Space Flight Mechanics

Prof. M. Sinha

Department of Aerospace Engineering

Indian Institute of Technology, Kharagpur

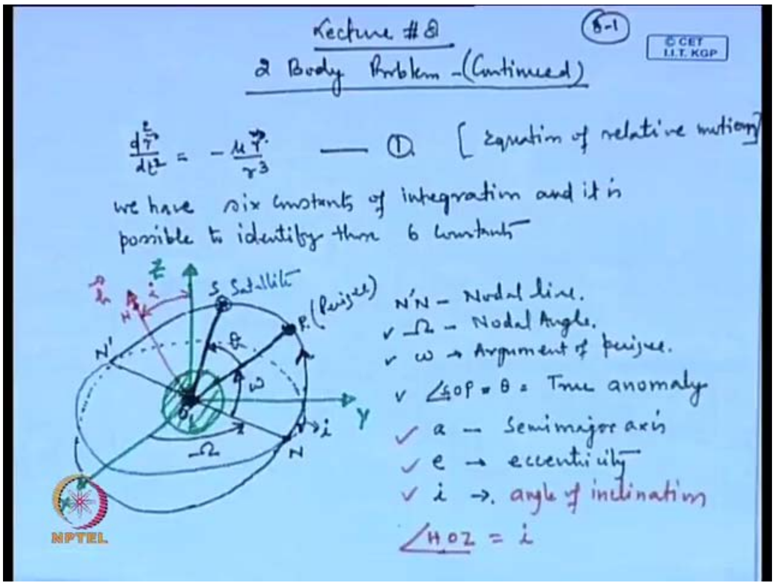
Lecture No. # 08

Two Body Problem (Contd.)

So, in the last lecture we had been discussing about the 2 body problem. So, in this lecture also we continue with the 2 body problem, now we derived few relationships last time regarding the body problem. So, what we worked out that we can only solve the relative motion of the 2 body not the absolute motion of the 2 body problem, because for the absolute motion we have total twelve constants of integration or what is called the constants of motion, but we were able to identify only ten constants of motion or integration. Therefore, the absolute motion description for the 2 body even it is not possible.

Now, for the relative motion of the 2 body means, if one body is moving about the other, if we want to actually the 2 bodies they are moving about the center of mass. But if we want to describe the motion of 1 body about the other body, that is the relative motion of 1 body with respect to the other, then it is possible to describe what kind of orbit it will be what will be the orientation of that orbit.

(Refer Slide Time: 01:35)



So, last time we have seen that it leads to the equation of motion which can be written as d square r by d t square, this is equal to minus mu times r bar r q. So, in this equation this is the second order differential equation. So, this will have total six constants of integration or the constants of motion and it can be solved. So, we have here we have six constants of integration, and it is possible to identify those six constants. So, this is equation of relative motion. So, we can identify the six constants of integration.

Now, what those constants are. So, in the two body problem itself we will come across the orbit determination problem. So, in the orbit determination problem we have six parameters involved. So, the six constants they are called the six or vital parameters. So, we will look into one set of the orbital parameters. So, we have here let us say suppose this is the x y z is the inertial reference frame. So, in the inertial reference frame about which we are assuming that last time as we have seen that if we have considering the relative motion, then it is a possible that one of the heavy-heavy the heavier mass can be considered as the about the center of the heavier mass we can fix an inertial reference frame because the whole equation it turned out in the in this way. So, in this equation if we multiply the left hand side by m 2, so what we were getting the equation of motion of the particle two with respect to the one.

So, we considered the particle one is at the center of this inertial reference frame, and the second particle is moving. So, in this case here let us say this main particle which is present at the center. So, we have earth in this case. So, we have earth here whose center at the center of the earth this triode is fixed this reference frame is fixed and about this our satellite moves in an orbit. So, we have this orbit here. So, in this orbit and this is the projection of orbit shown by the dotted line. So, this is the blue line, this is showing the projection of the orbit in the x y plane, and let us say some satellite is here.

So, of course, if we try to complete this orbit, so here the blue line its lying in the x y plane and while the black line it is above the x y plain and then it will go down below the x y plane and it will come and meet life here in this place. So, this is the line n n prime this is the point row. So, n prime n this is called nodal line the angle from x 2 o n. So, this is the inertial reference frame. So, this is the x direction. So, this angle whatever we measured this is called termed as nodal angle. So, let us suppose that satellite is moving in this way from top to from bottom to top on this side and the satellite is here in this place and there is one line which is called the periapsis line.

