Introduction to Aerodynamics

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Lecture No. # 19

Equations of Fluid Motion - Navier-Stokes Equations (Contd.)

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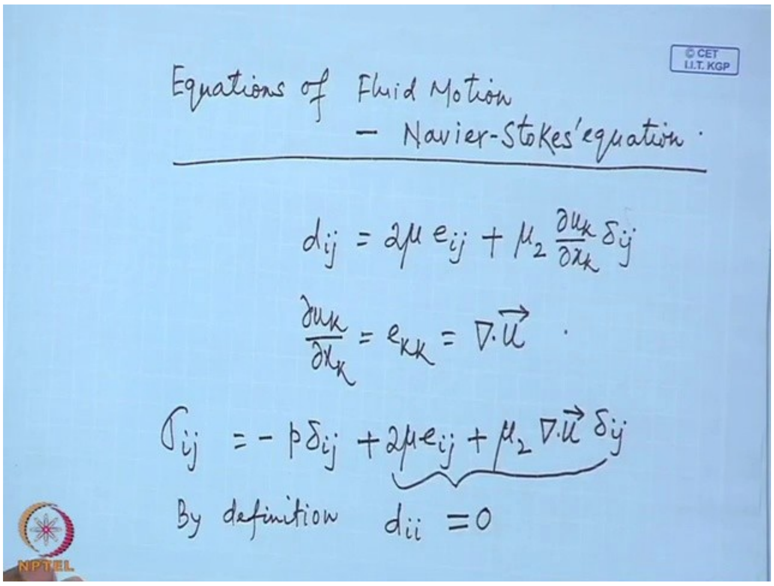

So, we will continue with our derivation of that Navier-Stokes equations or the equation of motion - equations of fluid motion. And the equation that we are now deriving as you mentioned is known as Navier-Stokes equations, because both Navier and Stokes they derived these equations independently.

Now, we derived an expression for the deviatiric stress tensor, if you remember d ij was 2 mu e ij plus, I think we wrote mu 2 du k dx k delta ij, and this du k dx k as you know is the rate of expansion or dilatation and in the conventional vector notation can be written as a divergence of the velocity with these the complete stress tensor we wrote sigma ij as minus p delta ij plus 2 mu e ij plus mu 2 divergence of u delta ij.

Now, looking to this deviatiric part of the stress tensor, it is the normal component in this deviatiric stress tensor are the remaining part of the normal stresses from which we have taken out this isotropic part, this minus p which is the 1 average of the sum of the diagonal, this we have taken out from sigma ij.

So, what remain in the say sigma 1, 1, sigma 2, 2 and sigma 3, 3, that is contain in this part d ij and since, some of the average is average of the sum is minus p then, the remaining that remains sigma 1, 1 minus p sigma 2, 2 minus p sigma 3, 3 minus p when you sum them that is 0, that means the sum of the diagonal elements of d ij is 0.

So, this is by definition that d ii is 0.

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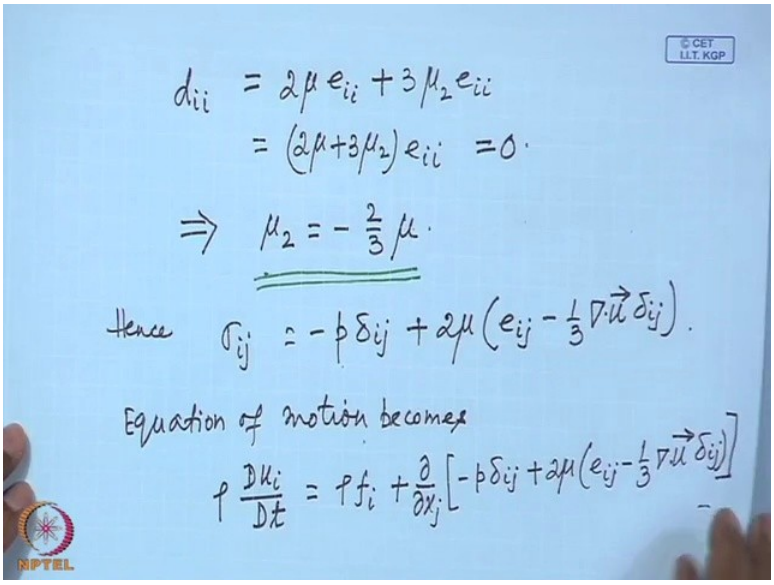

Now, let us see what does it mean? Now, what is d ii? Sum of the 3 elements 2 mu e ii plus what will be that? A second term, when you contact, when you make i equal to j, what will that be term be?

