Space Flight Mechanics

Prof. M Sineha
Department of Aerospace Engineering
Indian Institute of Technology, IIT Kharagur

Lecture No. # 09
Two Body Problem (contd.)

If we have been working with the two body problem, so we continue with that, so in that context, today we will be taking up the orbit determination problem of a satellite from the burn out data. So, what happens? We intend to put the satellite in some orbit, but in reality it is never possible to put the satellite in the same orbit in which we intend, because of the under or the over functioning of the rockets.

So, today we will look into that if the satellite is injected into the orbit, so at that time the inertial navigation system it will transmit the initial position and the velocity vector of the satellite, and from there we can know in what orbit the satellite is. But over a period of time then what is called the preliminary orbit determination; so in the preliminary orbit determination using the first few minutes data again the orbit determination is done to meet the orbit parameters very precise.

So, that tracking can be done properly over a period of time. But burn out data itself it gives you enough information, so that you can adjust the radar, and you can point it in a particular direction in this sky, so that the tracking will continue before the preliminary orbit determination data is available.

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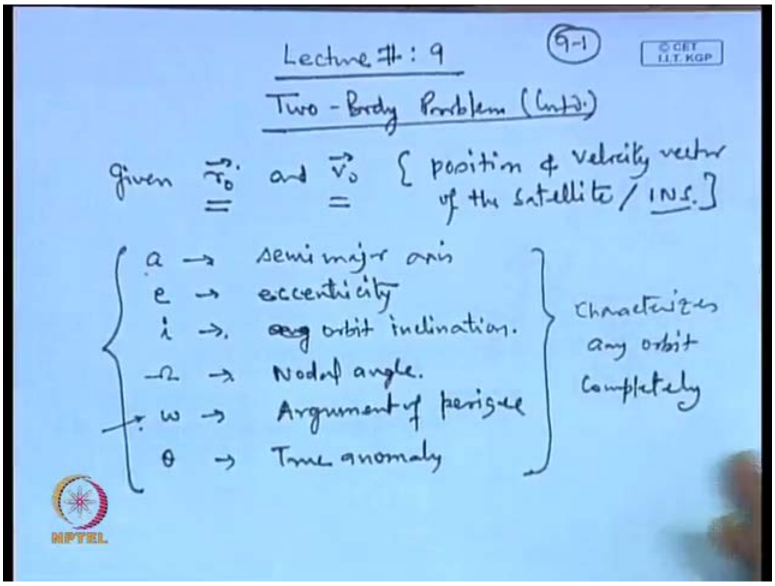

So to start with, we have the condition that given r 0 and v 0 that is the position and velocity vector of the satellite, which is available from inertial navigation system INS. So, using this information, what we need to find out is the semi major axis; e the small eccentric may be this is called the eccentricity; this is the angle of inclination or the orbit inclination; capital omega, this is the nodal angle; small omega, this is called the argument of perigee argument of perigee and theta this is the true anomaly. So, these six parameters together they characterized the orbit completely. So, instead of using r 0 v 0 you can use them to characterize any orbit, so it characterizes any orbit completely.

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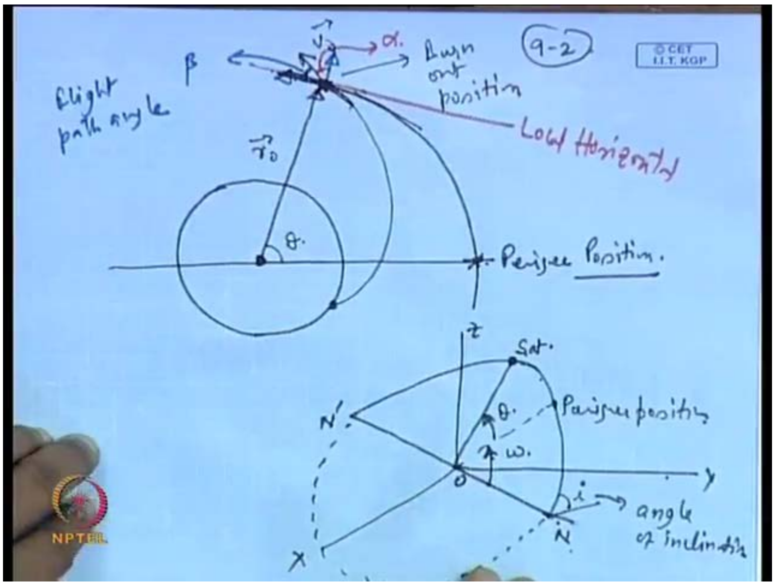

And today, we are going to work out, how to get these quantities from r 0 v 0 which is given to us. Now let us considered the earth is given and somewhere on the earth, you are lunching the satellite, say from this position somebody as lunch the satellite. So, satellite will go like this through the rocket will be lunched, and then ultimately it will be putting it in a orbit. So, here the rocket will come and it will inject into the orbit with certain velocity v 0, we draw radius vector from this point to this point. So, this is r 0 then the radius vector at burn out, so this is your burn out position burn out position, this is the angle theta.

So, let us say the orbit in which it has been injected that orbit it looks likes somewhat like this and this is the perigee position. So, if this velocity vector is broken in along the two directions and this angle which is beta this is call the flight path angle flight path angle and this angle the other one this is alpha. Sometimes, so the flight path angle is measure form the local vertical local horizontal also this is your local horizontal local horizontal. So, form here the flight path angle is measured and the angle alpha is shown here, some time let say many of the literature may be mentioning the alpha angle from this horizontal line to the toward the velocity vector. So, your radius vector is in this direction and velocity vector -vector is in this direction, so using this information now work out the initial orbit of the satellite.

As you may be aware of that in the inertial reference trend, we have been showing the orbit like this. So, this is the orbit, this is nodal line in prime m n, and somewhere this is the perigee position, this angle is argument of perigee satellite is say it is here satellite, then this angle is theta, which is the true anomaly and the angle the satellite mix with the x y plane. So, here this angle that it mix with x y plane, this is the angle of inclination angle of inclination.

