**Introduction to Aerodynamics**

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**Lecture No. # 21**

**Equations of Motions**

We will continue our discussion with the equations of motion governing the fluid flow, though basically we have completed that derivation of all the important equations or the equations that are required for different flow situations. We look for some very important or special form of these equations and for that let us start with the energy equation that we have already derived.

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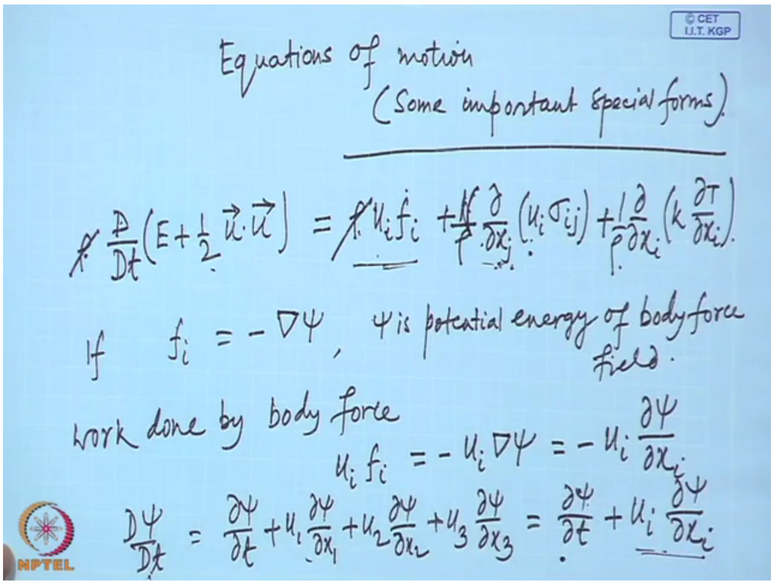


So, what we will do today is basically equations of motion, some important special forms. And as you will see, that in many situation these special forms, are perhaps more useful than the original or the complete form. Say first, let us start with the energy equation and we will write the energy equation as we started with that is in the energy equation we will consider first of all that the energy of a moving fluid for any elemental material volume is e plus half the kinetic energy is equal to the work done by the body forces plus the work done by the surface forces sorry plus the heat transfer term through conduction.

Then we saw that one term from this, this can be written in two terms and one term along with this will cancel this kinetic energy part on the left hand side and we wrote the equation in terms of internal energy alone. This time we are not going to cancel that kinetic energy part from the from both the sides here there is a kinetic energy and kinetic energy term comes from here also but, we are not going to cancel it now. Rather, we will try to write this equation in a different form. First of all we will assume that the body force is a potential force or what we are going to do is valid for the cases when the body force is a potential force. If the body force is a potential then it can be written as a gradient of a scalar potential function which actually represent the potential energy associated with the body force field. So, you write if f i equal to minus grad psi, that is if the body force is a conservative or potential field where psi is potential energy associated with the body force field and then just consider the first term let us see what happens to this first term. And for another case we will do this we will divide this equation throughout by rho, so that there will be no rho here, there will be no rho here, no rho here but, there will be one by rho here and here. So, we will write this equation in that form one by rho and one by rho.

Now, this term what happen the work done by the body force is what? u i f i that equal to minus u i grad psi which in index notation is minus u i d psi d x i. Now, can you tell me what is the material derivative of psi, d psi d t material derivative, d psi d t is what? d psi d t plus u 1 d d psi d x 1 plus u 2 d psi d x 2 plus u 3 d psi d x 3 and this can be written as d psi d t into plus these three terms together in our index notation is u i d psi d x I, which you see this is…

Now if we take it that this potential field is independent of time that is this function psi is independent of time then this term is zero and this itself is becomes this. If this is 0 if psi is independent of t then this d psi d t is same as this and psi is independent of time is see the at least true for the most common body force which is the gravitational force which is independent of time. So, at least for the most common body force this approximation is quite valid that psi is independent of t.