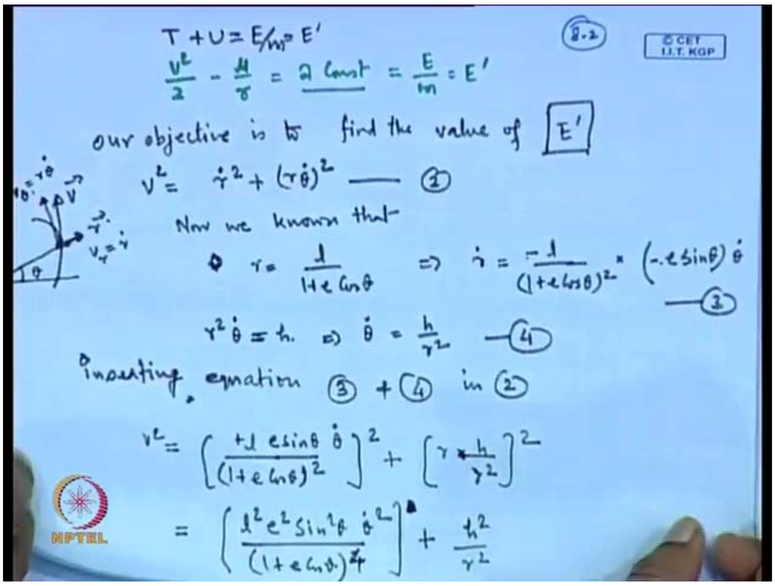
So, this is the point of perigee perigee is the point of nearest appose to the point o. So, the distance between point o and p is minimum. So, this is p is called the perigee point. So, this is perigee. So, the angle between the nodal line and the perigee line this is termed as a small omega. So, small omega is this is called argument of perigee argument of perigee while the angle between the periapse line, and the satellite where we have written o s. So, the angle is o p, this is equal to theta this is called true anomaly.

Thereafter, so we have three constants here capital omega small omega, and theta just we are left with three other constants. So, out of that the semi major axis, a is another constant semi major axis and then e the eccentricity and the last one which is i. So, i is the angle between the plane indicated by the blue line or the blue either ellipse or circle whatever it is... So, which is the projection of the black curve, so this angle is exactly here this one this is your angle of inclination i.

So, this is an inclination between this plane and this plane or either way we can say that this is also an if we have the angular momentum vector which is perpendicular to the orbital plane. So, if we indicate it by h here. So, the angle of inclination will be given by the line o z and o h. So, this is let us saying here this we write as capital h here. So, this is angle of inclination. So, this angle we can write as h o z. So, a total of six parameters we have been able to point out here. So, an orbit can be characterized by this total six number of parameters, so before we go into the. So, main or main objective of the two body problem will be to find out the in which orbit the satellite is moving at what time where it will be. So, our ultimate objective will be to resolve this problem in this two body problems two body system problem.

So, before we go into that the few things which we are continuing earlier. So, we will continue with that and find out few more relationship to have more insight into this problem and there after we will come to the orbit determination. So, orbit determination here in this case it will be we will define it as given initial position the six components of the position, and the initial velocity the six components of the velocity vector. So, find out the find out this six parameter, so which are a e i capital omega small omega and theta

(Refer Slide Time: 11:30)