So, we have discussed earlier this point. So, the figure we are taking right now is this it is the orbit, which is line give up the x y plane. So, this orbit will go down and you can complete by showing it by the dotted line, which is line below x y plane. So, we take this whole orbit and if this orbits you can show it in this way. So, you have a lunched the satellite, satellite is injected at this point, so at the time of injection the velocity is v 0 and the radius vector is r 0.

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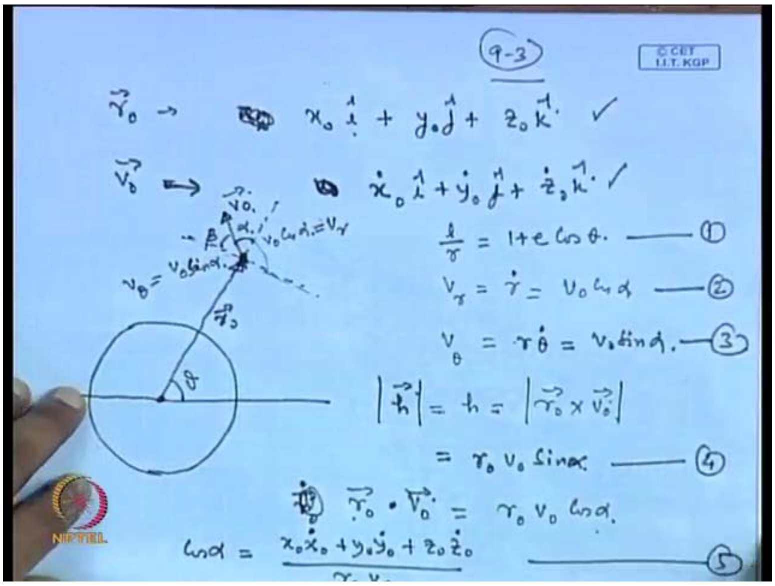

So, r 0 implies your given in the inertial reference frame say if we can write x 0 i cap and y 0 j cap plus z 0 k cap, where i 0, i, j and k cap these are the unit vectors along the x, y and z directions. Similarly, v 0 we can write as this this we can indicate by the components wise x dot 0 i cap plus y dot 0 j cap plus z dot 0 k cap, now in the previous figure that we have taken, now leaving out the lunching position only we are concerned with the velocity and position vector right now and this is say the velocity vector v 0 this angle we are writing as beta and this angel we are writing as alpha. So, from here you can see that this v 0 vector it can be broken in this direction as v 0 cos alpha and in this direction it is a component will be v 0 sine alpha. So, we are writing in terms of alpha.

So, same way you can also write in terms of beta it is not a problem or as I told you that many times it happens that the alpha angle is describe from this blue line, which I am extending this is call the local horizontal. So, it is measure from here to here sometimes either way if you write this alpha you measure from here or either from here it does not matter it will just alter your equation, but you can always get back to another equation by choosing different values of the alpha either this one or this one.

So, it is always possible to modify it and this angle we have indicated as theta. So, we have v r is equal to r dot which is the radial component we are writing this as a v 0 cos alpha and that theta component, which is perpendicular to the radius vector in this direction. So, this is your v r, and this is v theta, and this will be r times theta dot and we have already all turn this problem why v theta will be equal to r times theta dot. So, this is v 0 sine alpha also we know that l y r is equal to 1 plus e cos theta.

So, the this is the equation of the conic section equation number two equation number three now the angular momentum of this satellite this we can write as h equal to h magnitude is equal to h is equal to r 0 cross v 0 and this is per unit mass angular momentum per unit mass we here nowhere hem is appearing. So, this is basically your r 0 v 0 and the angle between r 0 and v 0 is a alpha here. So, we write here sine alpha equation number four similarly we can write r 0 dot v 0. So, this we can write as r 0 v 0 the angle between this two is now cos alpha here with angel is alpha.

So, therefore, this will be cos alpha and r 0 v 0 if you take the dot product of this 2. So, here we can write on the from this equation cos alpha this is equal to r 0 v 0 taking the dot product this will be x 0 times x 0 dot plus y 0 times y 0 dot plus z 0 times z 0 dot divided by r 0 v 0 and this is our equation number five. So, these are the preliminaries that we are building to solve the equation for the now the first equation that we are written l y r is equal to 1 plus cos theta 1 plus e cos theta. So, differentiate that.

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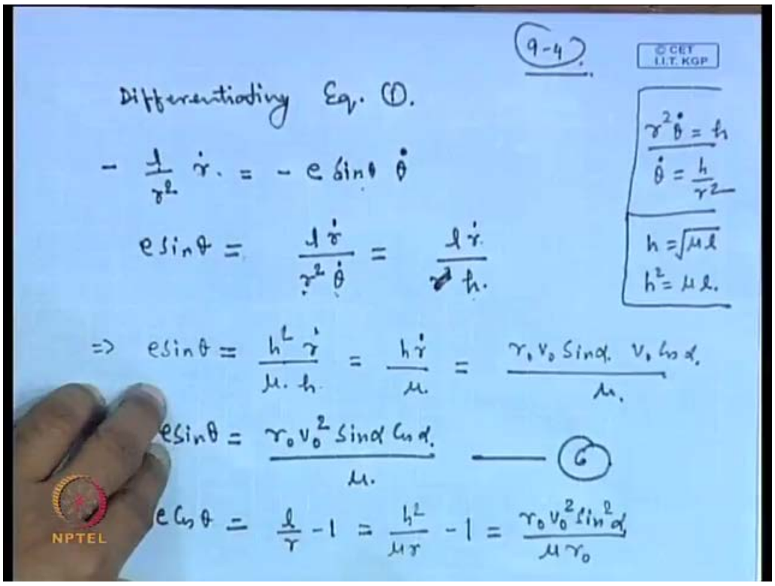

So, differentiating equation 1, this will yield l y the left hand side l y r s square and r dot is equal to minus the right hand side minus c sine e theta times theta dot. So, you can simplify it little bit further and we can write e sine theta is equal to l r dot by r s square theta dot and you know that theta dot you aware of from the previous discussion we know that r s square theta dot this is nothing, but h. So, theta dot becomes h by r s square for here. So, this quantity you can pick from this place we have taken here this one to one times r theta dot r s square times theta dot. So, directly we can write here the h from this place either taking this or either taking this you get this equation.