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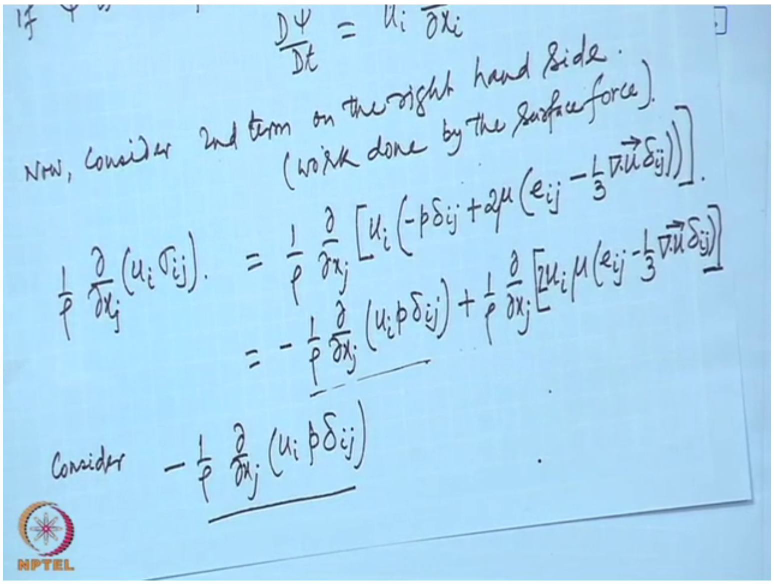


So, if psi is independent of t d psi d t is same as u i d psi d x i. So, the first term is in that energy equation first term on the right hand side on the energy equation can be written as minus d psi d t.

Now, consider the second term on the right hand side in the energy equation, second term on the right hand side what we have, we have the work done by the surface force. This is the work done by the surface force, the term is one by rho d d x j of u i sigma i j. Now write what is sigma i j? 1 by rho d d x j of u i sigma i j if you remember it is minus p delta i j plus, 2 mu e i j minus, one third either we can write divergence of u or e k k or d u k, d x k anything lets write only divergence of u into delta i j. This is the second term. Take only the pressure part what it is this becomes minus 1 by rho d d x j of u i p delta i j plus this term remain 2 u i mu into e i j minus one third divergence of u delta i j. Now consider only this term only the pressure term work done by the pressure force.

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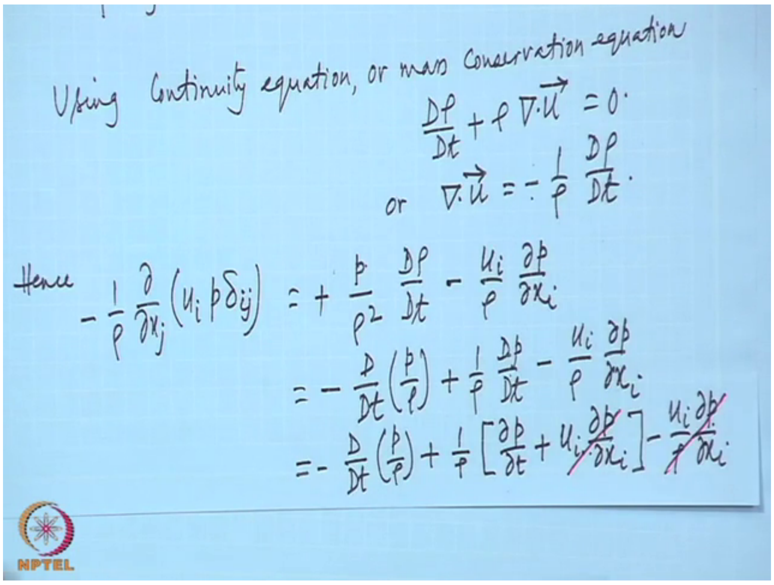


This you can write as, what we can write for that term minus 1 by rho d d x j u i p delta i j. We will write it separate differential term this product of two will write as a derivative of each so the first term will be let us say that p delta i j is one variable and u i is another variable, so this will be written as that minus 1 by rho will rho will remain of course, minus 1 by rho into p delta i j d u i d x j now delta i j d u i d x j that makes it d u i d x i. So, the first term will become p into d u i d x i and this second term will become u i d p d x i.

Now, this d u i d x i is divergence of u, if we use continuity equation. So, if we use continuity equation what happen using continuity equation or mass conservation equation, mass conservation equation if you remember says d rho d t plus rho divergence of u equal to zero. So, this is divergence of u so from here we can write it is minus 1 by rho d rho d t. Divergence of u. So, we substitute this here. So, the first term becomes minus p by rho square d rho d t. This becomes the first term becomes plus, one more negative here. This we now want to write this first part as d d t of p by rho then what it will be? We just want to write this as d d t of p by rho. What we will get then? p by rho square d rho d t.