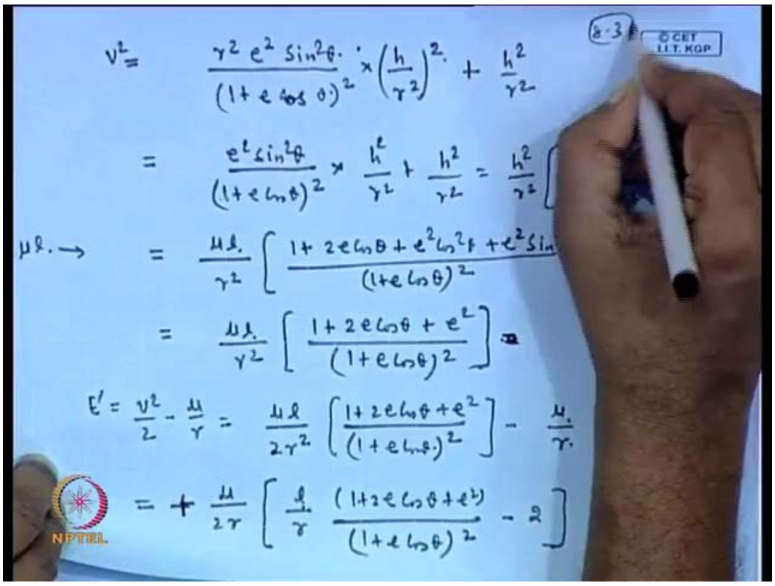
So, last time we found that v a square by 2 minus mu by r this is a constant and this was nothing but the total energy per unit mass. So, we could write it this way this we discussed last time. So, now, this time we have we want to know how much will be the value of e prime or either e y m. So, our objective here objective to find the value of e prime, so e prime can be worked out in various ways, but one simplest way will be two evaluate the left hand side and see what this is equal to because if we this we know this quantity is the kinetic energy while the this quantity with minus sign this is potential energy, and this we have written as the e y m or e prime which is the total energy per unit mass.

So, if we try to evaluate the left hand side. So, automatically we can evaluate this will be equal to the right hand side. So, this is 1 way of doing there are other ways of doing the same problem, but let us look into the problem in a very simple way. So, here we this is nothing but v square we can write as the radial velocity that is this is angle theta this is 0.2. So, if the satellite is moving in this direction this is the vector r. So, in this direction the velocity component v r it can be written as r dot and perpendicular to this can be written as v theta which will be r times theta dot. So, these are the two components of the velocity. So, we can write v square as r dot square plus r times theta dot square.

Now, we know that r is equal to l by 1 plus e cos theta. So, this implies r dot is equal to l by 1 plus e cos theta whole square minus sign, we put here and then differentiating this is minus e sine theta into theta dot, and we know that theta dot is equal to r square times theta dot is equal 2 h is a angular momentum per unit mass. So, from here we get theta dot is equal to s by r square. So, this is our equation number two and this is equation number three and this is four. So, inserting inserting equation three and four into, so left hand side we can write as v square this will be equal to minus l by 1 plus a cos theta square we put it plus to remove this minus sign here, and this is e sine theta times theta dot. So, this whole square and plus r times h by r square is whole square.

So, this quantity it can be written as l square e square sine square theta and theta dot square divided by 1 plus a cos theta whole square or if we put here four remove this a square from this place times h square by r square, so further working out this, so we know that l by 1 plus e cos theta is equal to r.

(Refer Slide Time: 17:14)



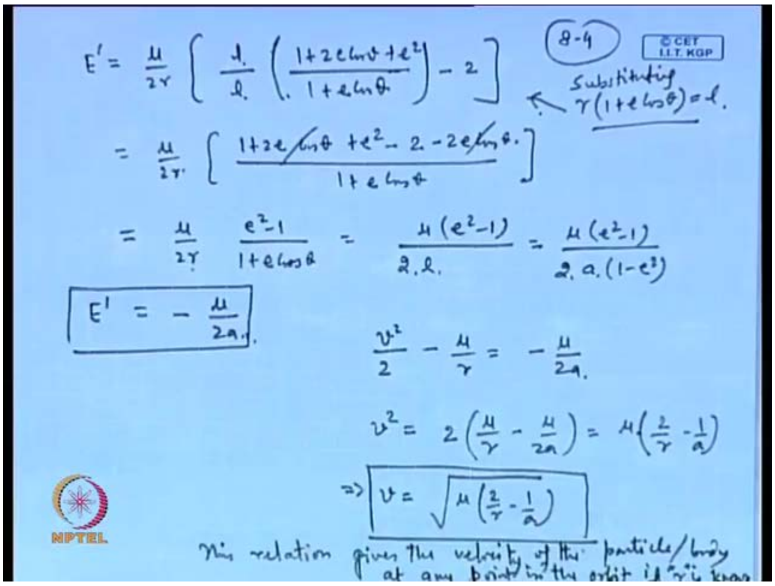
So, we substituting this value into this the previous equation. So, that will give us r square e square sine square theta times theta dot square also we need to replace by 1 plus e cos theta whole square. So, here we are replaced l by 1 plus e cos theta whole square by r. So, we are left with this quantity, and then that theta dot is there which is nothing but h by r square whole square and plus h square by r square.