So, this one times r dot by h now h also you know from our previous working the earlier lectures that h is equal to mu one under root that is h is square is equal 2 mu times one. So, therefore, this implies e sine theta this is equal to l we can replace from here h is square by mu and then r dot divided by h. So, this is h r dot divided by mu and h we know how much this is r 0 v 0 sine alpha and r dot nothing, but v 0 cos alpha and divided by mu. So, this keeps you e sine theta is equal to r 0 v 0 s square sine alpha times cos alpha divided by mu and this is our equation number 6.

Now, e cos theta you can write as using the equation number 1 e cos theta from here l y r minus 1 l y r minus one and l already we have written here l equal to h s square by mu and h we know this quantities r 0 v 0 sine alpha. So, this is r 0 v 0 s square sine s square alpha divided by mu times r now we are replacing with r 0 minus 1.

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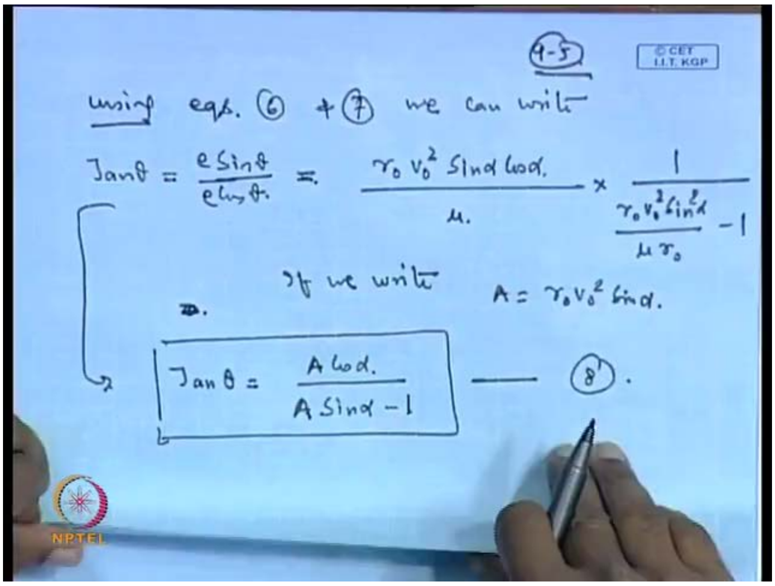

So, this keeps you e cos theta this is equal to we will keep it in the same form. So, let us write this equation as the equation number seven using equations six and seven we can write n theta equal to sine theta by cos theta e sine theta by e cos theta is equal to tan theta e get cancelled out and this quantity. So, we are using this equation number six and equation number seven here. So, r 0 v 0 s square sine alpha cos alpha divided by mu times 1 by r 0 v 0 s square sine s square alpha by mu times r 0 minus 1.

So, if we write a is equal to r 0 v 0 s square sine alpha then this equation gets reduce to tan theta this is equal to a cos alpha divided by here h is square this is square term we are missing. So, we supplemented mu r 0 is here and this is r 0 s square here in this place. So, this must be supplemented. So, here after cutting it you can write it that r 0 v 0 s square sine s square alpha, so this will become a times sine alpha minus 1 and this is our equation number 8.

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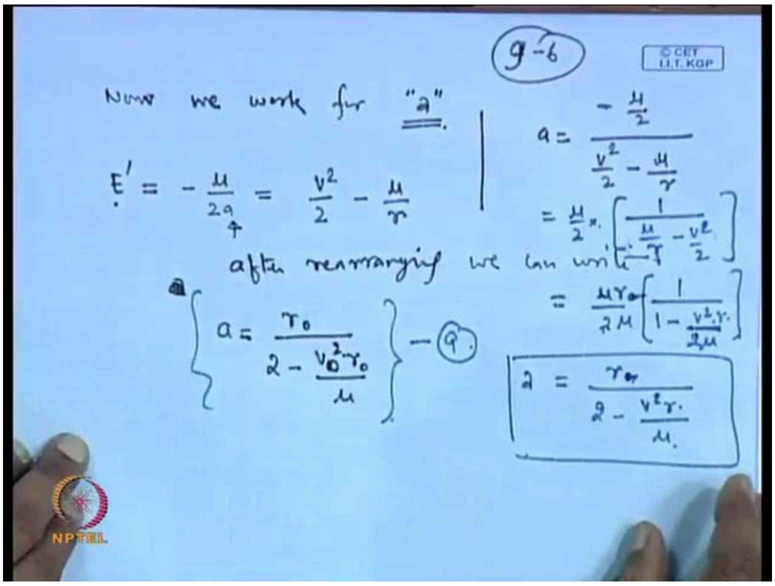

So, this keeps you tan theta now theta we have done. So, now, we are work for a semi major axis. So, we know from our earlier lectures that the specific energy of the satellite that is the energy per unit mass this is nothing, but mu y minus mu by two a and this we have written as v s square by 2 minus mu by r. So, this you can rearrange and get a from this place. So, what we can do after rearranging we can write here a equal to r 0 y 2 minus v 0 s square by r 0 mu. So, this is just the arrangement of this is required we are interested in this quantity. So, we will put it in this way we will write a equal to minus mu by 2 divided by v s square by 2 minus mu by r.

