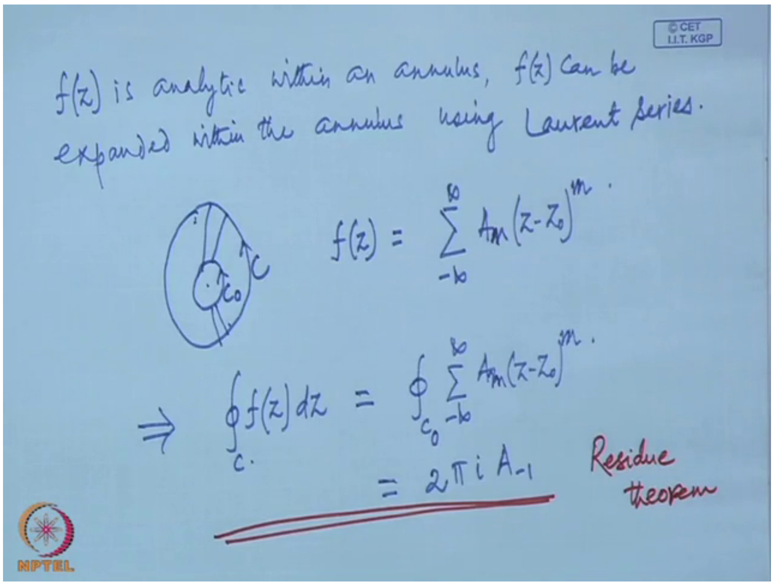

See why it is this is important? You often will find a analytic function which have some singularity in its domain, so near the singularity of course, we cannot expand function. Now, we can exclude the singularity by using again that same approach, consider a small circuit that includes the singularity, and then outside that circuit and within the boundary of the domain the function is analytic which now forms an annulus.

So, function with singularity this Laurent series expansion is very useful. So, within this annulus within this annulus you can expand the series is in this way, or you are expanding this function near point z zero once again. Now, this Laurent series expansion can now be used to find the integration, think about that we have a function which as some singularities, consider a function having some singularities and we want to evaluate its integral, then from here as before within see that there are some number of singularities, certain singularities.

This integration can now we performed using this and this z zero we can make it this point, so that this becomes let us say we give this c, and let us say this c zero so that this can be may write like this, and what will be this integration? For convenience consider just only one singularity at z equal to z zero, this result must be known to you. Earlier, we considered n as number singularity just to make a difference, let us make it m. let us make it m Just to make it that earlier here we used n as number of singularities.

So, we have to avoid that we will make it what will be this integration? This you have not done? You must have done it, something which is called a residue theorem, if you know remember it by name. It is the result is 2 pi i a minus 1, a minus 1 as you can see is the coefficient of m equal to minus term, the coefficient of m equal to minus term that is one by z minus z naught term. This result is called the residue theorem, because this coefficient a minus one is called the residue.

So, this theorem is called the residue theorem. This is of course, I have considered that there is only one singularity otherwise, for every singularity we will have a term like this, one term like this. If you have n number of singularity then there will be two pi i this a minus 1 for each of those n singularities, there will be one singularity for each of this, one residue for each of the singularity, and if we have n singularity then this result will be 2 pi i a minus 1, sum of m minus 1 for all the singularities.

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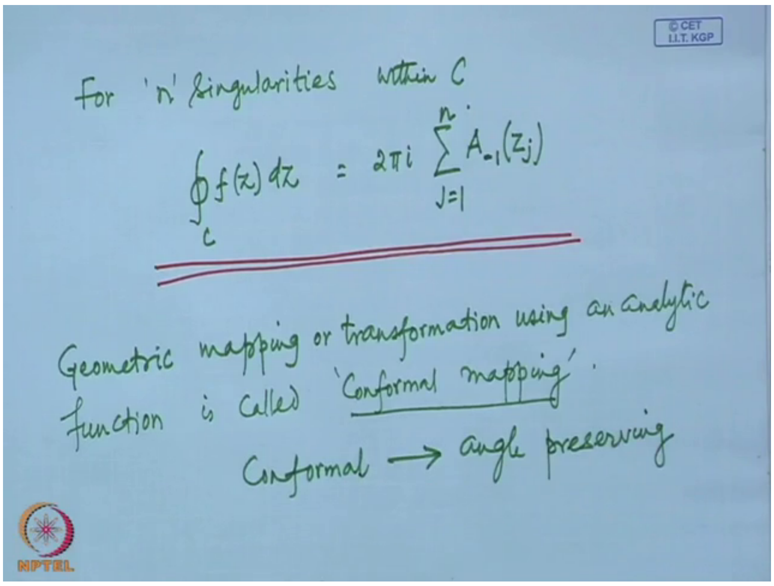

