**Introduction to Aerodynamics**
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**Lecture No. # 15**

# Irrotational Solenoidal Flow in Multiply Connected Region

So, last time we have considered irrotational solenoidal part of the flow, that is that part of the flow for which the divergence is zero as well as the curl is zero. Since, the curl is zero this velocity field can be expressed by a scalar potential and then that scalar potential satisfies laplace equation. We have discussed some general properties of the laplace equation and consequently the general properties of the irrotational solenoidal flow field. We have also discussed that uniqueness condition in a singly connected domain. And we have seen that the irrotational solenoidal flow field can be determined uniquely, if we know or if we specify the normal component of the velocity on all the boundaries or the potential function itself over all the entire boundary or a combination of the two.

That is the normal component of velocity specified on some part of the boundary and the potential is specified on the remaining part of the boundary. All together it implies that in any case the boundary condition must be specified over the complete boundary which is said also is a requirement for the solution of laplace equation. Or in general in any elliptic partial differential equation, that the solution of an elliptic partial differential equation can be obtained if the boundary condition over the entire boundary is known knowing only on the part will not be sufficient, we have to know it over the entire part of the boundary.

Next we will see it for not singly connected domain or rather the multiply connected domain, when the flow field is multiply connected. We earlier mentioned that in the singly connected domain the potential function phi is a single valued, but it is not show or need not be show in a multiply connected domain, this comes from the very simple fact that in a singly connected domain.

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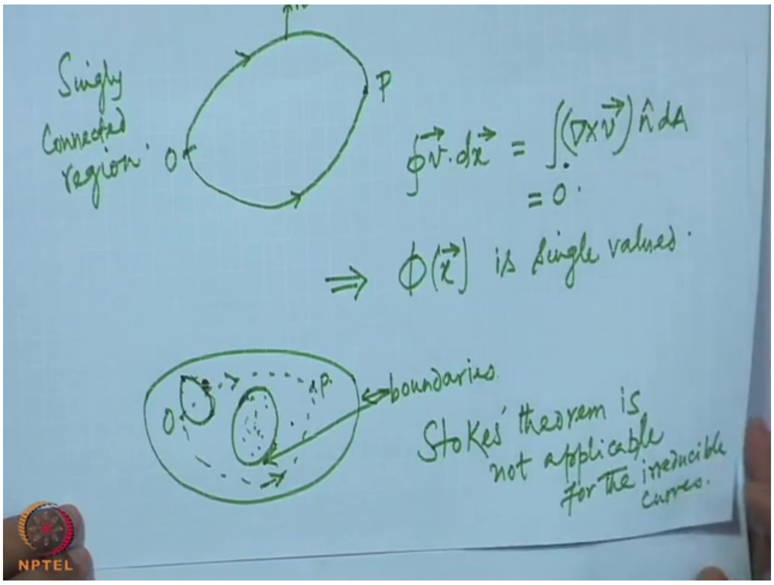


If we join two point by say two path say any two point o to p by any two path, since the domain is singly connected. This is a reducible curve that means we can shrink this curve this closed path to a point. And consequently these two curve together forms an open surface, and we can apply stokes theorem to it. There is a line integral of the velocity over this closed path is same as the curl of the vector field the velocity in this case over this surface. And since the curl of v for irrotational solenoidal flow is zero at each and every point that surface integral is zero, meaning that the complete line integral is zero.

That is what we took it that v dot d x is curl of v n d A, and this is zero and from this we concluded that since this v dot d x is same over all the path. So, v dot d x which is the difference in phi from one point to the other is same over all path. So, as you move from o to p whatever path v take the change in phi will be same consequently the p phi at p is also same, whatever path we follow or the potential phi is single valued. This is in singly connected domain or singly connected region. What happens? It is if this region is multiply connected? Then we cannot sink this closed curve to a point without going out of this region. Let us take that think about we have an infinite circular cylinder or any object.