Student: D d t of minus p by rho

Yes

Student: D d t of minus p by rho

Minus d d t of p by rho that is all or anything else? This is same as minus d d t of p by rho, No plus something p by rho square d rho d t is not simply minus d d t of p by rho. There will be 1 by rho d p d t also. So, this you can very easily check minus d d t of p by rho this is nothing new to you plus 1 by rho d p d t. This is what is p by rho square d rho d t. Write d p d t this term in full. What this becomes d p d t is this plus this is what is this material derivative. See this term and this term cancels, this term and this term cancels, this 1 by rho is here so this term and this term cancels.

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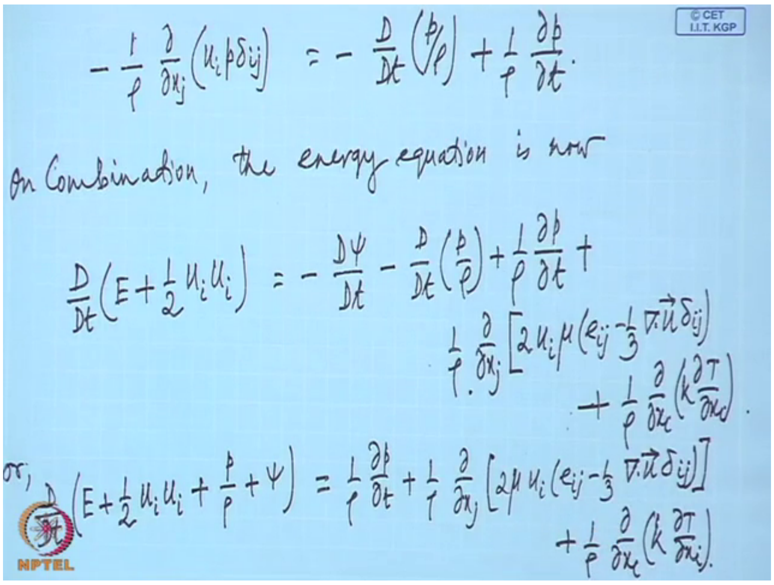


So this first term or the work done by the pressure force, now becomes what minus d d t of p by rho plus 1 by rho d p d t and if somehow the pressure field is steady then this second term will become zero, if the pressure field is steady the second term will become zero. Of course, it is not in general when if the pressure field becomes steady then only this second term will become zero and the work done by the pressure, pressure is simply given by this term. Which also if you look to that potential energy term this term is similar to that potential energy term as if the work done by pressure is similar to the work done by the potential force in this case the potential is p by rho, in the case of body force the potential energy associated is psi, in the case of work done by the pressure field we can say again that it is like almost like a potential energy the work done by the pressure field in which case the potential function or the potential energy function is p by rho but, associated with this force.

Now let us combine all the term on combination the energy equation now becomes d d t plus e plus of the work done by the body force, work done by the pressure force plus whatever remained in that 1 by rho d d x j of 2 u i mu e i j minus one third plus 1 by rho d d x i of these two term which contains the material derivative will also bring to the left hand side.

As before if we what we have mentioned that if we consider the pressure field to be steady just like this body force potential energy, the pressure field is also steady then this first term on the right hand side becomes zero in addition if somehow these two terms are also zero then the entire right hand side become zero, these are what, this is the work done by the viscous stresses and this is of course, the conductive heat transfer. So, if the work done by the viscous stresses and the heat transfer due to conduction are zero or negligible and the pressure field is steady then the entire right hand side is zero.

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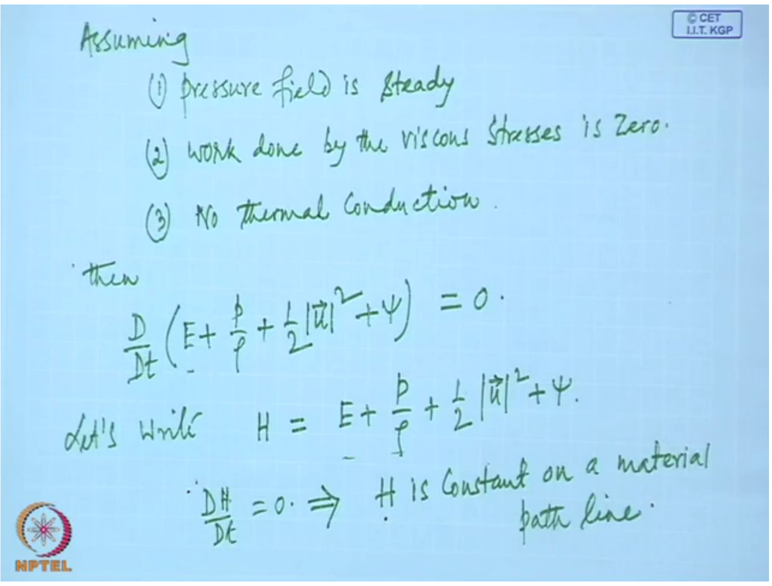