Or say, let us say if we have n singularities. And the word conformal means angle preserving.

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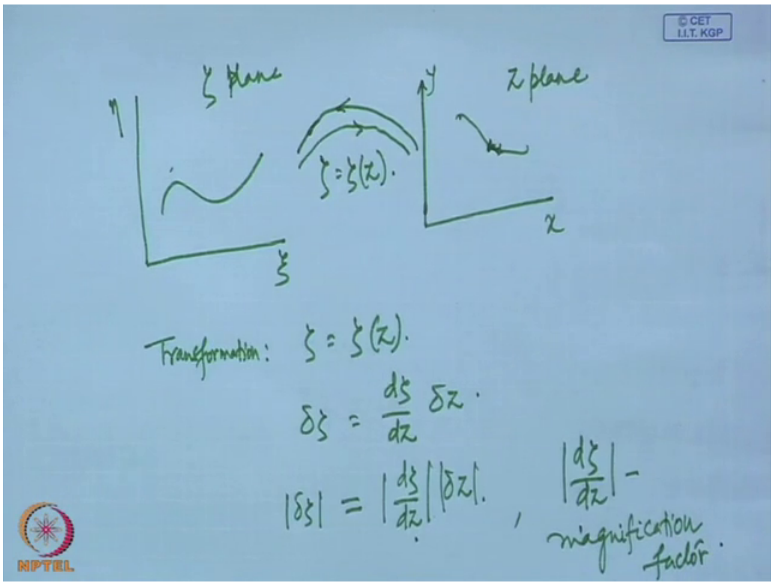

Think we have a curve in the z plane and we are using the transformation zeta equal to zeta as a function of z, and let us consider a one to one correspondence, that is for each point of in the z plane we get one point in the zeta plane for the time being.

Now, let us see what will happen, think about a small segment of this length in the z plane a small segment of the length in the z plane what will happen to this length segment because of this transformation? This small segment we can call it delta z, what will happen it in the zeta plane? Now, this relation, this transformation what is it imply that delta zeta equal to d zeta dz into delta z fine.

Now, how much you have the length of this small segment in the zeta plane? Its magnitude is delta zeta that equal to how much? So, what you see that the length delta zeta, what we have obtained after the transformation is multiplied by its original length, it has been magnified in the general sense, it might be shortened also.

So, this is what is a magnification factor, a length element because of this transformation is magnified by a factor d zeta dz which of course, as a finite definite value because the function is analytic, the derivative exist is nothing infinite or undefined quantity. So, this is called the magnification factor of the transformation.

Let us what happened to the argument? This is the magnitude length, what happens to its argument? What is the argument of the left hand side? We will simply call it, we cannot write some theta or something, so we will call it simply argument of delta z equal to what? Yes, argument of the right hand side d zeta dz into delta z.

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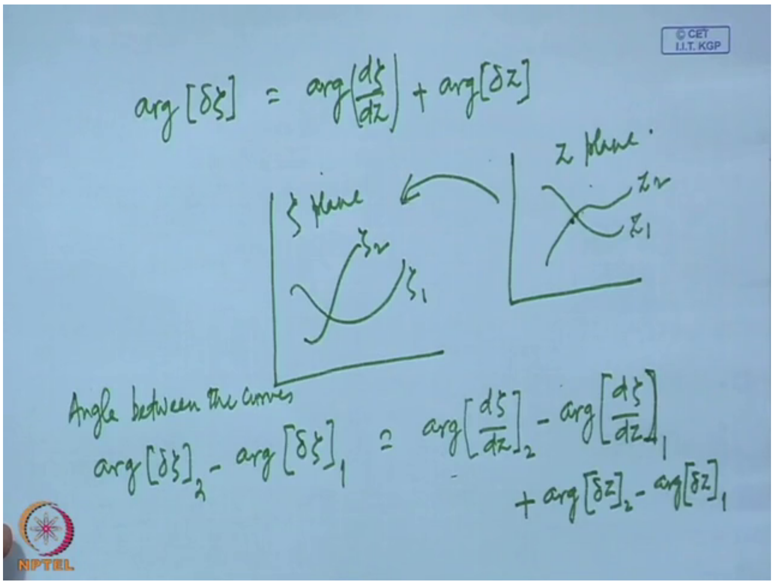