So, this is another say this is a boundary this is not just a curve this is a boundary. Now if we have any two points say o to p, say this is one path and this is another path, now if we try to shrink this curve and even if we translated to another side we can never shrink it to a point without crossing this inner boundary interior boundary. And when we are crossing the interior boundary of course, we are going out of the region, because this part is not within the region this part is not within the region. So, we are going out of the region and we cannot shrink it to a point this is not a single connected domain. Then even though we now have a complete circuit integral over this closed path, we cannot equate it to this right hand side and we cannot say that phi is single valued in this case.

We cannot form this open surface over which we can take this surface integral this right hand side of this equation, we cannot have such a relation here. So, stokes theorem this is what is stokes theorem is not applicable for the irreducible curves, but remember that still, if this path is reducible like think about another path. Here from these two point we can have this type of path so in for this case alone you can apply but for these paths we cannot apply it.

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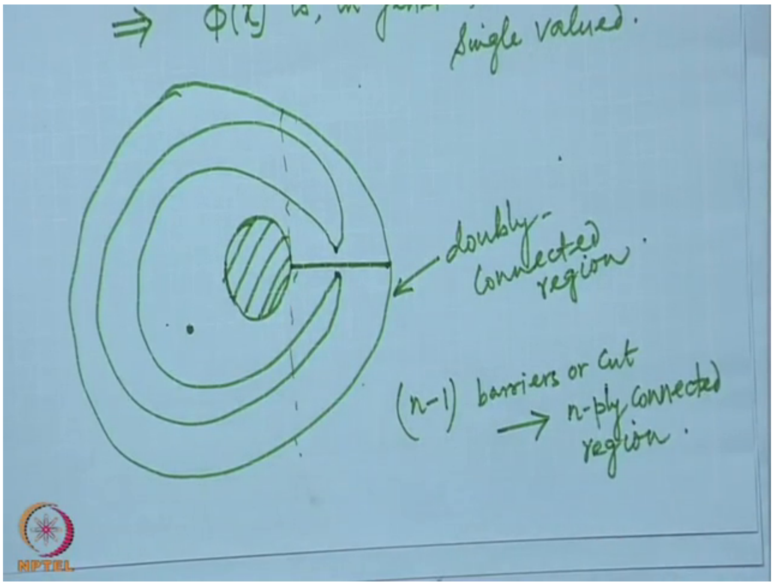


So, phi or the velocity potential is not necessarily single valued it may be single valued it may not be. Now before you proceed further let us say a few words about different type of regions and different type of curves. We are talking about reducible curves for some time and we have already discussed the reducible curves are those curves which can be shrinked to a point without going out of the region and in our case the region is simply the fluid. But quite often we will only call region or domain, because these are basically mathematical concept. This curve can also be stated that if there are two points within the region or within the fluid and if we can join them any two point, and if we can join them by a line without going out of the fluid or without going out of the region.

And then if any two such curves form a closed path this region is called simply connected or singly connected. That is the way properly this singly connected domain is defined, that if we take any two point within the region and can join these two points by a path without ever going out of the region. And if we can form two such path which make a reducible closed curve then that region is called simply connected if not the region is not simply connected. Now there is degree of connectivity, when it is not singly connected, it might be say doubly connected, it might be triply connected or it might be in general n-ply connected this is called the degree of connectivity. How do we define degree of connectivity? To that think about that we have any region a doubly connected in which there are some set of curves which cannot be shrunk to a point which are not reducible.

Now think about that we are inserting barriers within that domain the barriers, which has no thickness, but has only two sides say they are in the form of open surfaces and if by inserting these barriers we can make the region singly connected. Now depending upon that how many barriers are required the degree of connectivity is defined. If we need only one barrier to make the region simply connected then it is a doubly connected region, if we need two such barriers then it is a multiply connected a triply connected region. And if we need n minus one such barrier, it is simply connected region.

As an example look to this that we have think about this region we have an infinite cylinder and another boundary which is far away and there is fluid in between these, you can think about the situation flow over a cylinder. If the for a solid cylinder the solid part is of course, not fluid region that is out of the region. So, think about let us say this is and let us say that this is the region in the form of infinite cylinder. Now in this case you can see that any curve any closed curve that loops the circle they are irreducible. The curves which do not loop the cylinder they are reducible so here there are two different sets of curves some curves which loops the cylinder they are irreducible some curves which do not loop the cylinder they are reducible.

