**Introduction to Aerodynamics**

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**Lecture No. # 11**

**Kinematics of Fluid Motion – Velocity with specified extension and vorticity**

Today we will discuss the velocity field due to a specified rate of extension and specified vorticity. You have already seen that the fluid velocity can be thought of as summation of a uniform velocity; plus a velocity field associated with an isotropic expansion; plus a velocity field associated with deformation without change in volume; plus a velocity filed associated with a rigid body rotational motion, called vorticity. Now out of these two. Out of these different contributing factors, we will now consider the velocity field associated with a specified rate of extension that is when the isotropic expansion is specified everywhere. What type of velocity field it will be, or if the vorticity is specified everywhere. What type of velocity field we are likely to get these two. So, what remains is of course, a velocity field which has no expansion, no vorticity.

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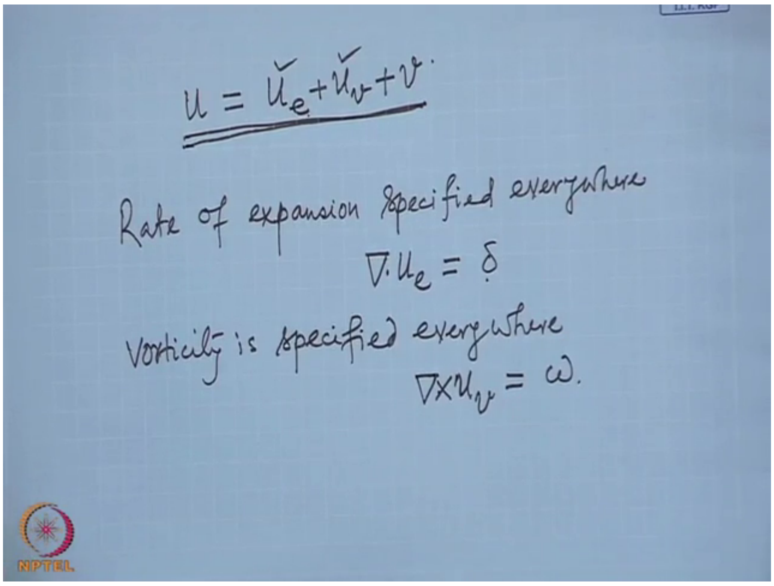


Let us say the fluid velocity at any point can be written as a velocity field which give an isotropic expansion, let us denote it by u e, the subscript e for expansion. A velocity field that gives an isotropic expansion, plus a velocity field which gives that rigid body rotational motion or the vorticity which we called u v; plus a velocity which has no expansion, no rotation, but may have deformation without any change in volume let us call it this v.

So, these two contribution we would like to now see u v which has a specified expansion distribution, rate of expansion distribution given everywhere and u v point on the… whatever we have got till now is based on pure kinematic consideration, just considering kinematics we have seen that the velocity field is likely to be this of course, simple kinematics, or pure kinematics cannot tell us in a particular situation how much will be the expansion, or how much will be the vorticity, or whether a particular contribution will be there or not there this cannot be obtained from pure kinematic consideration, that needs a dynamic consideration as well, but right now we are not going in to it we are considering that these are the contribution that is a total field velocity is simply sum of these and we would like to see one by 1. What type of velocity field they will be? Whether a particular given flow filed that part is there r not at this stage we are not going to going in to that.

Of course, further kinematics has given us that if the flow is incompressible in that case there will be no rate of expansion that we have already seen, rate of expansion is 0. So, that part will not be there, however; for a general case it will be there, let us say that we have a velocity field which has rate of expansion specified everywhere and let us denote it by delta, rate of expansion is specified everywhere, rate of expansion as you know is divergence of u e is we call that say delta of course, it is function of x y z, at each and every point the rate of expansion is specified divergence of e.