You want to think about the 3 terms separately say mu 2 du k dx k delta 11, plus mu 2 du k dx k delta 2, 2 plus mu 2 du k x, k delta 3, 3. So, what it will be mu 2 du 2 dx k into delta 1, 1 plus delta 2,2 plus delta 3, 3 what is delta 1 1 plus delta 2, 2 plus delta ,3 3? 3. So, that term becomes 3 mu 2 du k dx k du k dx k is same as e kk we can write as e ii; the second term becomes 3 mu 2 e ii, and this is equal to 0 but the definition of our deviatiric stress tensor this is 0, and this gives us mu 2 equal to minus 2 by 3 mu. So, this we have been able to even eliminate second coefficient of viscosity by the coefficient of viscosity itself.

So, if we substitute now then this sigma ij becomes minus p delta ij plus 2 mu into e ij minus one third divergence of u delta ij. Then this we can now substitute in the equation of motion. The equation of motion, if you remember was the equation of motion becomes what? By the way this is before you write down this equation, this relation that mu 2 second coefficient of viscosity is minus 2 by third mu, this is usually known as stokes hypothesis of viscosity.

Now, equation of motion becomes look back, it is rho du i dt equal to rho f i plus d dx j of sigma ij so that we can now write d dx j of minus p delta ij by the way as you can clearly see that this equation is written in terms of unit volume basis. It is clear; it is conventional to write the equation in terms of mass basis in solid mechanics, while in fluid mechanics it is conventional to write the equation in terms of unit volume basis.

So, in this equation, each term in this equation represents some force per unit volume as you can say, the first term on the right hand side is simply the body force per unit volume and this entire term this second term is the surface force per unit volume, the left hand side is of course, the inertial force which contains the local rate of change of momentum plus the convective transport of momentum. The left hand side contains as you know that this material derivative with sum of 2, a local derivative which is simply the partial derivative with respect to time.

So, that gives the local rate of change of momentum, and the remaining three terms u y du y dx j sorry u j du y dx j that gives the convective transport of momentum. The way momentum is being convicted, you know now looking here this first term within the bracket d dx j minus p delta ij.

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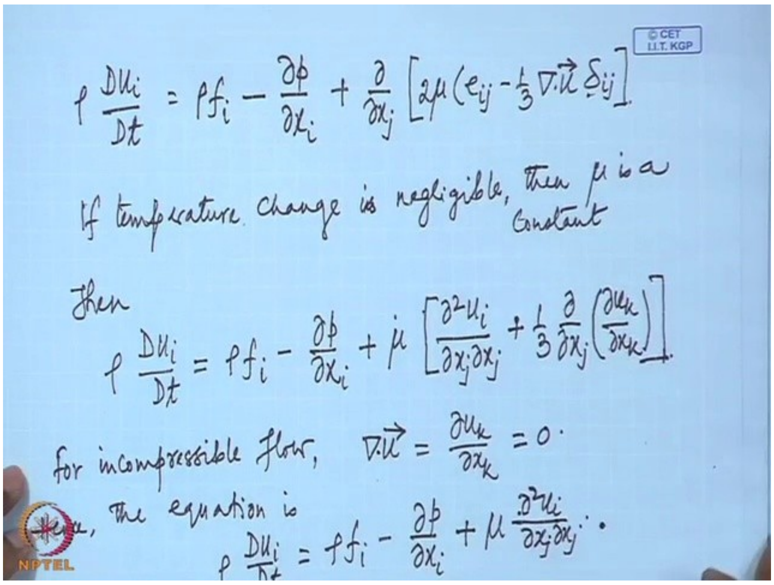

We will take it out from this bracket, and for then that term will become? That term can be written as minus dp dx i d dx j of minus p delta ij is minus dp dx i plus the remaining term d dx j of 2 mu e ij minus one third divergence of u delta ij, this dp dx i is basically the gradient of pressure dp dx i.

Now, see this coefficient of viscosity is in general a function of pressure and temperature, and not only that its dependence on pressure is extremely weak, so that all practical purposes coefficient of viscosity can be thought of as a function of temperature alone, so if this coefficient of viscosity as a function of temperature alone and there are many flow problems in real life or in practical applications, where the temperature variation is in really not large, or not considerable, temperature variation are really negligible.

So, if in that situation this mu can be thought of just a constant, if there is no change in temperature, or practically no change in temperature; then mu can be taken as constant. So, you can write that, if temperature change is negligible, and then this mu can be taken out of the derivative, and the equation will become. where you have replaced that e ij is half of du y dx j plus d du j dx I, we have replaced e ij by its definition, and that divergence of u once again you have written in this form du k dx k.