Now think about that we have inserted one barrier in this form so that this is also a boundary of the region. Insertion with this boundary still does not make this domain unconnected different regions of the domain still are connected they are not unconnected. This is also another requirement that when we insert the barriers we must not make the region unconnected like we cannot think about a barrier like this then these two part. If we think about a barrier like this that is not allowed, because that will make these two region completely unconnected, this barrier will also be thought of as a boundary it is no longer fluid.

Now see that just by inserting one such barrier this region has become singly connected. You can now think of any curve that can be shrunk to a point any closed curve like see if you take about these two points any two curves joining these two points will be these curves. These curves not crossing this because you cannot cross that is out of the boundary out of the region. So, just by a singly simple barrier single barrier we have been able to make this region singly connected so this is called a doubly connected region. If we need two such barriers it is triply connected region if we need n minus one barrier it is simply disconnected domain so n minus one barriers or some time simply called cut entire curve cylinder multiple times the entire curve circles (( )). How it can more once that is quite simple?

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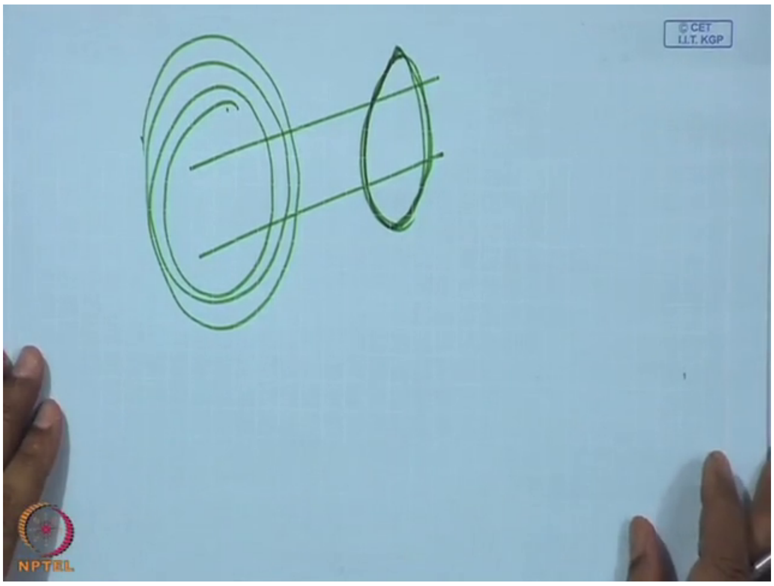

Think about again say this is the cylinder as many time as you want it will go on looping.

Student: (( )) it is not closed.

How do you closed? You end these two points they are closes here, I have a do not that and it can trace the same curve again and again that is also possible, but it is looping if we just consider say this the curve is now looping this cylinder many times.

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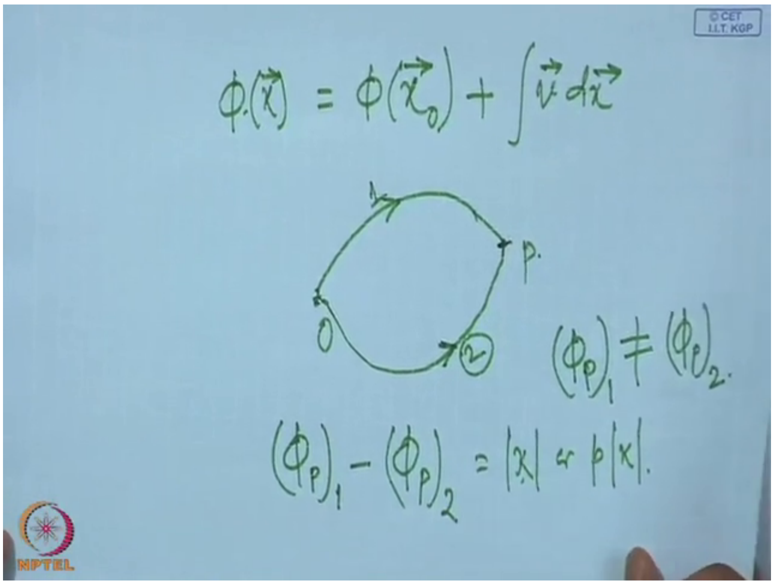