And if you rearrange it here now this is mu by two times minus sine we will take inside, so this will become 1 by mu by r minus v s square by 2 and if you take r from common from here. So, this will become mu by 2 times r 0. So, r I am taking common from the denominator here of the lower force of the denominator from this denominator force all denominator-denominator force. So, this becomes and mu also we taken outside. So, here what we get 1 minus v s square by r we are taking outside. So, r will go on this side and mu will appear here and two remains here in this place. So, this mu means cancels out what we get here r 0 2 v we can take inside. So, this becomes 2 minus v s square by r by mu this is your x.

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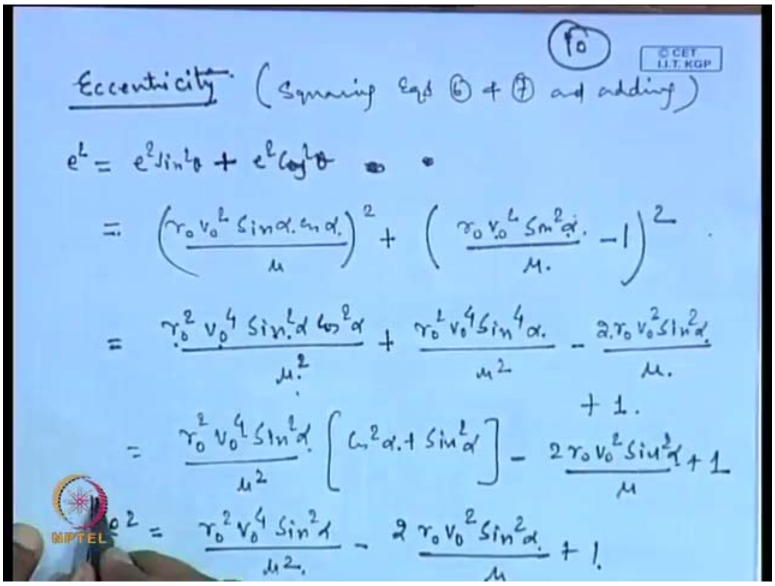

Now, finally, this r and v need to be replaced by r 0 v 0. So, we get this equation and this is our equation number nine. So, this is the equation for the semi major axis. So, we have calculated till now theta and a now we need to work out further eccentricity next we are going to work out the eccentricity and this we are all doing from the burn out data. So, again looking back into our equations the equation six and seven this is written in terms of e sine theta e cos theta on the right hand side you can see that r 0 v 0 is known to us.

And therefore, alpha will be known to us, because the direction is given for the r 0 v 0 mu is known to us. So, right hand side is known to us and therefore, the left hand side if you square and add. So, we get the eccentricity. So, here e s square this becomes e s square sine s square theta plus may here we write s square a squaring equation six and seven and adding mu s square cos s square theta. So, this becomes r 0 v 0 s square sine alpha cos alpha by mu whole s square plus r 0 v 0 s square sine s square alpha by mu. So, r 0 here one thing, what we have missed out that we need to supplement this while writing time theta is equal to e sine theta by e cos theta.

So, without this equation now r 0 v 0 is a square sine alpha cos alpha and this is mu here this term already we canceled r s square term was f would be canceled. So, this term is not present here. So, this correction must be given and then what will do here will put the mu in the denominator of this. So, a equal to r 0 v 0 s square sine alpha by mu, and then you write this. So, then it becomes a cos alpha this term becomes a, and then the cos alpha and here also you can look into this here you will get a times sine alpha minus one then, it become square otherwise this term was missing here. So, we are supplementing it here and take this as a correction.

So, your r 0 v 0 s square sine alpha divided by minus this is the correction provided here. So, we have here r 0 v 0 sine s per alpha and divided by mu this is the quantity we are taking r 0 v 0 s square sine s square alpha divided by mu minus 1 and then whole s square. So, expanding it cos s square alpha divided by mu s square plus r 0 s square v 0 to the power four sine to the power four by mu s square minus 2 times r 0 v 0 s square sine s square alpha divided by mu plus 1.

Now, here from you can take sine s square alpha r 0 s square and v 0 to the power four as common and we can write here r 0 s square v 0 to the power four sine s square alpha taking this as a common and mu s square in the denominator. So, after taking common v i v s cos s square alpha sine s square alpha minus 2 r 0 v 0 s square sine s square alpha divided by mu plus 1. So, this gets reduced to r 0 s square v 0 to the power four sine s square alpha divided by mu s square minus 2 r 0 v 0 s square sine s square alpha divided by mu plus 1 and this is your e s square.

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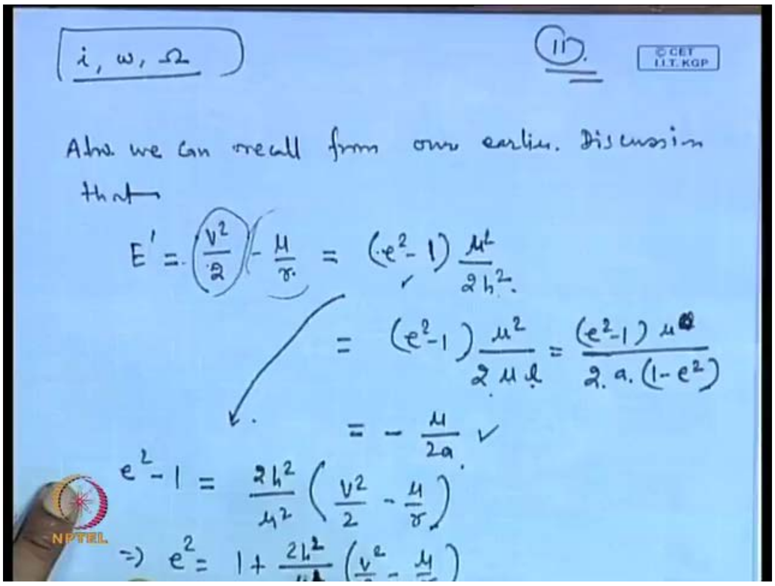

So, this gives you the value of e and this is the equation number nine. So, thus we see that still now we have worked out that theta the semi major axis and the eccentricity. So, this three quantities are worked out next we we will try to find out now what is remaining our I small omega and capital omega. So, this we need to work out also we can recall from our earlier discussion that is in the earlier lectures we have worked out this and we have written there v s square by 2 minus mu by r. So, this quantity is nothing, but e s square minus 1 times mu s square by 2 h s square and you can verify you can check this this is e s square minus 1 and mu s square h s square is nothing but mu times one. So, this is mu times one and again you can replace. So, this mu may will cancel out and you get here to an one is a times one minus e s square.