Now, if the flow is incompressible, if the flow is incompressible we know by conservation of mass, or by equation of continuity, divergence of u is 0 that means, this second term is 0 there, and this equation becomes So, this is the final form of the Navier-Stokes equation, if the flow is incompressible and this will be taken as the final form of the Navier-Stokes equation, when the flow is not incompressible, and mu is not constant mu cannot be take temperature changes are not negligible.

What is the relation between mu and (( )) the equation? There are many relations the most popular relation used is called as Sutherland relation which is a can be expressed as something like this mu by mu 0, is t by t 0 to the power 3 by 2 into a constant; the constant is usually written as t 0 t plus s by t 0 plus s where, s is again a constant, t 0 is reference temperature, and mu 0 is a reference viscosity a Sutherland relation but particularly in this course and also the next course on aerodynamics, we will consider mostly incompressible flow.

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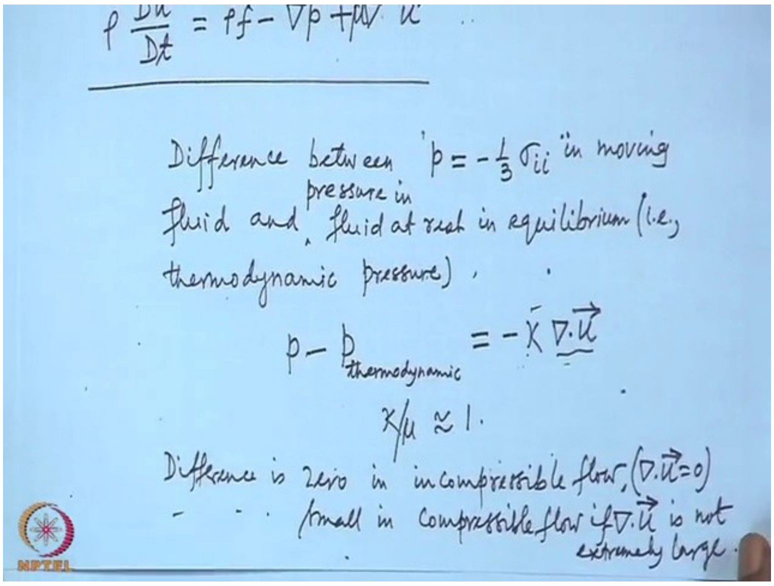

So, this is the equation that we will be using quite often or even perhaps some simplified form of this equation. In the standard vector notation and this equation we can write as rho in the standard left hand equation, and we have to remember this equation always in all form.

Now, this is of course, a vector equation means it can be written for three components, three separate equations. Now, let us summarize at this stage as far as governing equations of fluid motions are concerned. We now have two equations of course; one of them is in vector so, basically four equations the continuity equation, and these three components of the Navier-Stokes equations, so we have four partial differential equations.

How is the (( )) drop there when you do lambda square u square why is the mu drop? Sorry it is not dropped; it was just over sided mistake, so these four partial differential equation of course, the continuity equation, or equation of mass conservation is a first order partial differential equation, while this is a second order partial differential equation. The equations are non-linear because, in both cases this term is present whether the continuity equation, or this Navier-Stokes equation this term is present which as you know can be written as du dt plus u j du y dx j.

So, that term u g du y dx j that is non-linear, the coefficients are themselves unknown functions, or the actual variables so the equations are non non-linear partial differential equation, anyway and these four equations and how many unknown we have here at this stage we? Have three component of velocity so 3 unknown plus pressure and density.

So, we have five unknown in four equations, so we need 1 more equation to supplement it, and the easiest equation that perhaps we can think of is the equation of state, which relates any particular state variable in terms of two other state variables, and the most often the density, or pressure in terms of say let us a pressure in terms of density and temperature, but if we use that immediately we get 1 more unknown the temperature then, we need 1 more equation, and also at this stage perhaps I should mention something about this pressure again, if we use the equation of state the thermo dynamical equation of state, the common example, the equation of state for a perfect gas as you know p equal to rho RT.

Now, we have already mentioned that the pressure has defined in thermodynamics as a state variable is not the same the way we have defined the pressure, here which you have taken as simply the average of the sum of the surface or normal surface stresses. The basic difference comes in from here that in thermodynamics all state properties are defined using a thermo thermal and mechanical equilibrium and of course, in a fluid motion they are not exactly in thermal and mechanical equilibrium.