Now, this transformation function zeta is analytic, assuming that it is analytic at the point of intersection also. Now, if this function zeta is analytic at the point of intersection then the derivative there is path independent, that is evaluating d zeta dz following curve z one or following curve z two should be same, that is by definition of analytic function. So, for an analytic transformation or transformation through analytic function these two terms cancel each other.

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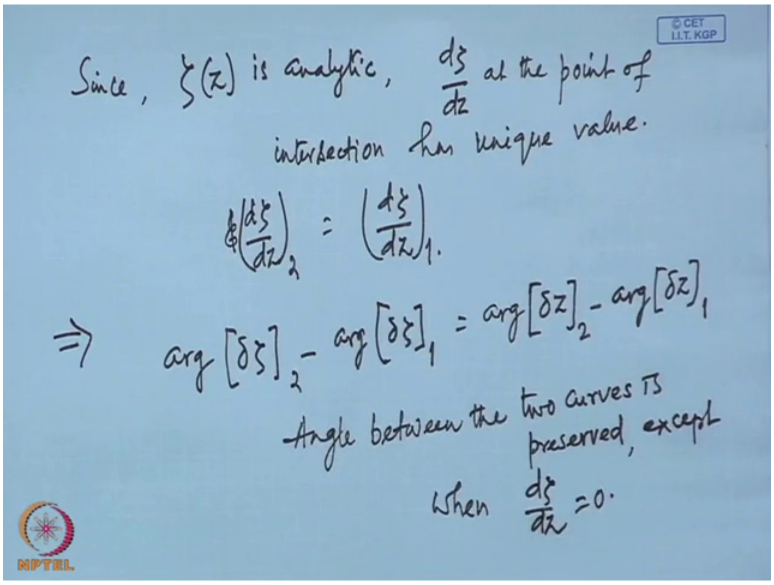

And what we get? The angles remain same however, there is one exception in the particular case if d zeta dz become zero that is the function is analytic there, but somehow d zeta dz become zero then of course, you cannot find its argument. So, we can put it that, except it is not that zeta that at that point where d zeta dz equal to zero, it does not mean that the function is not analytic, the function is still analytic that is not a singular point. However, its derivative is zero it may happen because zero is of course, a finite value the derivative is well defined, but its value is zero and since for zero we cannot find what its argument is, we cannot say that triangle preserved or other the angle will not remain preserved, angle may change. The point where this derivative becomes zero are called critical points the point where the derivative is zero is called the critical point of the transformation or critical point of the function anyway.

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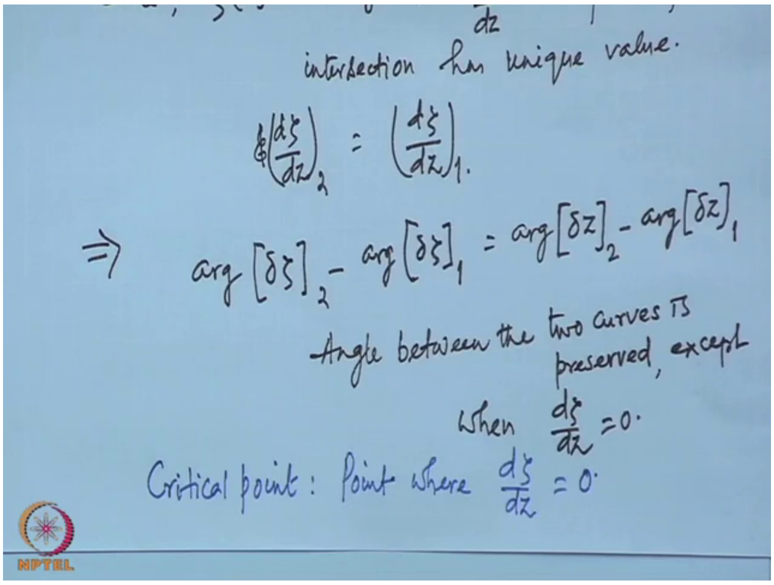