Now once again we define that the potential function as before that potential at point x, at position x is the potential at position initial point zero plus. In the case of singly connected domain if we have two point o and p then this whatever path we follow this integral would have been given this potential would have the same value singly connected. But in this case it may not be so by if we let us say that o to p this is x zero and this is x. If, we take this path or if we take say this path remember they are joined the phi p at this point may not be same the value will differ by the cyclic constant kappa. That is if we evaluate phi following this path and if we evaluate phi following this path we will get two different values of phi at point p but which will have a difference given by that cyclic constant.

Assuming that this paths are such that it is making one loop of closed curve, if we follow this path say we will call this path one path two. So, we call it phi p following path1 is not equal to phi p following path2 and phi p as the case may be as the case may be. If the two path make only one simple closed curve only one loop of the closed curve it is simply this kappa. But if it forms a complete p loop p number of loops then it is p timed kappa. However the gradient of phi that is the velocity part the irrotational solenoidal velocity part that is still single valued that is quite obvious because this cyclic constant. So, gradient of phi will not be affected by this cyclic constant gradient will remain single valued.

So, even though phi is many valued so this we have considered for two dimensional case, if we have a doubly connected region we have two values of phi at one point or even many valued also. If we think about the curve is looping and looping and in but the gradient of phi is still single valued, because this constant the cyclic constant is not affecting the gradient the derivatives are not affected.

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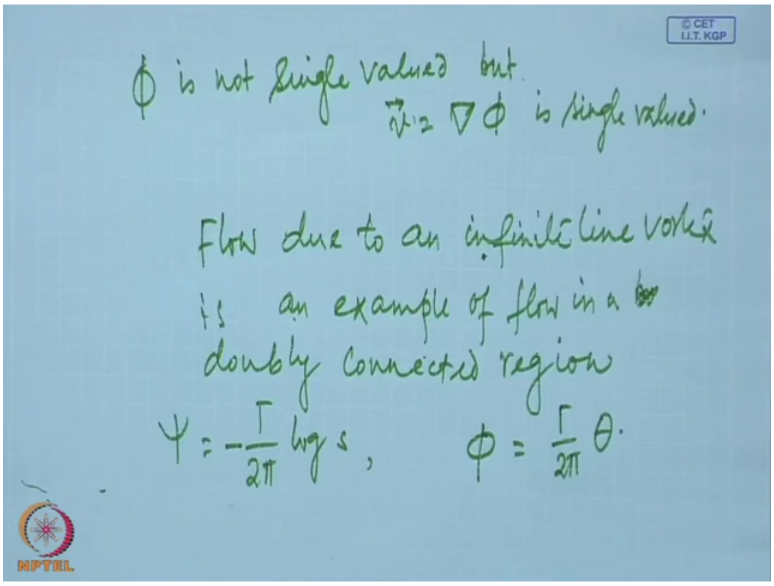


So, phi is not single valued, but grad phi is or we will say v equal to grad phi is and obviously then the once again that laplacian phi will still remain the equation or other behaviour which you have associated with velocity field. They will remain as it is the only thing is that phi is no longer single valued it is associated with the cyclic constant if a in a singly connected domain it is only one cyclic constant. If it is n ply connected region we need n number of cyclic constant, because we had n number of irreconcilable curves and applying this stokes theorem to the all those n number of irreconcilable curves will have n number of cyclic constants. In a doubly connected region we have only one single cyclic constant of course, in some cases this cyclic constant may be zero. In case it is zero if we find such a flow in which this cyclic constant is zero then that flow is called a cyclic or otherwise it is called a cyclic.