The rate of expansion that we have seen dilatation, or the rate of, or the vorticity is specified everywhere, curl of u v let us call it omega. The vector sign I am not using, but you must remember that these are all vectors u v u, v. Now we would like to see what are these velocity fields, what type of velocity field we will have. So, that the rate of expansion is delta everywhere and vorticity is omega everywhere. First of all consider that the only the rate of expansion that is specified, but the velocity field is such that rate of expansion is everywhere delta but the vorticity is 0 everywhere.

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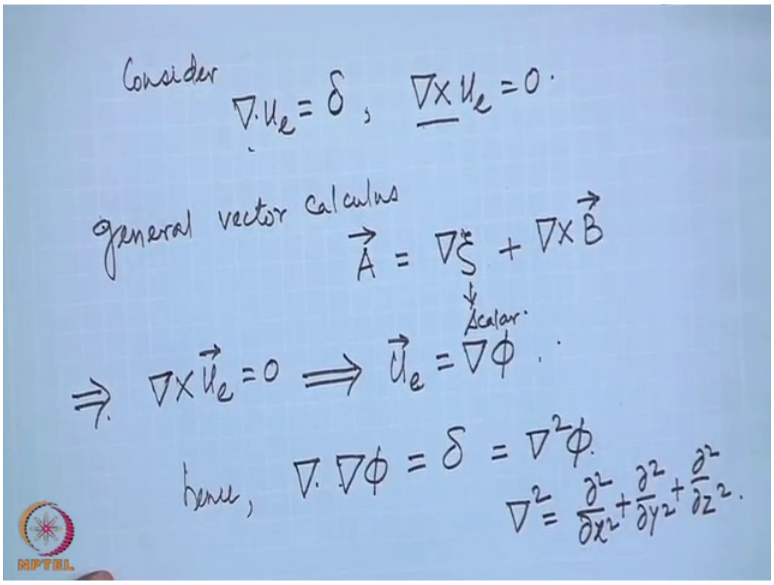


We will consider one of one at a time, not two together. So, consider first this case divergence u is delta, but u e is such that it has no vorticity associated with it. Now, it is also a very simple thing in vector mechanics that any vector can be written as sum of one variant of a scalar plus one curl of a vector, this is of course, a result form general vector algebra for vector calculus, that any vector a any vector a can be written as gradient of a scalar say gradient of a scalar let us call that scalar to be xi a scalar gradient of a scalar plus curl of a vector, we call that vector B this is a general result true for any vector that all vectors can be written as term of the gradient of a scalar plus curl of another.

This then gives that, if the curl of the vector is 0 if the curl of the vector is zero then that vector is simply gradient of a scalar that is when curl of a vector is 0 as in this case curl of u is 0; this gives that u e is simply a gradient of a scalar let us call that scalar phi with this substitute u e in this relation gradient of phi is delta, divergence of gradient you can try by writing this in standard ddx, ddy form and you can see that divergence of gradient will become simply laplacian, laplacian of phi, divergence of gradient is always laplacian, laplacian is a second order operator d 2, dx 2 plus d 2 dy 2 plus dz 2 in three dimensional Cartesian space. In three dimensional Cartesian space laplacian is d 2, dx 2 plus d 2, Dy 2 plus d 2, dz 2 anyway.

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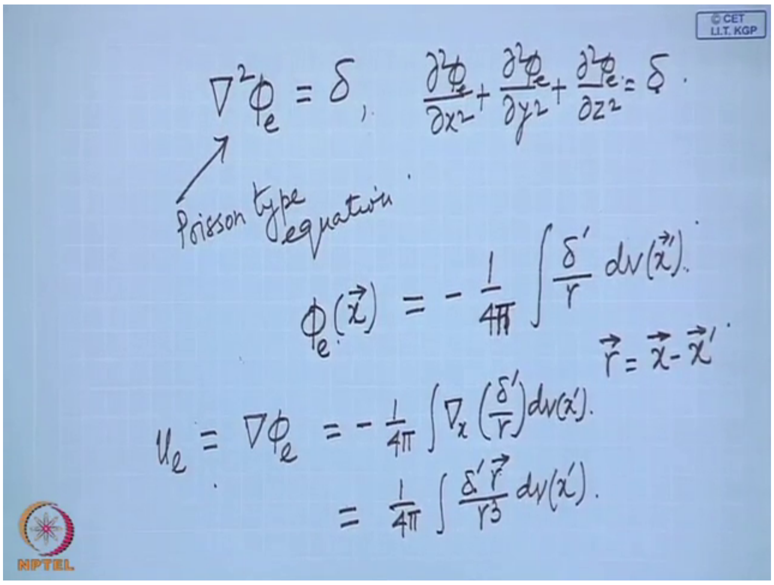