So, the quantities which are defined as state variable in equilibrium thermodynamics is likely to have certain differences when they are not exactly in equilibrium, and without going for any mathematical proof or anything, I will just state that the difference, what is the difference of this pressure with the pressure in defined in thermodynamics, or the fluid static pressure. The fluid static pressure is same as the thermo dynamical pressure because, when the fluid is at rest it is also in mechanical or thermal equilibrium, what is assumed in case of thermodynamics.

So, the fluid static pressure is same as the thermo dynamical pressure or the equilibrium pressure but pressure in moving fluid which is one third of the sum of the diagonal terms of the stress tensor is not there the same. The difference in quantitative form we will not go for any proof or anything, or even derivation, we will just simply state the difference between, this miss thermodynamic or equilibrium pressure we are just the difference between these two pressures is given by this. The derivation is to some extent similar, the way we have derived that deviatiric stress tensor considering isotropic fluid and all those things we did last time.

So, similar in the similar manner this also can be derived, but we will exclude that from here, we will not consider and the magnitude is given by this where this is as you can say is equal to or very close to the coefficient of viscosity, and divergence of u is the rate of expansion or dilatation. So, you see if the fluid is incompressible or if the flow is incompressible then diversions of u is 0 and the difference does not exist.

So, in case of incompressible motion even though the fluid is moving and not in exact thermodynamical and mechanical equilibrium, this difference is 0, that means; both can be taken as the same thing the pressure in a moving fluid is same as the thermodynamical pressure, or the static pressure, however, when the rate of expansion is non 0 as in case of a compressible flow, there is a difference, but again this quantity being very small this quantity being very small even though rate of expansion is larger still the difference is very small.

However, there are some cases where the rate of expansion approaches very large value, extremely large value, then of course, this is considerable, but in most usual cases where the rate of expansion is not that phi, is some average or moderate value again these are negligible. So, in usual cases in we need not consider a difference between these 2 pressures and hence, forth in all our discussion we do not have to consider that discussion that difference. We will take it for granted that whatever pressure we are finding that is the same as the thermodynamic or equilibrium pressure as well as the pressure in a moving fluid. Now, using a vector identity Laplacian, the operator Laplacian can be written as gradient divergence of u minus curl of curl of u.

Now, since we are dealing with incompressible flow this first term is 0 the first term is 0 because, divergence of u is 0 for incompressible flow, so this becomes then minus mu curl of, this curl of u is omega the vorticity. So we see that viscous flows per unit volume is obtained or given by the curl of vorticity curl of vorticity.

Now, while deriving this equation we have already seen that this term has come from viscous stresses, or the shear stresses, and the shear stresses come only because, of strain rate and rather we have from there that the vorticity part has no contribution to a stress. Vorticity does not contribute anything to the stress to the shear stress. So, vorticity has no contribution towards shear stress, and shear stress is a reason of the viscous force which is given by the vorticity.

How is it possible? We have seen correctly this is not only a mathematical, pure simply mathematical derivation, it was even physically correct that the rigid body rotational type of motion or the vorticity has no contribution towards the shear stress. The shear stress is given by strain rate alone and only shear stress is what is giving the viscous force, and then we are seeing that the viscous force is given by the vorticity.

So, it cannot give the stress but it gives the force, how is it possible? Any answer? So, you think that imagine a situation the flow is irrotational, a flow which is irrotational there are many such flows which are irrotational. Then this viscous force is 0 even not only irrotational even, if the vorticity is uniform vorticity is uniform everywhere, then also this is 0. So, is it that whenever the flow is irrotational there are no shear stresses, because viscous force is 0 that of course, we cannot say because, you know that vorticity has no contribution towards shear stress, that comes because, of the strain rates a particular is the tangential strains then, what is the problem? What is this discrepancy?

This equation to be looked into, this particular equation is to be looked into just a mathematical relation because, of the identity of this vector operator this is not a cause and effect relation this is not a cause and effect relation this is just a mathematical coincidence that the viscous force has become curl of the vorticity just a mathematical coincidence that the components of viscous forces are obtained from the components of the curl of vorticity vector, and has come because, of this vector identity by which Laplacian is expressed as gradient divergence minus curl of curl.

So, this will be true that, if the flow is, if vorticity is 0, or vorticity is uniform then the viscous force per unit volume will have 0 magnitude, but the viscous stresses will be non 0, will may be non 0, but the viscous stresses will give a total viscous force of 0 magnitude. So, while strain rate of strain and viscous force are cause and effect, this vorticity and viscous force equality is not cause and effect relationship, just a mathematical coincidence, a mathematical relation.