Now if we look back we have all ready found such a flow, in which there was a single cyclic constant or rather we consider the flow we can say that we have considered the flow in a doubly connected region. Very nice example that infinite line vertex that we consider, if we have a infinite line vertex of line is basically on limit of cylinder a line is a limiting cylinder. So, flow about that infinite line cylinder infinite line vertex is basically an infinite cylinder you can think of similar situation. And if look back to that velocity field that we obtained for a line vertex gamma by 2 pi s i think we wrote gamma by 2 pi s where s is the distance from the line vertex distance of the point from the line vertex, gamma by 2 pi s that is essentially an irrotational solenoidal flow in a doubly connected region. And in that case the strength of that line vertex gamma is the cyclic constant. We earlier expressed in that time we express that that flow has an stream function associated with it which was gamma by 2 pi log r or log s

What it was minus gamma by 2 pi s or just gamma by 2 pi log s potential or stream function we earlier.

Student: Gamma by two gamma by two.

The stream function we wrote as gamma by 2 pi s gamma by 2 pi log.

And now you can see that we can write can you write, what should be the potential function? For that flow a potential function can be written in that case flow due to an infinite line vertex the flow was only azimuthal. So, if you think in terms of r theta coordinate we had only u theta velocity u theta was that gamma by 2 pi r. And in terms of potential function. What is u theta? 1 by r d phi d theta the velocity is gradient of phi the velocity is gradient of phi and in r theta coordinate the gradient is d phi d r the r component is d phi d r and the theta component is 1 by r d phi d theta so 1 by r d phi d theta equal to gamma by 2 pi say s we can call r.

Student: (( ))

Minus which is minus.

Student: (( ))

Minus gamma by 2 pi.

Student: Log s.

This one is this 1 was minus gamma by 2 pi log s so what will be phi for that s we can write r also its r limiter.

Student: (( ))

Phi is how much gamma by 2 pi 1 by r d phi d theta is gamma by 2 pi r, d phi d theta is gamma by 2 pi, so phi is gamma by 2 phi theta 1 by r d phi d theta is gamma by 2 pi r gamma by 2 pi r is what we have derived last time. So, for that where this r theta plane is normal to the line vertex and theta is the angle taken anticlockwise in that plane. If you have an infinite cylinder rather theta is normal to that and theta is taken anticlockwise in that plane theta equal to zero is of course, arbitrary. Now I do not know whether a question should come to your mind that we have mentioned or we have seen that this potential phi is defined, when the flow field is irrotational?

We have said if the curl of the velocity is zero, then the velocity is gradient of a scalar potential so the velocity potential is basically defined when the flow is irrotational. If the flow is not irrotational then you cannot define a velocity potential. It is not that for all flow there is a velocity potential, no if the flow field is irrotational only then you have a potential all flow field are not potential flow field. Then line vertex you know that vorticity is associated with rotation, the rotational part of the velocity that is what we called vorticity, then if the flow is due to a line vortex which is associated with rotation.

How can we have phi of velocity potential? the answer is simple if you remember that, what is this line vortex? We have consider that there is a singularity in vorticity distribution and the singularity such that there is no vorticity anywhere in the flow, but only on that line only on that line and for an infinite line it is equivalent to a two dimensional case. So, in a two dimensional only at that point everywhere else there is no vorticity but only at that point there is the concentration of vorticity. So, essentially this poten[tial]- the flow field is not rotational only at the not irrotational, only at that point everywhere else it is irrotational. Just because that there is a vorticity does not make that the flow field is rotational everywhere, the flow field is rotational only on that line or in two dimensional case only at that point. Everywhere else it is irrotational of course, the flow field is solenoidal. So, flow due to a line vertex is solenoidal and irrotational everywhere except on that line or on that point in two dimensional flow everywhere else.