So, this we can write as e square sine square theta r square r square here cancels out leaving at r square in the bottom. So, 1 plus e cos theta whole square times h square by r a square by h square by r square h square by r square, if we take it common. So, we are left with 1 by 1 plus a square minus square theta 1 plus a cos theta whole square the quantity h is square which is nothing but mu by l if we remember from our earlier derivation. So, here we are substituting h square is equal to mu times l we are substituting into this. So, this is mu l r square and then we further solve it by this is 1 plus 2 e cos theta plus e a square cos square theta plus e a square sine square theta divided by 1 plus e cos theta whole square mu l by r square 1 plus 2 a cos theta plus e square divided by 1 plus a cos theta whole square.

Now, it is a very easy to see from this place the complete result. So, what we do now we need to evaluate we can further simplify this little bit. So, mu times let us go little further bit and then to work it out. So, we have e prime is equal to v square by 2 minus mu by r this we wanted to evaluate now putting this quantity here this is mu l by 2 r square times 1 plus 2 e cos theta plus e a square divided by 1 plus e cos theta whole square minus mu by r. So, evaluate this quantity and this quantity will ultimately turn out to be minus mu by 2 a.

So, now let us take minus mu by 2 r common from this. So, if we or plus mu by 2 r, if we take this common from here what we are left with l by r times 1 plus 2 e cos theta plus e square divided by 1 plus e cos theta whole square minus 2 this quantity can be solved further, now we know that r is equal to l by 1 plus e cos theta means r times 1 plus e cos theta will by l.

(Refer Slide Time: 21:51)



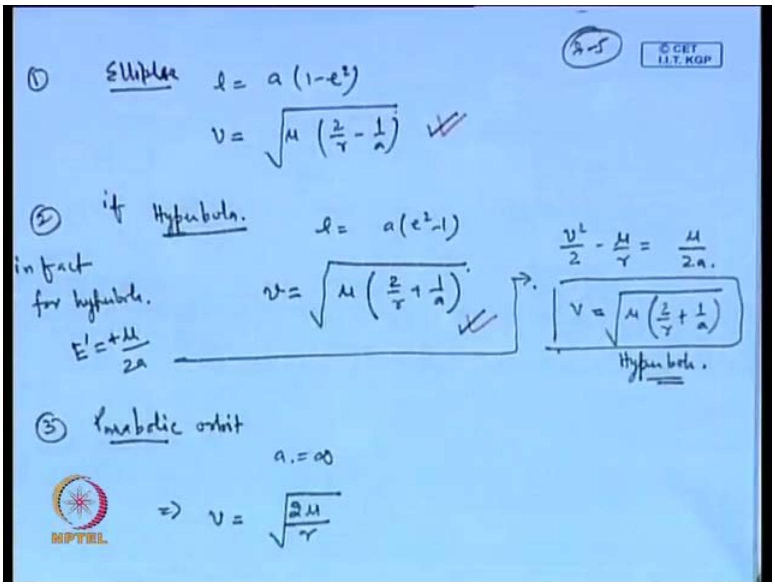
So, here there is possibility of cancelling out few of the terms. So, we concerned it, we write it here. So, e prime is equal to mu by 2 r l by r plus 1 e cos theta this becomes one. So, we are left with 1 plus 2 e cos theta plus e square divided by 1 plus e cos theta minus 2.

So, we have made here very simple substitution r times 1 plus e cos theta this is equal to l substituting we have got this result. So, this becomes mu by 2 r 1 plus 2 e cos theta plus e square minus 2 minus 2 e cos theta divided by 1 plus 2 cos theta. So, this is mu by 2 r times this this cancels out and we get here e square minus 1 divided by 1 plus e cos theta.