So, once you cancel it out. So, this quantity cancels out and leaving you minus mu y 2 a. So, this is what you have written as e prime equal to v s square this is the kinetic energy per unit mass and this term is a potential energy per unit mass. So, this is the total energy per unit mass and this is minus mu by 2 a. Therefore, we can write here e s square minus 1 from this equation we can write e s square minus 1 equal to 2 h s square by mu s square. We are using this equation times v s square by 2 minus mu by r and this implies e s square is equal to 1 plus 2 h s square by mu s square times v s square by 2 minus mu by r.

So, now h is known to us this quantity is known mu is known v you can replaced now by v 0 and r can be replaced by r 0. So, also you can get this eccentricity directly in terms of this equation.

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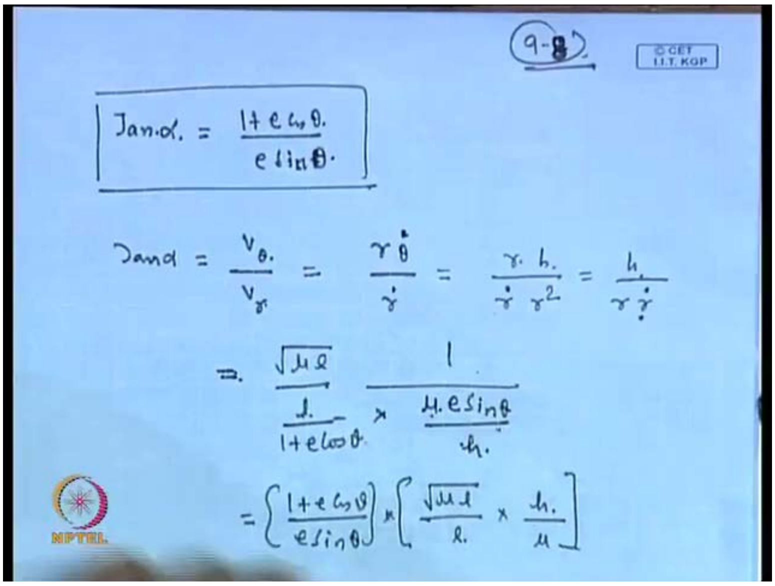

So, we have two equations for finding out the eccentricity either we use this or either we can use this ten this is 9 6 this is nine seven once we have done this. Now we go to the next step this ten alpha is a quantity that we can write as one plus e cos theta by e sine theta, and this is equation is very useful and quite frequently used, and you can prove it very easily ten alpha. In fact, you can write as v thetas by v r just look into the previous figure that we have drawn. So, from here v theta is nothing, but r times theta dot and v r is r dot.

So, theta dot you can replaced in terms of h. So, this becomes h by r times r dot now h you can replace in terms of mu one. So, this is mu one under root where one is the semi lotus spectrum and r you can write as l y 1 plus e cos theta and r dot; obviously, we need to replace here. So, we recall from our earlier discussion how much r dot is. So, look into this equation this r dot you can write form here mu times c sine theta divided by h.

So, here r dot is appearing; so we can write mu times e sine theta, and this divided by h now this need to be simplified little bit. So, what will get from here once plus e cos theta divided by e sine theta, and this times just other things, we have to bring in one place mu one under root and this is divided by one and from here we get h in the nominator and this gives us mu in the denominator.

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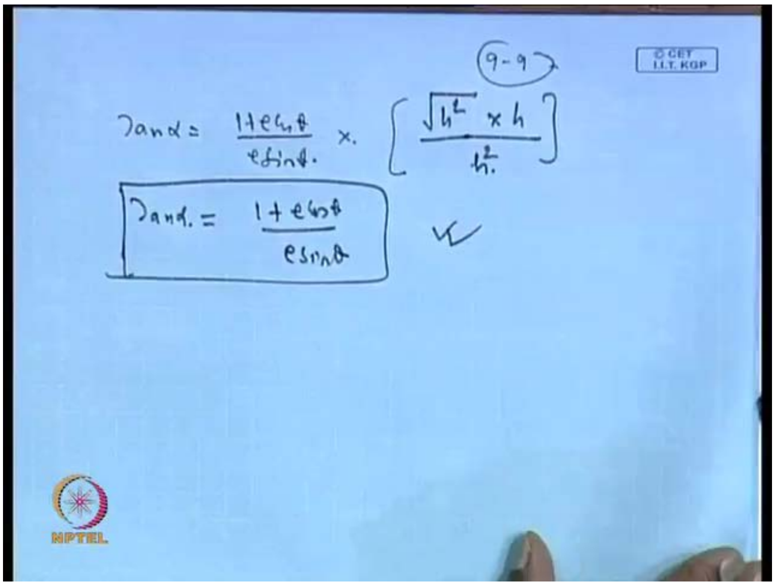

So, we can write here ten alpha equal to one plus e cos theta by e sine theta times now mu one under root is nothing but h s square. So, this quantity mu one under root is nothing but h s square. So, we take the under root of this this is h, and again in the denominator you have done mu times l which is nothing but h s square.