So, we now have the velocity field is such that is satisfies this equation laplacian of a scalar function phi is delta and the gradient of that scalar function gives us the velocity field. This type of equation this is of course, a partial differential equation if you write this equation completely this is a partial differential equation, this can also be written in Cartesian system and remember this delta that expansion distribution rate of expansion distribution is function of space and time though we are not writing explicitly all the time, this type of equation when the laplacian of a function is a non zero function then that equation is called a Poisson type of equation.

So this is a Poisson type of equation… if delta is 0 then the equation is called Laplace equation. If laplacian phi equal to 0 then the equation is called Laplace equation, if the if it is nonzero then the equation is called Poisson type equation.

Now, this equation has… since, this phi will also use a subscript e, because this phi is associated with this velocity u e. So, we are using 1 e here. Now, the solutions of this equation… are you familiar with the solution? of this equation the solution of this equation is given by phi e again a function of x and t, if you want you can write as a function f x and is minus 1 by 4 pi sorry 1 by 4 pi delta prime by r, dv, x prime.

However; x prime is nothing but a general point, another point and r is the distance from the point x to x prime; that is this r is the distance between the two points is r delta prime is this delta prime is the value of delta at x prime delta prime is value of delta at x prime This integration is carried out over the entire fluid this integration is carried out over the entire fluid the velocity field then can be obtained as u e equal to grad phi. How much will this be? Gradient of 1 by r is simply 1 by r square minus and since, the gradient is a vector it has a direction which is again in the direction of r so that to denote that direction it is written as r by r cube. So, 1 r by r will simply give the direction and r square will remain the integration is carried out over the infinite fluid. Now, look to the integrand only delta prime r prime by r cube dv x prime. What does it mean then that? This is the velocity distribution due to infinitesimal volume element dv located at x prime.

The rate of expansion associated with point x prime is delta prime. So, this velocity field can be thought of as simply the velocity field due to a rate expansion delta prime at point x prime, but there is no other rate of expansion, if we consider one small finite element sorry infinitesimal element at x prime which is given by dv prime volume then the velocity field due to that rate of expansion delta prime is this quantity 1 by 4 pi into this quantity.

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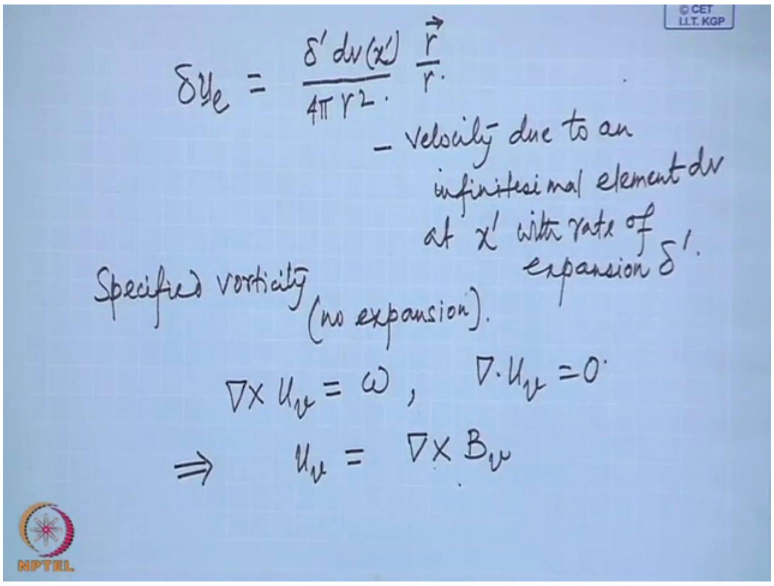