So, if you now assume then what we are getting is that u dot u or u i u i is the square of the magnitude of the velocity. So, we can write it this way. Now, what is the meaning of this equation? What is the meaning of this equation? Let us say that entire term within the bracket lets write it by any one notation something what should we say? Call it H, then we have this, So what? What does this mean? This variable H is?

Student: (( ))

Independent of time and space. Is it so? It will be independent of time and space if the partial derivative of this quantity with respect to each of these with respect to time and all three coordinates of space where identically zero then this is independent of time and space but, does this equation mean that? It may not be like say the partial derivative with respect to time or partial derivative with respect to x y x coordinate or y coordinate may not be zero but, that sum because this d h d t has a special meaning, it is d d t plus u into d h d x plus or u 1 into d h d x plus u 2 into d h d y plus u 3 d h d y d z that is zero not individually d h d t or d h d x or d h d y they may not be zero, that combination is zero.

So, what does that mean? It is constant but, constant with respect to what? It is not independent of t x y z. This H is not independent of t x y z. See this d d t, it comes from the name itself. This d d t which we called as substantial derivative or material derivative, that means the derivative following a material element. So this function, this function h is independent or constant with respect to a material as it moves with respect to a moving material. So, what is it that represents moving material x y involving x y z and t of a moving material that is its path. So, this function H is constant on a path of the material element or a material path or what we called as a path line.

So, this implies H is constant on a material path line but, you see this is not always this needs all these conditions must be satisfied that the body force must be a potential force and that potential must be independent of time, the pressure field must be independent of time, the work done by all the viscous stresses is zero and there are no thermal conduction. Of course, we did not consider any outside direct heating. In the energy equation we have not considered any out direct heating to the fluid so that is of course, not there. So, there is no direct heating, no thermal conduction, work done by the viscous stresses is zero, the pressure field is steady, the body force is potential and the potential is independent of time.

If all these conditions satisfied then this quantity H which is given by this is independent of the material path line or constant on a material path line but, it is different on different path line. On a one particular path line it is remaining constant but, as we move to another path line again it is constant but, it is at different value of the constant. So on each path it has separate value.

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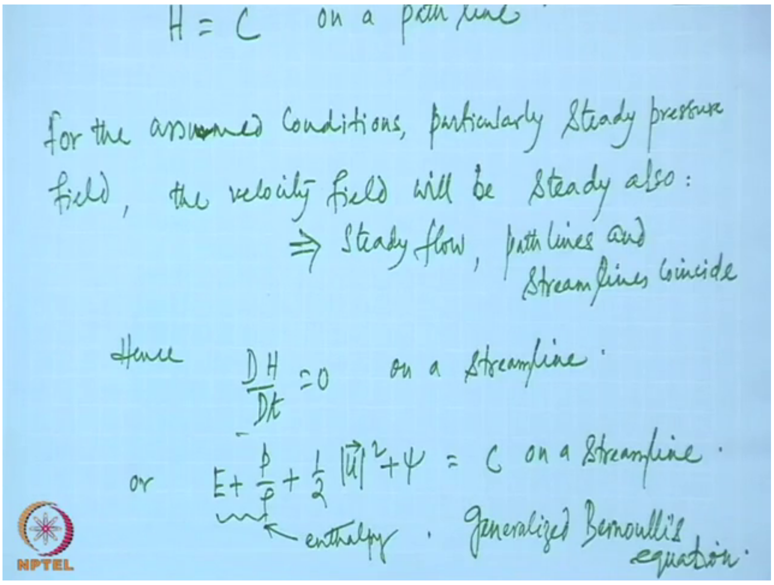


So, this can also be written that H equal to constant on a path line. Now, we see that these necessitates that the pressure field must be steady. Only then this is, that is one of the most important condition that the pressure field has to be steady. Now, if the pressure field is steady it is most likely that the velocity field will also will be steady most likely along with the other condition that the stresses by the viscous work done by the viscous stresses are zero, all these are all there and in addition the pressure field is also steady then it is expected that the velocity field is also steady. So, for the given conditions or for the assumed condition.