So, this is nothing but mu times c a square minus 1 times 2 times r times 1 plus e cos theta is equal to l, and now l we know that the semi lattice vectum this can be written as 2 times a times 1 minus e square. So, the quantity becomes equal to mu by 2 a thus e prime is equal to minus mu by 2 a. So, once we know this quantity that e prime is equal to minus mu by 2 a. So, our original equation we can write as v square by 2 minus mu by r is equal to minus mu by 2 a, and from here if we resolve it. So, this can be written as mu by r minus mu by 2 a times 2 taking out the mu outside taking two inside. So, this is 2 by r minus 1 by a, and this implies v is equal to mu times 2 by r minus 1 by a.

So, this relationship gives the velocity of the particles like body at any point in the orbit if r is known. So, if r is known then we can find out the velocity of the particle or the body at any point in the orbit.

(Refer Slide Time: 25:23)



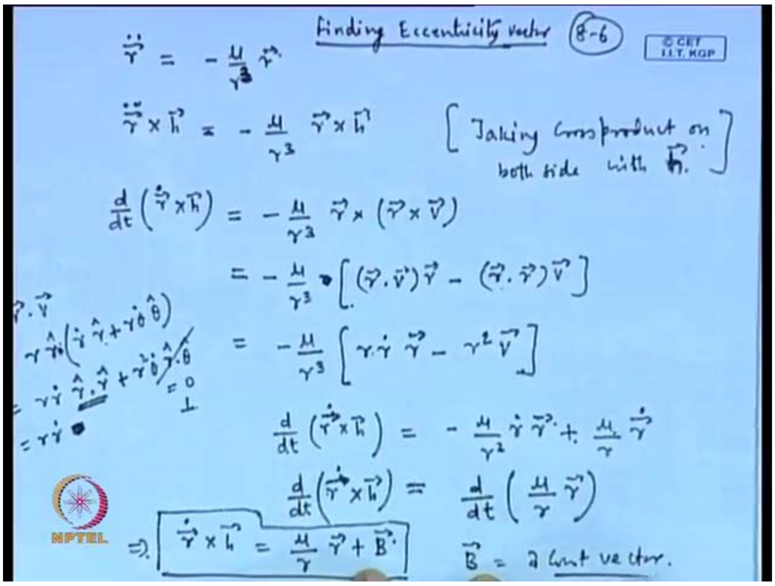
So, if the equation that we have derived this is for an ellipse for an ellipse we have a is equal to l is equal to a times 1 minus a square; therefore, for an ellipse the v will be given by mu times 2 by r minus 1 by a if the we have hyperbola. So far that the one will be given by a times c square minus 1, so the result will get reverse and here the change of sign will take place.

So, we can write this as v is equal to mu times 2 by r plus 1 by a under root we can see this from this here itself because the one we substituted as a times 1 minus e square if the same thing we write as a times a square minus 1. So, e square minus 1 will cancel out and we get a positive sign here. So, here instead for hyperbola instead of a negative sign this will be positive sign. There this will be positive sign. So, v square will be mu by r plus mu by 2 a. So, we get a positive sign here in this place. So, here I can write it that for hyperbola v square by 2 minus mu by r is equal to mu by 2 a, so in fact that here we… In fact, for hyperbola e prime is equal to mu by 2 a this is with positive sign; therefore, we get this figure.

So, from here this can be resolved to give mu times 2 by r plus 1 by a under root this is the relationship for hyperbola, if we have parabola if the satellite or the particle is moving in the parabolic orbit for the parabolic orbit a is equal to infinity therefore, this simply implies that v is equal to 2 mu by r under root. So, so we have seen here how to find the velocity of the particle in the orbit now this relationship that we have derived here this relationship it is very important and serves our many purpose, and it is the most handy equation.

Now, till now we have seen that the eccentricity of the particle in eccentricity of the orbit. So, this has a scalar quantity.. In fact, the eccentricity this can be defined as a vector. So, magnitude of that vector is nothing but the eccentricity. So, let us look into this. So, though we will be repeating few of the things, but the derivations the method we are trying to apply and workout it gives further insight into the solving of the problem and it brings more insight like eccentricity is a vector which is replaced towards the perigee. So, how to think about this kind of thing, so this can be worked out mathematically. So, let us try to work it out and look into what exactly the eccentricity vector looks like.