So, we can call that as delta u e and let us, write it this way delta prime dv x prime by r square into r by r sorry 4 pi. Velocity due to… Now, what is this quantity delta prime dv prime? See delta prime is rate of expansion, and dv prime is the volume of a small element. So, delta prime dv prime is again a volume, because delta prime is rate of expansion. So, when you multiplied by it the volume then it gave the total volume expansion.

So, this much is the volume expansion over that dv prime meaning that if we consider say a spherical volume then the rate of expansion within that spherical element is given by this quantity. We can also think that as if there is some sort of source which produces volume within that small element at the rate of delta prime. Something is there which is creating the volume, a volume source, a source of fluid volume. So, if there is a source of fluid volume which has a rate of expansion delta prime and then over the element dv rime then it will induce a velocity field given by this, outside that volume element everywhere, then the expansion remains this delta prime.

So, when we integrate it we get the specified volume element so that is this can be thought of as a volume source placed within that volume dv prime and the complete rate of expansion. Can be thought of as sum of infinite number of volume sources located at each and every point with rate of expansion delta prime, and this delta prime dv prime can be called as the strength of the source, strength of that volume source, that is the volume it is increasing. So, we now have got the velocity field associated with a specified rate of expansion.

Now, let us see what the velocity field will be? If the rate of if the vorticity is specified and once again we will consider that the vorticity is specified such that the expansion is zero. The velocity field is such that it has the specified vorticity, but no expansion. So, consider now the specified rate of vorticity, sorry specified vorticity but no expansion. That is we have the velocity field such that curl of u v equal to omega, but it has no expansion that means divergence of u v equal to 0.

This then satisfies from that basic vector relation this imply that u v is a field, vector field such that u v is curl of a vector field u v is a curl of a vector field then if we satisfy sorry if we substitute this in this equation what do will get?

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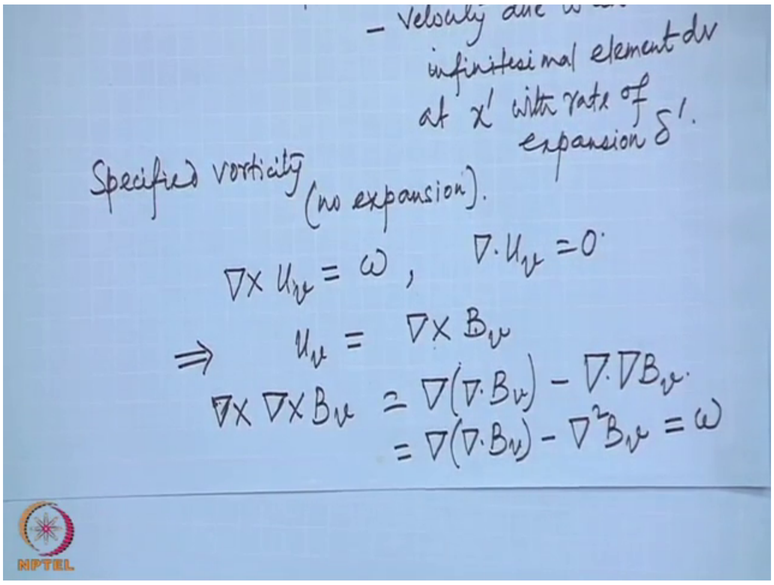


Curl of curl of B equal to… you know what is curl of curl? curl of curl this is again a vector identity. It is gradient divergence minus divergence gradient gradient divergence minus divergence gradient so it is gradient divergence of B minus divergence gradient of B. The divergence gradient we have already mentioned is simply laplacian so this can also be written as gradient divergence of B minus laplacian of B and this is equal to omega. Gradient divergence of B minus laplacian of B equal to omega the equation is this case is not as simple as in the earlier case in the earlier case we simply had laplacian of phi equal to something, in this case there is two term there are two terms even this to simplify just let us assume that B is such that divergence of B is 0 just assume and then the equation will be simplified as before.