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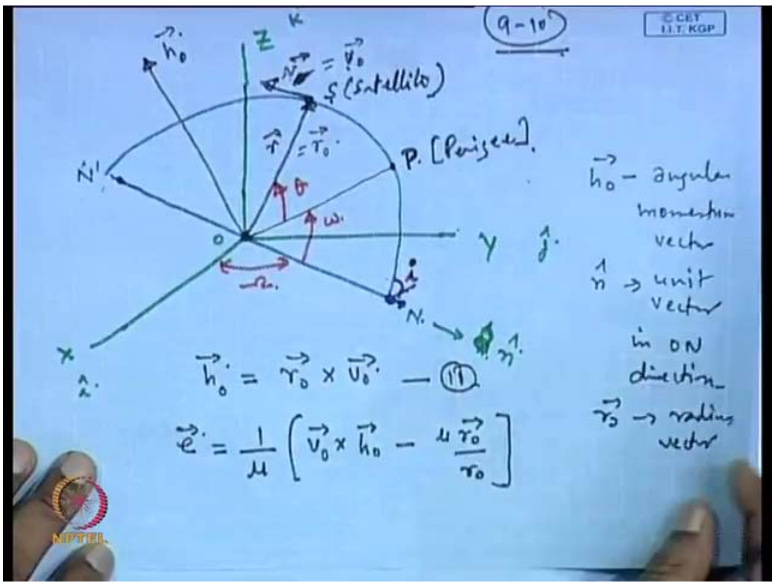

So, this will cancel out leaving with 1 plus e cos theta by e sine theta in this relation in gastro dynamic it is used quite frequently. So, I am worked it out here now we can go for working for the I small omega and capital omega, which are the remaining parameters of the orbit to be worked out now we take in a inertial reference frame indicated by y x y and z unit vector in this direction are k cap I cap and j cap this is the origin here, this is nodal line in prime n, this is call the ascending nodal, this is call the descending node and satellite is moving in this orbit.

So, the orbit that it mix inclination with this x y plane this is your angle of inclination I this is angle I this is your nodal angle which need to be worked out and somewhere on this orbit your perigee position is located and angle to this from the nodal line we are measuring and this is the argument of perigee, and then your satellite may be located in this orbit in any other place on this is your orbiter and here the satellite is moving. So, this is your satellite with certain velocity v 0 or may be for generalized case you can write v and this is the perigee position.

So, the angle from here to here, these also need to be worked out and this is true anomaly. So, these are the three angles that we are theta we have already done this, because we measure theta from the perigee position in the beginning and we have worked out. So, capital omega small omega and this I will be working. So, now, we take a unit vector in this direction and in this direction write this as n cap in this direction we have already shown k, now perpendicular to the orbit is the angular momentum vector, we know that angular momentum of vector is perpendicular to the both radius and velocity vector.

So, if the and radius and velocity vector they lay in the orbital plane this discussion we have already done. So, you are the angular momentum vector it will be lying here perpendicular to the plane of the orbit and here we put h 0 or h or similarly v equal to v 0 for the generalized case. So, once we are seeing this velocity as the burn out velocity. So, this becomes v equal to v 0 and r becomes is equal to r 0. So, here h 0 is the angular momentum vector and n cap is the unit vector in o n direction r 0 is your radius vector, so which this information, we can work out if you more thinks.

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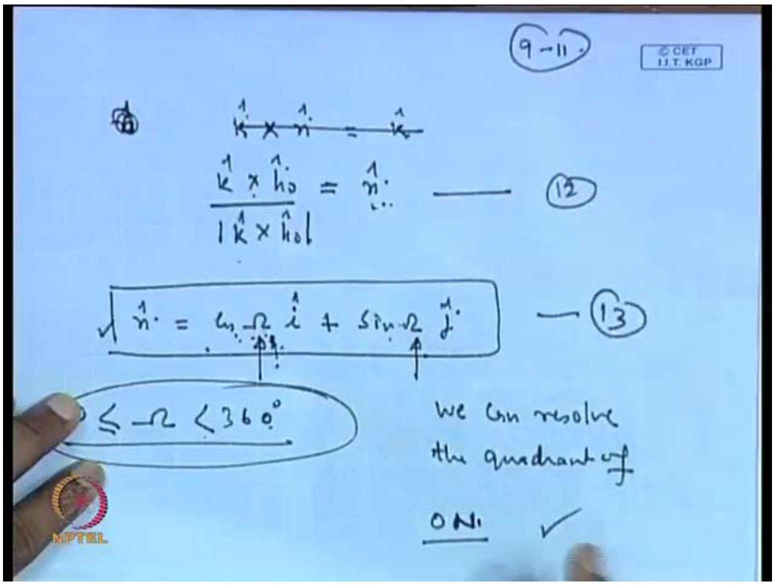

So, let us write h 0 equal to r 0 cos v 0 and we have all earlier written also this equation what once we write it again then it the eccentricity vector if you remember from our earlier lectures we can write eccentricity vector is e is equal to 1 by mu times v 0 cross h 0 minus mu r 0 by r now the unit vector if we take the product of k and h 0. So, we take the product of the unit vector k dot or say k cross h, we have to define n. So, n is a vector which is lying in the orbital plane, but this is perpendicular to the k vector and also h vector which is perpendicular to this orbit and as we incline the orbit.

So, it moves from here to here and this goes from this place to this place by angle I. So, this is also your angle of inclination. So, this vector is also perpendicular to the point vector or the n cap vector. So, we can write this as k cross k cross h 0 k cross we are taking the cross product of this k cross h 0 and divided by magnitude of this. So, k cross h 0 magnitudes and this it will give you the unit vector. So, this is our equation number twelve this gives the unit vector in the direction of o n.

Now, this is the nodal angle, which is a we are interested in finding out. So, you can also express this unit vector in terms of this nodal angle and we can write here n cap equal to cos capital omega I cap plus sine capital omega j cap. So, this vector is available to you here the n vector is available to you and from here also n vector is available to you their capital omega and capital omega we are interested in finding out, so if we get the expression from here and compare the sine omega and capital omega here.

So, we will able to find out the capital omega because this will help the component you can break them like this and after this you just compare the components and that will give you the capital omega and the sine omega now the capital omega it varies from 0 to three sixty degree it will lay from 0 to 360 degree and you will be able to. So, and it is a possible using this 2, because this is a extend formation available towards that n can be describe in this way also. So, therefore, it is you can fix the in which quadrant your this nodal line is line.