Once again we I think we left that discussion in the in between that at this stage when you have the conservation law for mass and momentum giving rise to four equations in five unknown, and two sub give at one more equation, we can hold back upon the equation of state p equal to rho RT, and then of course, we derived that difference between the 2 p the p in the momentum equation and the pressure in the equation of state.

However, so we need one more equation to have because, as we had a short of one equation, we added equation of state, but when we added equation of state; we added one more variable. So, obviously we need one more equation, but look to the case of incompressible flow, in incompressible flow we have already stated that by definition incompressible flows are those flows, where density does not change due to pressure where density does not change due to pressure.

In addition now if we consider that the temperature changes are very small or negligible that is, practically temperature do not change then, density remain constant, density can change because of pressure, and temperature, incompressible flow means the density does not change because of pressure. So, if temperature is also a constant then density does not change at all, and most often by incompressible flow we take that, the density is a constant. So, when density is a constant now we have only four unknown, density is no longer an unknown; it is a constant; a constant also can be an unknown, but we will take that constant is known constant, or even, if we want to find that constant we can consider any one point where, other conditions are known, some conditions somewhere we must be known.

So, from that point we can find the density and that remain constant so, density no longer unknown for the differential equation. So, we have four unknown now three components of velocity, and pressure. And we have four equations; the equation of state in this case simply becomes rho equal to constant. So, we can solve these equations with assist theoretically that you can solve these four equations for the four variables and

See mu is unknown because, we are adding we will be having some relations, but we are not actually considering it as a unknown for the differential equations that is already supplemented by some equations, if the temperature, when the temperature will change when the temperature will change we will need one more equation, and not only that we will need a relationship for mu also, this we are considering where, m is constant. The temperature is not changing. So, mu is nearly constant and incompressible flow so density is also constant now.

So, this density and viscosity they all of them have become constant. So, the unknown that we have are; the three component of velocity, and pressure, and these four partial differential equation, the three component of these equation, and the equation of continuity these four equation together are sufficient that is mathematically to solve for these three unknown variables four unknown variables, 3 components of velocity and pressure, at all points; at all time theoretically.

However, there are certain other things that must we have like we must have specified boundary conditions, you must have some specified initial conditions, and those are requirement of solving any partial, any differential equation. Even ordinary or partial whatever it is. That you must have some boundary conditions, you must have some initial conditions as well.

So, provided we have all those the boundary conditions, and initial conditions then these four equations are sufficient to solve, however, unfortunately the equations as you are can see are quite old 200, 300 years old. These equations those Naviers and stokes they were in those time, unfortunately till date where in a given general boundary condition, or initial conditions; no solution has been found by mathematically, means only some simple cases only some simple cases the equation can be solved and we will discuss such simple cases.

However, for complete solutions ultimately one has to fall back upon numerical methods, but anyway that is quite far away as yet, we go to that numerical solution of these equations, quite late in your curriculum, but before that we will try how to solve, or how to get the solution instead of solving this equation completely? How to get the solution? But that we gradually build up.

Now, one or few more word about the case when the flow is not incompressible then, as we mentioned that this will be the strategy; that these four equations with five unknowns we can bring in one more equation by the equation of state usually, pressure expressed as a function of density, and temperature from thermodynamics. And we can have the difference between these two pressures, the pressure in the momentum equation, and the pressure in that equation of state, we can even neglect them because the difference is small.

However, introducing that equation; we have introduced one more unknown; the temperature, or may be internal energy. Sometime pressure is expressed in the form of or internal energy is a preferred variable than, temperature. And one more equation is needed however, we have one more conservation equation, which we have not discussed as yet; the conservation of energy, of course, a scalar equation and assuming the that is not giving any further unknown, further field variable; that will complete the set of equations, of course, in that case temperature being a variable we have to have certain relation for expressing the coefficient of viscosity, and anything else if we bring in that also must be expressed in as a function.

See, when there are partial differential equation it is customary to consider them to be the equation and any other equations as not as equations they are just relations. So, that if we have a viscosity relation as an example; that Sutherland relation we are talking about mu by mu 0 equal to say t by t 0 to the power 3 by t into t plus s by t 0 plus s this mu 0 and t 0 is at reference temperature and the viscosity; at that reference temperature. This s is a constant usually taken as 100, 10k, this temperature is in Kelvin. So, this is called Sutherland relation; so these types of relation are not called an equation, by equation then, we will be meaning only partial differential equation.