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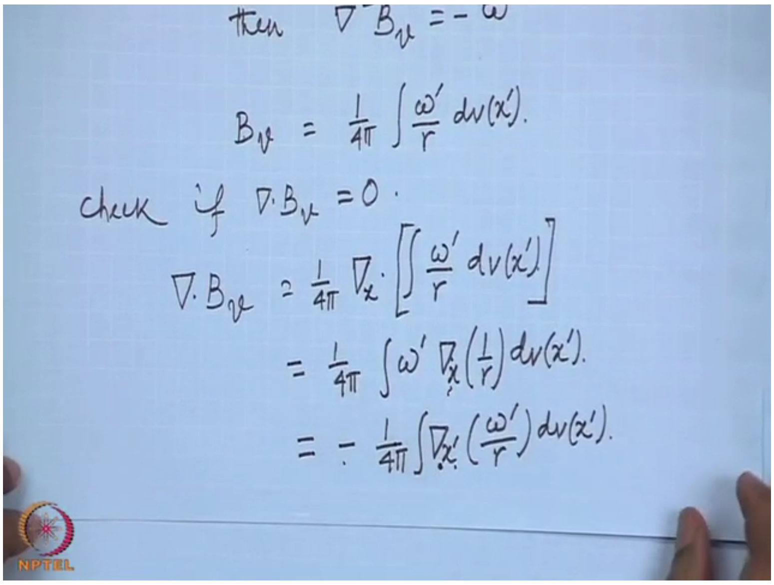


Let us assume and then we have again the same equation laplacian of B is minus omega. We have the same equation and again we can write the solution, another poisson equation and the solution is… again integration over the entire fluid or the infinite fluid however, before we accept this to be a solution of our original problem we have to see that whether this has really the divergence 0 that is what we are assumed. With that assumption only we have got this equation.

So, this will be solution only if the divergence of it is zero; otherwise not so. First we must check then whether the divergence of this is 0. So we need to check now what is divergence of B? This we have made a change of coordinate here this divergence x, meaning while the derivative are with respect to x, here we have changed it to derivative with respect to x prime which has given as a negative sign here. Now, we can apply that divergence theorem and change this volume integral into a surface integral, change this volume integral into a surface integral.

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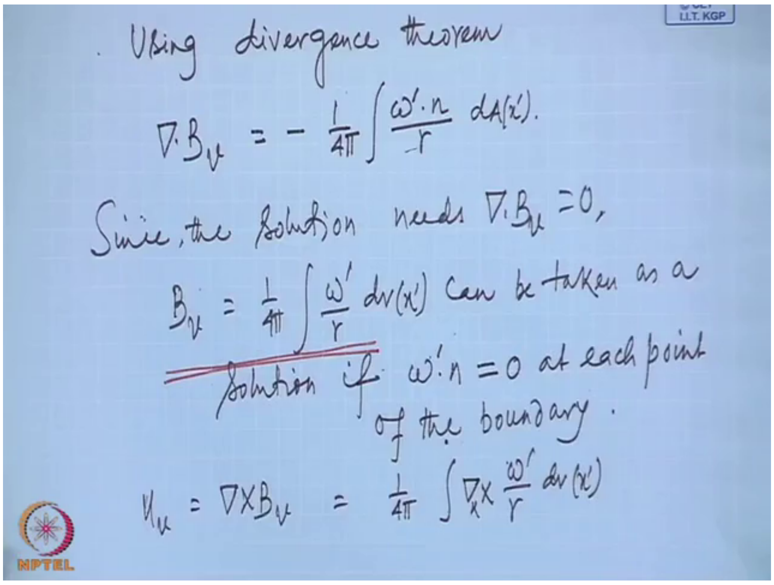


Using diversions theorem we now have what it will be the volume integral will become surface integral where the surface is the surface of that infinite fluid. Yes this is what is no divergence of B, but we have obtained this solution assuming diversions of B is 0.