This is the first quadrant we are here we are showing x y. So, where it is going this, where the capital omega is lying it is less than ninety degree or more than ninety degree, but less than hundred eighty or it is a line between the in the third part or in the four quadrant that is possible to work out in this case. So, we can resolve the quadrant of o n in which quadrant it is laying it can be resolved now once we have done for the capital omega. So, we are left with the small omega and I, so again we take help of the figure that we have shown here I is the angle of inclination. So, h is the vector here and k is also a vector. So, if we take the dot product of k and h. So, between the angles between this two vectors we will be known to us.

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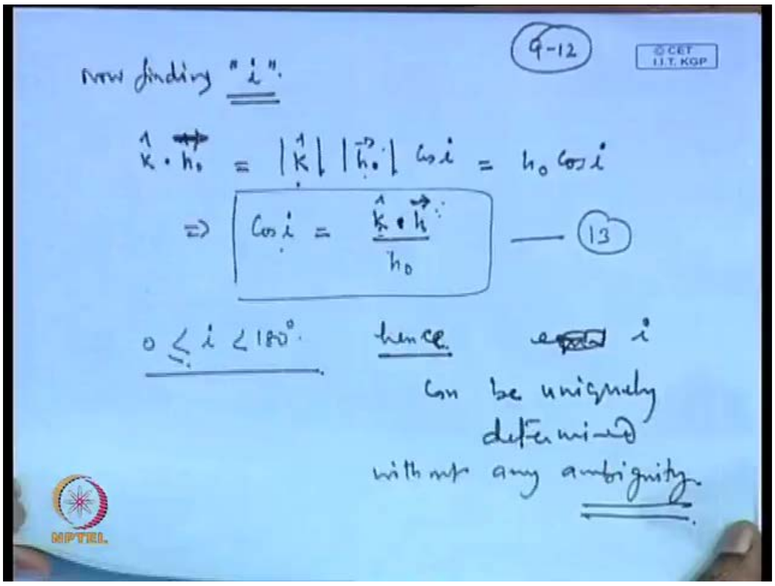

So, let us write here now finding I the angel of inclination the orbit inclination. So, we have k cap dot h 0 cap this is k cap magnitude h 0 this is not cap this is a vector and this implies this is the of unit magnitude and therefore, we get here h 0 cos I and this implies cos i equal to k cap h dot k cap dot h k cap and dot product with h h arrow and then divided by h 0 this is our equation number thirteen.

Now, in this equation we know that I here this is your orbit which is shown here. So, say the orbit is lying in this plane. So, how your orbit inclinations, we will vary say of the orbit can suppose this is the x y plane this is the x y plane. So, the x y plane the orbit is coinciding with by x y plane, and then it can vary. So, this is the angle of the orbit; so it will keep on changing there, and the lower position of this, I can make a break of this I am showing that this line is breaking and showing the lower position below the x y plane and this is up the x y plane.

So, as you move it from the x y plane, this is coinciding right now with the x y plane and this is also in the x y plane. So, it will go like this it comes 90 degree and once you come to hundred eight degree again the orbit is in the x y plane. So, I will vary from 0 to 180, hence there will be no big duty in finding I from this equation, because cos is uniquely determine between 0 and 180 degree. Hence, I can be uniquely determined without any ambiguity.

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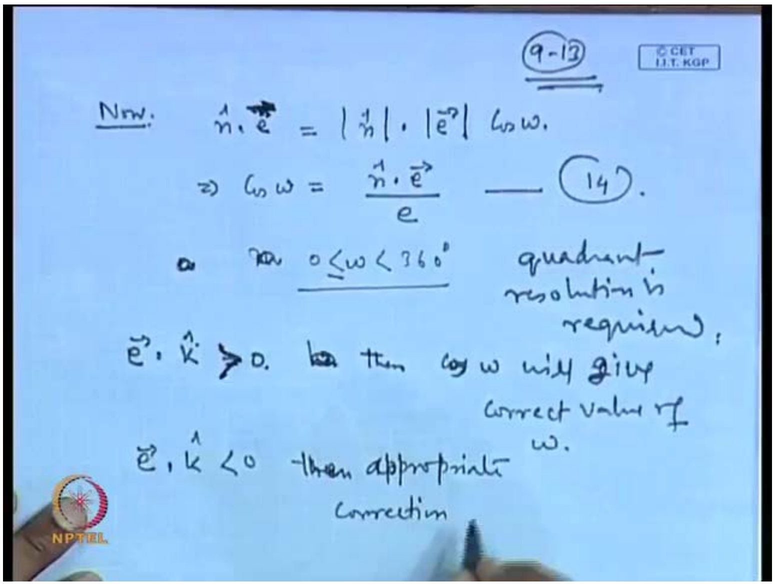

So, if a we are rubbed off for I next we can work out for capital omega is already done for I we have already done the small omega is remaining. Now again we take this figure this is the argument of perigee we from our earlier lecture we know that eccentricity is a vector, which is lying in this plane along the perigee line o p this is the perigee line. So, e vector is lying in this direction and n vector is given to you.

So, if we take the dot product of n and e, so the capital angle omega will be available to us from now n dot n cap dot e cap this is equal to and v indicated by a vector. So, taking the dot product of n cap with e we write this and this implies cos omega this is equal to n cap dot a divided by e this is our equation number fourteen, but you remember the omega this is just deciding the position of a line and it will vary from here to here.

So, it is a total three sixty degree of rotation you can give to it. So, argument of perigee in the in the orbit plane if you look. So, if this is the nodal line. So, from where with respect to this nodal line where the omega will lay either e long this is the perigee here or is the down in which direction the line. So, 360 degree angle is possible therefore, omega here is lying between 0 and 360 degree and quadrant regulation is required. So, if e dot k this is greater than 0 then cos omega will give correct value of omega if e dot cap less than 0 then if a propagate correction must be given.