Now, that means the flow is steady and we know that on a steady flow the path line and streamline coincides. So, this means are steady flow, so we can now say that this H equal to this constant is also on a streamline. That is on each and every streamline this H is constant but, the value of the constant changes from one streamline to the other. It is not that everywhere it is same, it is constant on a particular streamline but, if we change the streamline it will change but, again then it will change and remain again constant on that new value at on that streamline. You have come across these equation or you have seen something like this?

e plus p by rho, is it familiar? That term is familiar to you what is e plus p by rho? e is internal energy and p by rho is what? Internal energy plus p by rho? Density is reciprocal of specific volume, density is reciprocal of specific volume. Have you come across these quantity in thermodynamics e plus perhaps in thermodynamics you have written e plus p v.

What is that? See the left hand side now each term is per unit mass e plus p p by rho is enthalpy. So, this is enthalpy. We have come across these equation. What is this? You have not come across this equation. Let us see if you drop this e part, have you come across this equation?

Student: (( ))

Bernoulli's equation. The way Bernoulli derived this equation that was Bernoulli's equation. So, here we can call this a generalized Bernoulli's equation or else it is little more general. So let us call it as generalized Bernoulli's equation or more complete form of Bernoulli's equation which Bernoulli himself did not derive. Bernoulli derived without this part. We will come to that equation and see why but, you can see from here itself that if the situation is such that the change in internal energy is negligible, if the change in internal energy is negligible that mean yours change in velocity will give you change in kinetic energy.

That may cause the change in internal energy provided sufficient amount of the kinetic energy goes into that internal energy but, if the internal energy is very small and then its changes something that is hardly going to affect the internal energy, that is a very common situation. If the kinetic energy concerned kinetic energy of this bulk motion or this preview work they are so small that they hardly affect the internal energy then we can drop this internal energy from this equation because that is so huge that it hardly changes because of this. In that situation this equation become that.

We will come to that equation also but, let us now first of all consider what is now happening. We are saying that this constant will change from one streamline to the other. We will see what that change is and is there any condition where that constant will be same throughout, that is what we will now try to look and for that let us see this equation.

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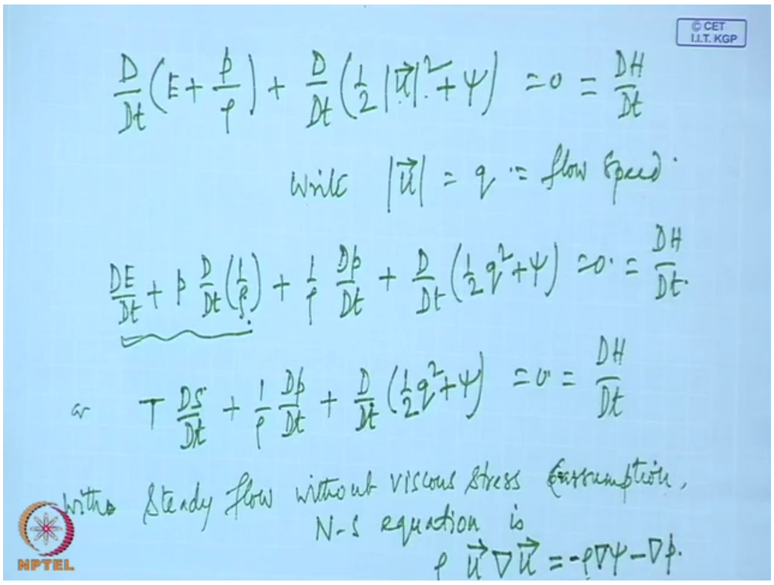


Once again come back to this equation say d d t of e plus p by rho plus this part we will write separately d d t of, I think we should change write something separate for this quantity, speed of the, this is actually the square of the speed so for speed I think we should write something, lets write q for speed so that we do not have to write every time this x. So, we will just write this the speed. Now, this first term again we will separate you can write it say d e d t plus.