(Refer Slide Time: 29:57)



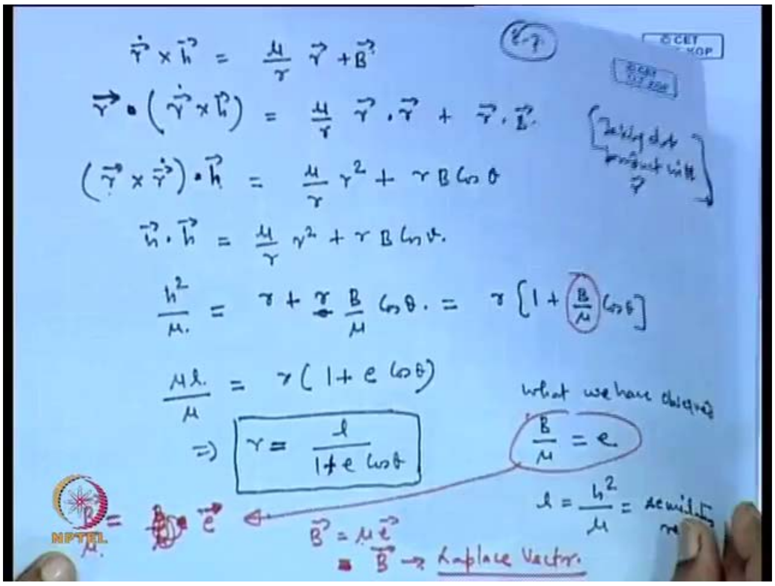
So, we already know r double dot this is equal to minus mu by r square times r q times r now, if we take the cross product on both side with the s vector. So, we get this equation what we are trying to. So, what we are trying to do here finding eccentricity vector there is an another name to this. So, we will come to this once if we derive then you will see how the thing looks like. So, this can be written as d by d t times r dot cross h, if we take the derivative of h because h is a constant.

So, that will be zero. So, we if we break it this will give you d square r by d t square means the r double dot and similarly we have on the right hand side this can be written as r cross r cross v, now little bit of working on this it will resolve this problem it will make it simpler to work out. So, let us write the this as breaking this bracket here r dot v times r minus r dot r times v mu by r q now r dot v this we have to evaluate this quantity is nothing but r times r dot I will show you how this. So, r times r dot times r minus this is nothing but r square and v.

So, now let us into this quantity r times v dot how this is equal to r times r dot. So, r we can write this as r times r cap the unit vector in the r direction, and v we can write as r dot times r cap unit vector in the r direction plus r times theta dot times unit vector perpendicular to the r vector which is theta cap. So, this becomes r times r dot this is the dot product here times r cap from r dot plus r square theta dot times r cap theta cap. So, this quantity is zero, because they are perpendicular to each other therefore, what we get r times r dot and this together multiplied this is the unit in magnitude. So, we replace it r by r times r dot.

So, this is the simplifying relationship here. So, finally, what we get d by d t r times in this is r dot here sine is missing is r dot cross h. So, r dot cross h is equal to minus we can cancel out r here. So, minus mu by r square times r dot r plus mu by r times v is nothing but r dot. So, we write it here in this way. So, if you look this quantity is nothing but d by d t times mu by r times r then differentiated this quantity will turn out to be. So, if we integrate this equation. So, this is d by d t r dot cross h. So, this gives us mu by r times r plus another vector v, now b is a constant vector here.

(Refer Slide Time: 35:39)