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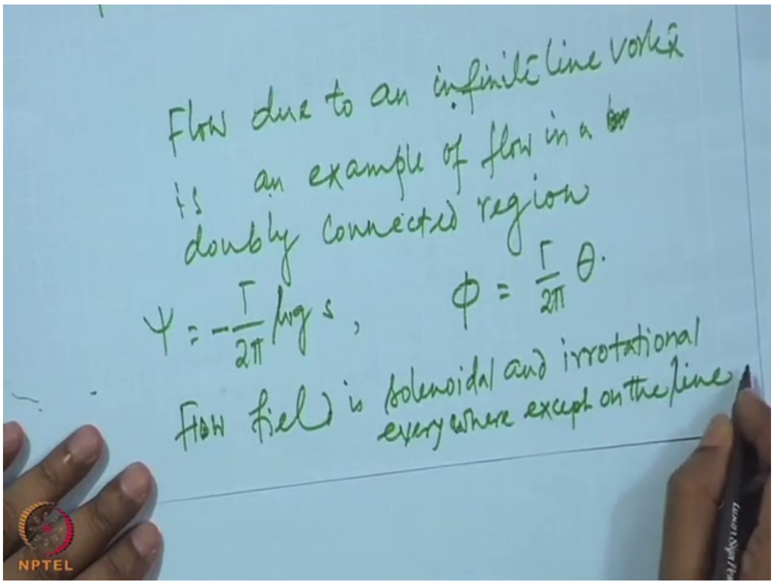


Now we will see now that, what is the necessary condition for a particular flow field to be determined uniquely in a doubly connected region? We have already seen it for a singly connected region, but we would like to see, what will happen in case of a doubly connected region?

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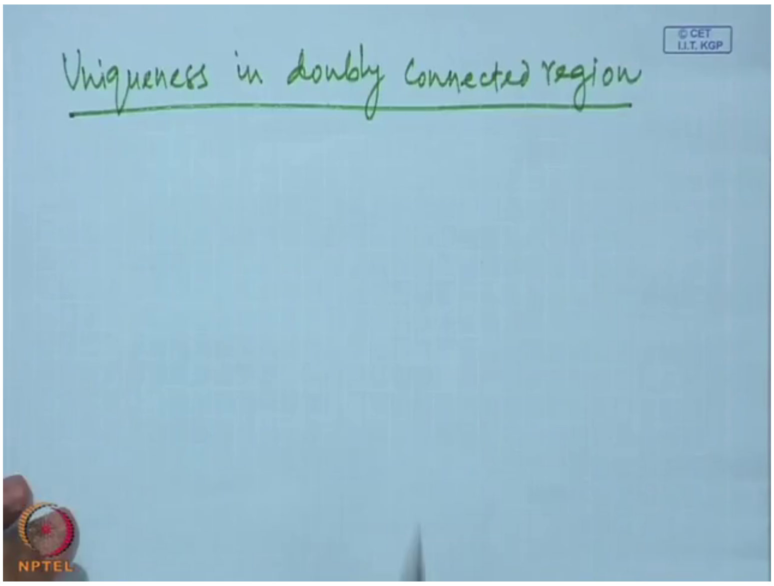


And uniqueness we can very simply state at this way like say that we have two solution. And let us say that they have same value of the cyclic constant both the flows have the same values of the cyclic constant. Now since the laplace equation is a linear equation what we have mention already the some of these or difference of these two solution is also a solution. So, if you have two solutions which have same values of cyclic constant same values of cyclic constant then the difference of these two solution is a cyclic. There is no cyclic constant associated with that we have phi1 which is associated with a cyclic constant kappa, another solution phi2 which also has the cyclic constant kappa.

Then if we subtract these two then that phi1 minus phi2 will have no kappa, and again we know what would be the condition that phi1 minus phi2 that will be satisfied that should be satisfied again the same condition. What you have found for singly connected domain? Because in that case the solution is like a singly connected domain solution if there is no cyclic constant the solution is same as the solution in a singly connected domain. So, if you have two solution from there with same cyclic constant from there we can see that the same condition what where necessary for flow to be determined uniquely in a singly connected domain they are also required if the region is doubly connected.

But we have already imposed another condition that the two solutions have same value of cyclic constant. So, if the two solution of same cyclic constant or thus if the cyclic constant is fixed then we need the same condition what we needed for singly connected domain. And we can say that in a doubly connected region we need an additional condition other than those condition that we had in a singly connected domain we have an additional condition that the cyclic constant must be specified. If, we want to find the solution uniquely for a particular problem we must know the cyclic constant specifically along with those other boundary condition.