Are you familiar with this equation? What is called the combined first and second law t d s equal to d plus p d v in thermodynamics. So look to now this first two terms what it is and in thermo dynamical relation treat everything, all terms has a rate term. In thermodynamics usually we do not write it as a rate term, we simply write t d s delta s, change in entropy, change in internal energy but, simply think about rate of change. So, t delta s can be thought of as t del d s d t equal to d e d t plus p d v d t. See this d e d t plus p this is the 1 by rho is specific volume, so p d v d t, so this is t d s d t, this is entropy..

Now, all these three terms together can you say what they that is? Compare this equation with the Navier-stokes equation, compare this equation with the navier-stokes equation and you see already here we have considered the work done by the viscous stresses are 0 or the viscous stresses are zero. Same approximation you apply there also in the Navier-stokes equation. So, forget about that viscous stress term on the Navier-stokes and all time dependent term. We have already taken that the flow is steady, pressure field is steady only then you have got these equation assuming that the pressure field is steady, the velocity field is steady or over all the flow is steady and also that viscous stresses are not there those assumptions already we have taken with that assumptions what happens to the Navier-stokes equation?

So, with steady flow, steady flow without viscous stresses assumption that navier-stokes equations is the left hand side d u d t or rho d u d t the first term d u d t vanishes. So, this remains equal to the body force which already we have written minus rho grad psi and the pressure force which is minus grad p, the body force term is written as rho grad psi as here also and the viscous terms up of this were there, they are also vanishing and on this left hand side that d u d t partial derivative of the velocity with respect to time that term vanishes from the material derivative. So, this is what is the equation.

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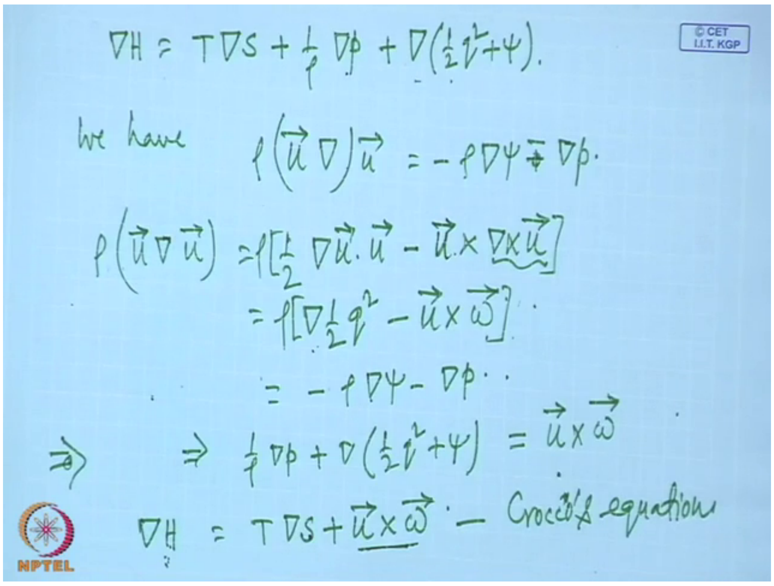


Now from this equation let us forget again this rate term and go back to that conventional thermo dynamical term. What it gives? grad H equal to t s plus 1 by rho grad p. Now, in that navier-stokes equation. We already have this grad psi as here and if we multiply this by rho that comes, grad p is there. Only this term is little different. So, let us see what we can do. We have or let it be sorry this is minus. Now this is a vector identity and this vector identity this u, u can be written as this is product of 3 vector one of them is of course, an operator. So, using that vector product rules this is what it comes and this is of course, half q square and this can be written as u cross, this is what is the vorticity?

So, this so if you substitute this here what do we get and this equal to minus rho grad psi minus grad p. So, this equation had rho also here sorry. Now you can substitute these here what do we get then? These two combine these two, this equal to this, grad of half q square plus c plus 1 by rho equal to u cross omega. Combining these, this gives what? 1 by rho grad p plus grad of half q square plus psi and we can substitute it here.

This equation will be useful later on but, at this stage only this will be the only utility here that this equation is known as Crocco's equation. Crocco's equation and as you can see, this is not a new equation this is a simplified form of the energy equation we have obtained it from the energy equation itself but, for a very special cases and what you find here the change is change in this quantity H is due to change in this entropy and this. So, if the entropy does not change the flow is isentropic and if omega is zero then the right hand side will become zero and this h will not change, so that constant which varies from streamline to streamline will not vary if the flow entropy does not change and the flow is irrotational.