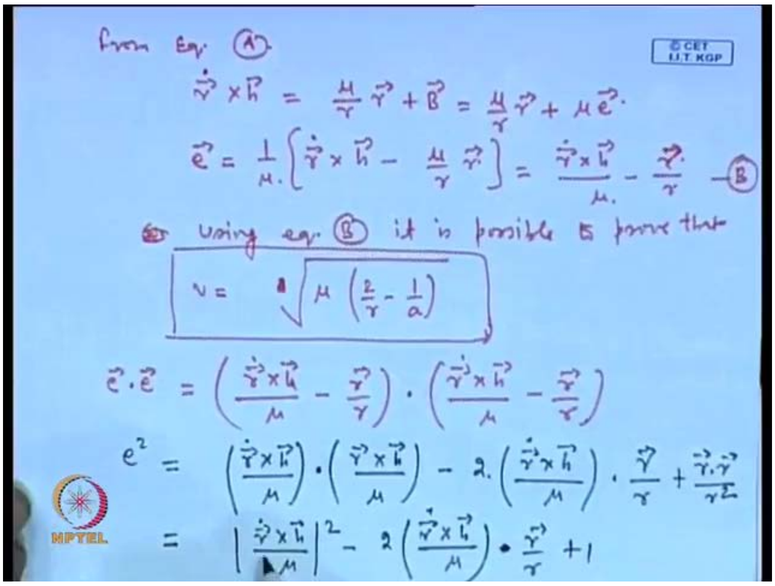
Now, we can get a lot of insight from this equation. So, whatever the earlier relation we have derived that is r is equal to l by 1 plus e cos theta we can get from this equation also. So, if let us have a look of this. So, we have r dot cross h this is equal to mu by r means r plus b now taking dot product with respect to r dot, So, with taking the dot product with respect to r. So, here r dot times r dot cross h mu by r b taking dot product with r. So, this gives us we can write it another way this will be r cross r dot exchanging the dot and the cross this is the property of the vector this vector product. So, we have here on the right hand side mu by r r square r r dot square, and this we can write as r b cos theta where theta is the angle between r and b.

So, what we get from here we can see this r times r r cross r dot this is nothing but h. So, this becomes h is dot is equal to mu by r r square plus r b cos theta that is h square let us divide this by mu the whole thing. So, h square by mu this becomes here r r, we can cancel out. So, this becomes r plus r b and divided by b times mu cos theta now look into this relationship if we take it out r outside what we get 1 by 1 plus by mu cos theta, and we know from earlier that h square by mu h square is nothing but mu times one we have earlier defined this as. So, if we write it in this way see what this gets reduced to. So, this can be written as one plus and here b by mu if we write as e. So, this implies r is equal to l by 1 plus e cos theta. So, what we have seen here that b by mu what we have observed that v by mu this is nothing but equal to e.

And l we have defined as H Square by mu which is semi lattice rectum this is parameter of the orbit. So, this is the relationship that we derived much earlier for the central force motion and again we have got the same relationship by treating in some another way, but here if you look what is most important this is the quantity b by mu b is a vector therefore, if we write b as quantity b by mu and times. So, b is a vector. So, if let us write this b by mu as b time let us define in this way b by mu we have defined as e. So, if we write as like this. So, this will be simply b by b by mu.

B is b by mu is nothing but e b by mu is nothing but e and therefore, b r by mu will be nothing but a vector which will be given by e e vector therefore, what we get from eccentricity vector and this is. In fact, the b we can write as mu times e bar and this is known as this b vector this is known as lap lace vector. So, if he multiplies the eccentricity vector by mu we get the lap lace vector b or other way we define divide the lap lace vector by mu we get the eccentricity vector e. So, eccentricity is a vector which is directed toward the perigee and it is a very easy to prove.

(Refer Slide Time: 41:39)

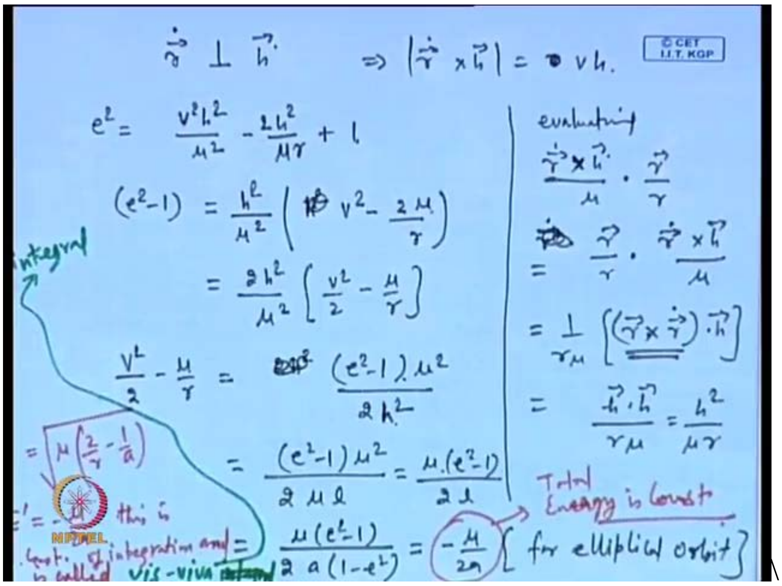


Now, so, let us define this equation as the earlier equation number let us write this equation as. So, from equation a equation a now this this equation as a we are defining this equation as a not found this one as a. So, from here we have r dot cross h. So, r dot cross h this is equal to mu by r times r plus b, we can write this as mu by r times r plus b is equal to mu times e; therefore, the eccentricity vector we can write as r dot cross h minus mu by r times r divided by one by mu is equal to r dot cross h divided by mu minus r by r.