So, under what condition this can be 0, this is possible that if the fluid at infinity is that rest if the fluid at infinity is at rest. Then this omega dot n will become 0 for an infinite fluid this is quite possible that we can consider under a given situation that the fluid at infinity will be at rest, and then that omega dot n will be 0 there, but in many situation it may not be so, that the boundary at some of the boundary this omega dot n may not be 0 the fluid may not be at rest at all the boundary.

However, in that case we may again consider that we will expand that boundary and again take it to infinity, and by that we will make it 0, in those cases where we will be able to make it only then this is the solution otherwise not. So, wherever this particular integral that is at the boundary is the fluid is at rest and vorticity then this condition will be satisfied and we can consider that this is the solution. If it is not then if we can extend the boundary and have a boundary where again the fluid is at rest, then also we can take this as a solution, but in those cases where we would not to be able to do it in that case this not the solution. Then of course, the solution is not known, because we have not been able to solve the complete general equation, we have solved the simplified equation which holds only if this equation is 0.

So, we can write it that since the solution needs divergence of B v equal to 0. That is the vorticity normal component of the vorticity at the boundary must be 0. The specified vorticity is such that it is normal component is 0 at the boundary then this will be the acceptable solution, this will be the solution only when the vorticity distribution is such that it has 0 normal at the boundary at the boundary of the fluid.

Now, assuming that we have the vorticity distribution such that, as we said that this is this we can obtain in most cases, at least wherever there is an infinite fluid and the fluid at infinity is at rest this is satisfied, in case we can expand the boundary to infinity even though the boundary is not infinite, but we can expect extend it such that again the normal component at that extended boundary is 0 then also this is acceptable this is a solution and we will see that most cases we will be able to do it. So this is then even though not general solution, but in all practical cases this will survives the solution. Now if this is the vorticity vorticity distribution then what is the associated velocity field u v that you have to find, u v is curl of v so, we have 1 by 4 pi curl of omega by r can you evaluate that curl eventually it will be curl of 1 by r.

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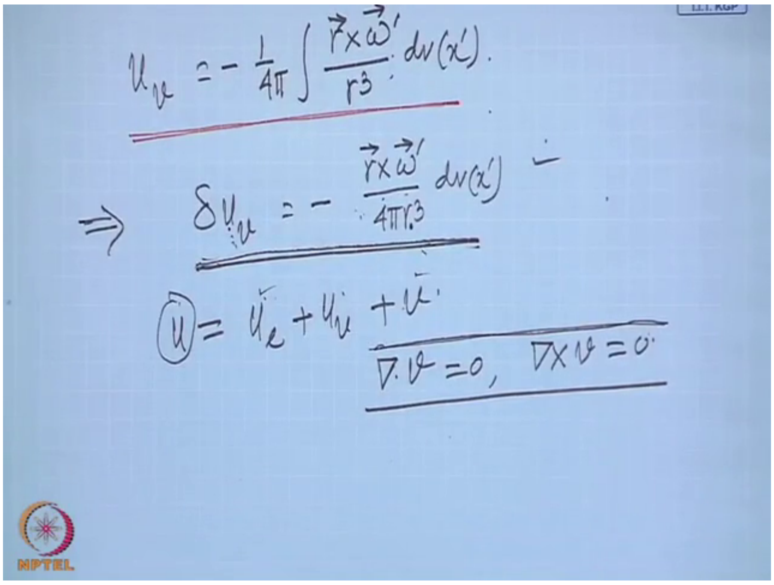


What it is then u v equal to minus 1 by 4 pi r cross omega by r cube. Once again we can consider any infinitesimal element and write the velocity field associated that with the infinitesimal vorticity distribution. And once again here also you can interpret in the similar, manner that if this is the vorticity associated with the infinitesimal element, then the velocity that it will produce or induce whatever you call it at any other point will be given by this at a distance r. And when you consider the entire vorticity distribution this is the the integral sum will be the total velocity field.