Do not confuse them with singular point, at the singularity the function itself is not analytic, the function is not well defined while at the critical point the function is analytic, its derivative is well defined but it is zero.

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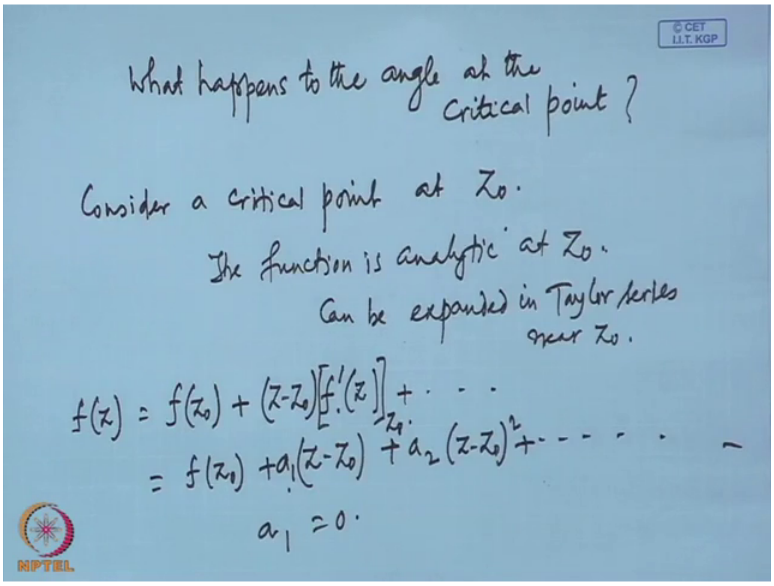

Now, what will happen to the angle at the critical point? Can you answer what will happen to a function if it has a critical point say at z equal to z naught? So, you now oppose this question, what happens to the angle at the critical point? Because as you will see that critical point plays a very important role in our transformation, I will tell you why, which is quite easy to comprehend.

So, if the function is analytic at each and every point, then after the transformation the curve that will obtain will be almost or similar to what we had earlier, our aim here the with the, so long why you are studying this is that we want our result for flow about a circular cylinder to convert it to more useful result about flow over airfoils, that is our main interest or other only interest here, that we want to transform our result for flow over a circular cylinder to flow over an airfoil.

Now, a circle is a very smooth, body and if we use an perfectly analytic function or the function which is analytic everywhere with all regular points by the way, the point critical the other points are called regular points, where the derivative is not zero, the function is analytic, the derivatives are not zero they will be called regular point of transformation, so at the regular point of transformation. So, we will mark it here also that other points are regular points

See, if all the points on the circle is regular point of the transformation that we would like to use, then the resulting geometry will also be smooth everywhere, but we know the airfoils are not smooth, airfoils we usually keep a sharp trailing edge. So of course, with a fully analytic function or fully regular transformation we would not get an airfoil, so we need this type of critical points within or transformation. So, that is why the critical points are very important, and we must know what happens to the angle at the critical points. It will be zero or pi, It will be zero or pi.

Let us see what happens, let us say that we consider a critical point at z naught at z naught. Remember critical point is a at the critical point the function is analytic; critical point is not singular point. So, near the critical point we can expand the function in Taylor series, critical point is a point where the function is analytic.

So, we can use a Taylor series expansion, and we write… Instead of writing all those, f prime, f double prime we will write it by some constant. Now, since z zero is a critical point we can clearly say that a one basically zero, because df dz at z naught is zero meaning a one is zero, because that is what is f prime. What is our a one? a one is this f prime z at z naught, and since the we have critical point at z naught means a one is zero, but we have no idea about the higher derivatives, a two is of course, the second derivative of f evaluated at z naught, a three is third derivative of f or third derivative of zeta evaluated at z naught and so on, and if you do not know anything about them.