So, you can see from this place this is the vector k here along the green line and this is the vector e. So, if you take the dot product of this. So, this basically showing the e vector that the component of the vector e along the k direction. So, it will have a positive intersect. So, in this case, so till your vector the argument of perigee it is lying give up this. So, you get this product here and at this dot product and where the intersect will come out to be positive. On the other hand, if your argument of perigee goes below the x y plane, it is a going below the x y plane, then you can see that your e vector will be lying down while that k vector is in this direction, and the intersect will come on the negative side of the k direction. So, this is k cap negative.

So, in this direction, so what you are getting a negative intersect. So, therefore, it is a indicating that condition that if e dot k dot if dot product this is greater than 0 then progressively omega is lying up if this turns out to be this quantity turns out to be less than 0.

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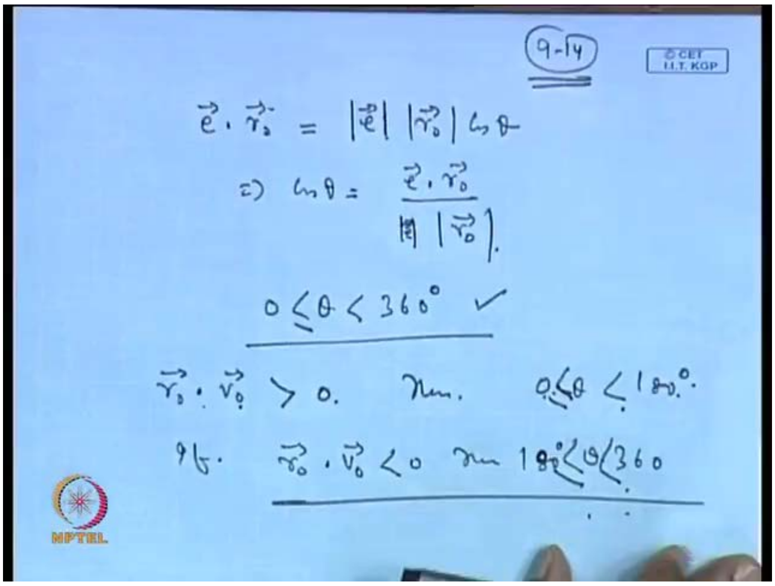

So, this simple indicates that omega is lying below this line and accordingly you can take the decision similarly if we will have the problem with theta also, but theta can be in the same way it can be resolved. So, if you look into the again this theta is the angle between the radius vector and the eccentricity vector. So, we can write here e dot r 0, this is equal to e magnitude times r 0 magnitude and the angle between is cos theta. So, this implies cos theta equal to e dot r 0 by e magnitude and r 0 magnitude.

Now, here that theta again in this case as earlier we have discuss that it lies between three 60 degree, because in the plane of orbit this is the orbital plane. So, your theta argument of perigee from the perigee line, it can lay from 0 to 360 degree. So, this condition is mentioned here now then using this only you cannot resolve it you need further condition and this can work out if you look into the sine of this r dot v dot. So, if r dot v dot is greater than 0, then the theta is lying between 0 and 180 degree; and if r dot v dot is less than 0, then theta will be lying between 180 degree and 360 degree. So, we can put the boundary here like this if we write zero here. So, another we go up to this place from here to here we can write like this.

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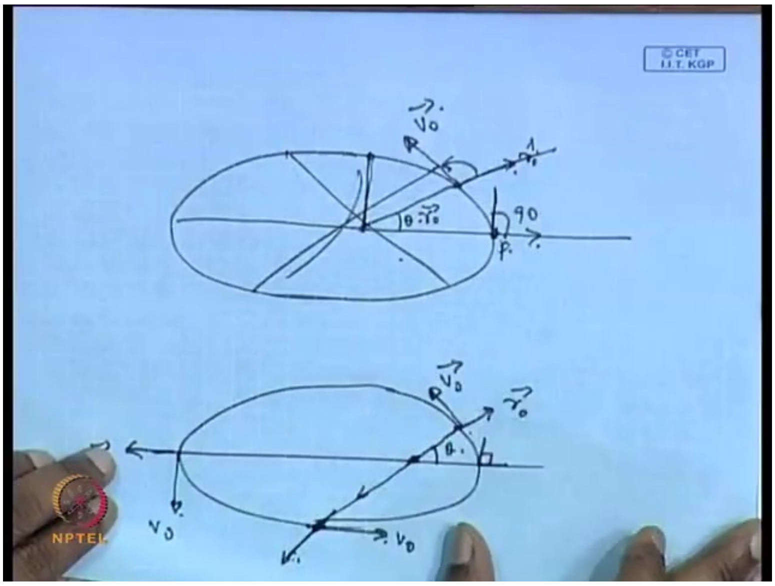

Now, what this exactly, it is a telling this is your perigee position, and this is the orbit r 0 vector this is the angle theta and velocity vector is in this direction. So, you can see that this is that if r 0 is a vector in this direction. So, this is the angle here and now as you keep going from this position. So, here if you look into this position, so here this angle is ninety degree, then it increases thereafter it will increase the figure is not correct, we will let me make it little phase this is the r 0 vector and v 0 vector is lying in this direction.

This is the angle theta. So, at this position this is ninety degree, if you go here you can see that this is getting reverted r 0 will come here, r will come here and v 0 will go in this direction. On the other hand, if you take this just opposite of this, so r is here and v 0 is in this direction. So form here, it will be able to for you will able to resolve this, if you take this r 0 dot product of r 0 and v 0, the we will get exactly whether if it is greater than 0, then if you write it that theta is between 0 and 180 degree and if r dot v 0, this is less than 0, then it is a lying between 180 and 360 degree. And you just reason it on the same basis as we have done earlier for the other angles time is getting over. So, we do not have much time to elaborate further on this. So, we continue in the next lecture, thank you very much.