However there is a difference with the earlier case. It is in the physical nature that at infinitesimal that source distribution our expansion rate of expansion distribution is physically possible, you can have in a fluid a small region where there is some non 0 expansion distribution or non 0 rate of expansion everywhere else it is 0 it is possible, but in case of vorticity it is not possible. That an infinitesimal element of volume there is a some vorticity non 0 vorticity distribution, but everywhere else the vorticity distribution is 0 that is not possible, because such a volume vorticity distribution will not have a 0 divergence such type of vorticity distribution will not have zero divergence.

So, this velocity distribution on it on it is own cannot exist in a fluid. However it is counterpart for the expansion distribution can exist it is possible, this alone cannot exist this can exist the integral integrated result that is everywhere the vorticity there is vorticity distribution such that the divergence of that vorticity field is 0 while, but this is not possible. However that does not prevent us from doing that mathematical interpretation. I think you have come across this relation at least the mathematical relation elsewhere. If we replace this vorticity distribution by current density is that relation looks familiar, to you yes is not it or no no. You have not come across this relation in electromagnetic where the vorticity is replaced by the current density and the velocity field related replaced by magnetic field. And you use to call that relation biot savart law.

So, mathematically this is what biot savart law is and eventually they solved this equation only. The same mathematical equation that same Poisson type of equation that is what they solved and got that relation. Eventually that relationship between electric current density and magnetic field also has similar, relation and the mathematical solution is same. And following that sometime we may call it that this is the velocity field induced at a point r distance away due to a vorticity present at that point or vertex distribution given by omega prime dv. Like we call that if there is an electric current flowing through a conductor that induces a magnetic field, in same manner it is also called that if there is a vorticity with this vertex vorticity distribution it will induce a velocity field at that point.

So, now look into the complete velocity field in a flow. We have already seen that the complete velocity field is some total up velocity field due to a specified rate of expansion. We have known what is that velocity field, plus a velocity field due to a specified vorticity distribution; we have also known what that velocity distribution is and remain. The third contribution which then of course, has no rate of expansion associated with it no vorticity associated with it. So, the third contribution which we initially denoted by if you remember look to that first relation that we wrote u equal to u e plus u v plus v. And we know that u is associated with a specified rate of expansion, a specified vorticity and that that part u e and u v we have already evaluated what remains is v and v is such that it is not associated with any expansion that is divegence of v is 0 and vorticity of this is also 0.

So, our next task will be to find this velocity field. However before we try to solve this problem, and see what type of velocity field we are likely to have, we will consider a few special cases of expansion distribution and vorticity distribution special cases as an example. We have found the velocity field due to a general distribution of a expansion or general distribution of vorticity, but we now like to have more concrete more explicit relation for some specific expansion distribution or vorticity distribution. The first one would like to consider where perhaps the simplest one is the expansion distribution is such that, at one point there is a non 0 expansion everywhere else it is 0 which is possible as we mentioned that in case of expansion it is possible, in case of vorticity it is practically not possible, but in case of expansion that is practically also possible. We can have at one point where the rate of expansion is non 0 and everywhere else it is 0. It is some sort of a singularity distribution so it is known as a singularity distribution or singularity in expansion.