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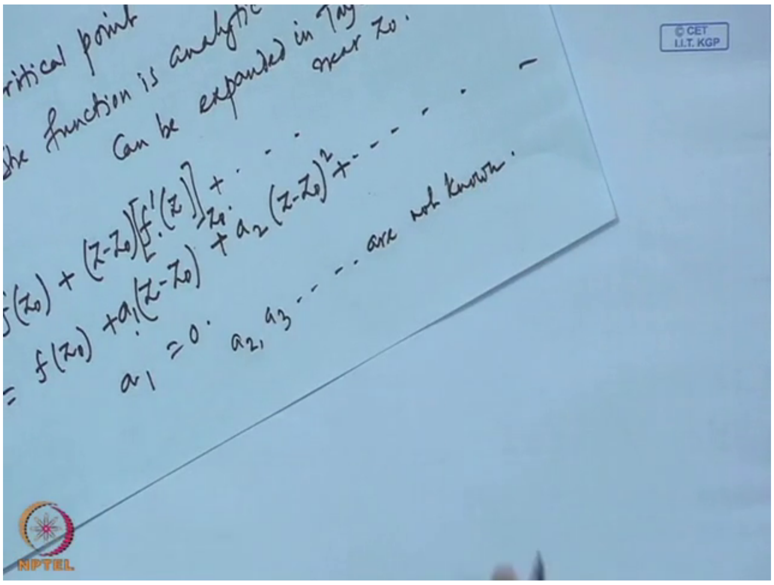

Let us assume that up to nth term all are zero, up to nth derivative all are zero, it can be anything.

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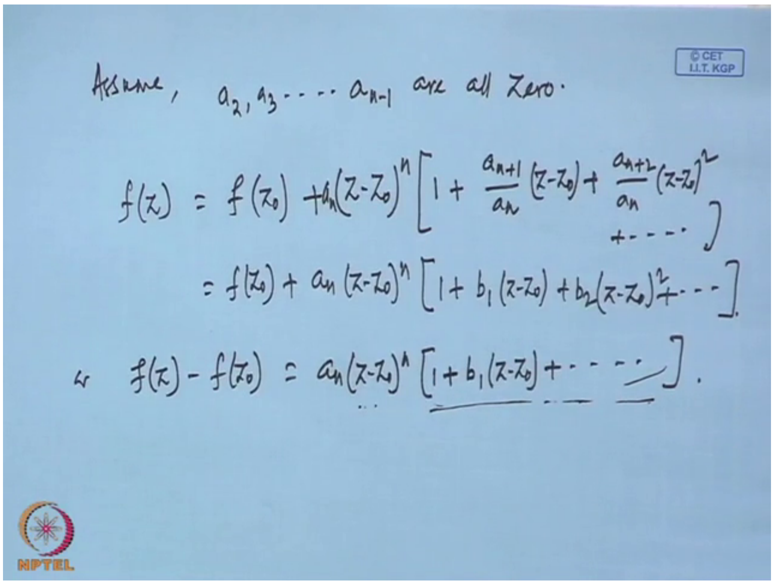

Just assume then what we will have, then we have hz equal to fz naught, and the first non zero term that we have is at the nth term, and from there we take this or let us take a n also… So, we see that the transformation near the critical point is now expressed by this, this is the same transformation we are now expressing it by this. Now, looking to this part you can see that here there is no critical point here b 1 is not zero, because that is the way we have taken it, so this part there is no critical point.

So, the critical point or the effect of the critical point is evaluated here only in z minus z naught to the power n, the transformation by this part will preserve the angle, only this part we have to check what will happen to the angle. Now, if you apply a transformation z to the power n, what happen to the angle at z naught? The angle increases by or changes by n times, z to the power n. If you have re to the power theta i theta in z, and zeta equal to z e to the power n it is up to the power n e to the power i n theta.

So, the angle becomes n times, what is n? The first derivative which is non zero that is what we have assume that up to nth derivative all are zero, so the first non zero derivative is n nth derivative, so the angle is multiplied by n times or the first derivative which is non zero.