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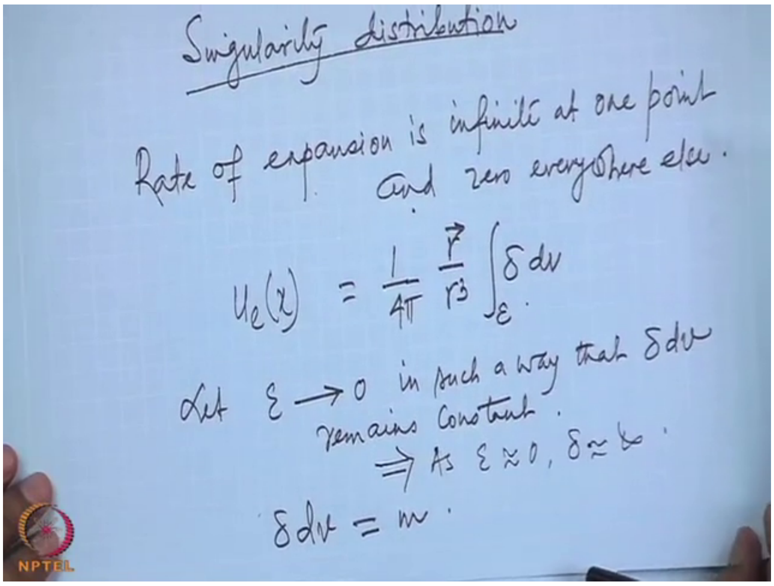

Let us look into this way think about a small volume element and let us say again that the rate of expansion within that small volume is delta prime. Now think that you are shrinking that volume in such a way that the total volume flow rate that remain constant. The total expansion is delta prime dv prime, as we or we will call delta dv forget those primes delta dv delta into dv, d v is that volume element and delta is the rate of expansion. So, total volume or total expansion is delta into dv. Now we shrink the volume dv in such a way that the product delta into dv remains constant. That is when you reduce dv by half simultaneously we increase delta by two proceed this way. So, when dv approaches 0 this delta will then approach infinite.

However the product is still remaining that delta into dv whatever originally we had, that is what we are going to consider that at one point the rate of expansion is infinite everywhere else it is 0. So, rate of expansion is the velocity field obtained is at any point x the velocity field is we have already obtained it what 1 by 4 pi into now you can write let epsilon approaches 0 and this constant delta dv constant let us denote it by m, then we can write u e equal to m by 4 pi.

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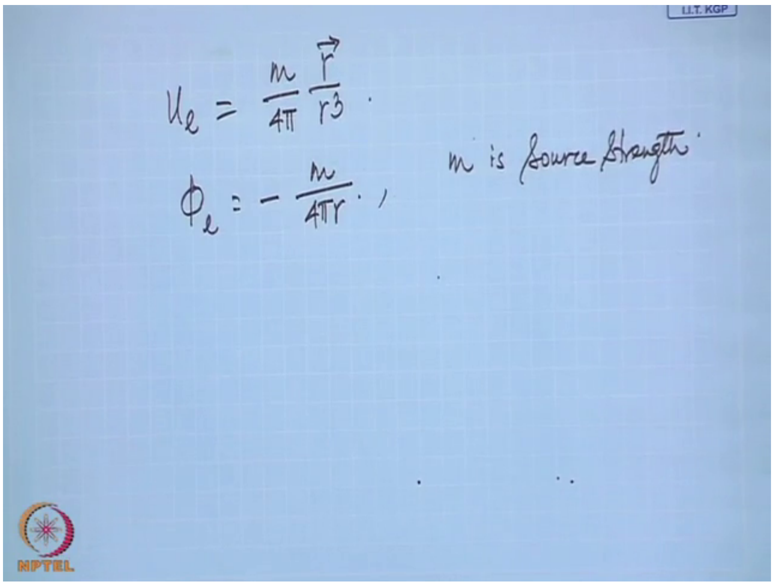


And we already said that delta into dv that is the total expansion. In this case the total expansion is associated with a single point, and since it is increasing or expanding the volume we have called it a source of volume. And this m is then like delta dv is we defined as the strength of the source, in this case this m becomes the strength of the source. Physically you can think that at one point we are increasing the volume by m just at one point. You can very easily do it by discharging through of say pipe or tube in a fluid again you can have some discharge by another tube or pipe at the rate of m. So, that is it that is a point source in this case it is a point so it is point source.

So, this is a the velocity field associated with point source of strength m, and this is the associated potential opposite of source is called sink where the strength is negative. We will continue further next class.