So, term this as equation b. So, using equation b we can work out the relationship that is using equation b it is possible to prove that v is equal to mu times two by r minus 1 by a under root. So, we see that the same relationship can be derived in multiple ways. So, here what we need to do that you take the dot product e dot e. So, this will b equal to r dot crosses h by mu minus r by r in dot product of this r by r. So, the left hand side we can write as e square and right hand side we have to expand. So, if we do this expansion it is a very easy to work it out.

So, first what we will do here we write this in a resolved way expand it and write it by mu minus 2 times r by r and plus r dot r by r square. So, you see from here that this is r dot cross h this is a vector which is lying in the plane of the orbit, because r dot is a vector in the plane of the orbit and h is a vector perpendicular to the orbit; therefore, the cross product will lay in the plane of the orbit, and because this is a dot product. So, we can simply write as r dot cross h magnitude mu whole square, and minus this quantity here 2 times r dot cross h mu dot r by r plus 1 this quantity equal to 1.

(Refer Slide Time: 45:25)



So, this is nothing but you are taking the magnitude of this. So, the angle between the r dot. So, vector r dot is perpendicular to h and therefore, this implies r dot cross h magnitude this will be equal to nothing but v times h. So, the equation can be written as e square equal to v square h square divided by mu square minus now expand the quantity r dot. So, what we need to evaluate evaluating r dot cross h divided by mu times r by r. So, here again we can take advantage of the properties of the dot and the cross product. So, here we can write this as r dot times r by r dot r dot cross h divided by mu, and this is nothing but 1 by r mu times r cross r dot r dot h and this quantity is nothing but h. So, this is becomes h times h dot by r mu, so for this becomes h square by mu r.

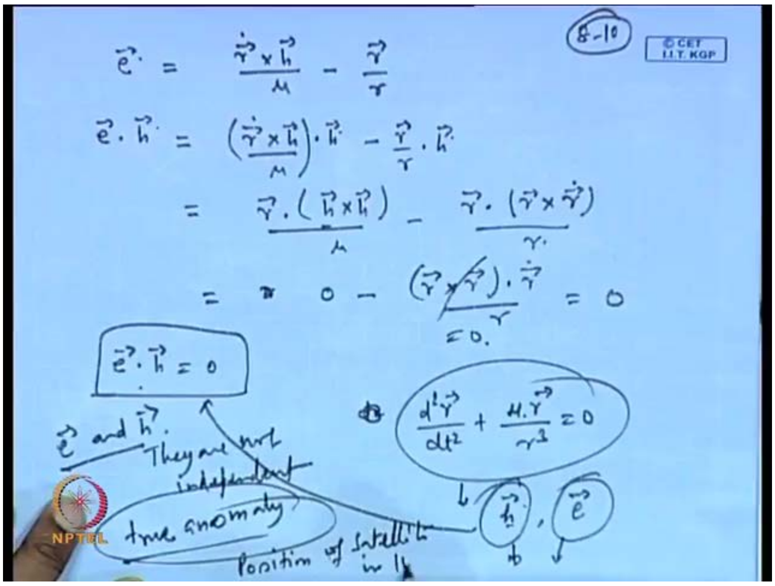
So, here we get H Square by mu r and multiplied by two and this is plus 1 and now taking this one on this side. So, we have a square minus 1 this becomes equal to and further we can take few things outside. So, let us say we take the mu square outside here. So, we take h square outside here on the right hand side and then the mu square also and what else we take further let us work with this. So, e square minus 1. So, this becomes h square h square sorry this is v square and on the right hand side we have minus 2 h square we have already taken out mu square taken out. So, there we write mu and divided by r.

So, if we take further two outside. So, this is 2 h square by mu square and this becomes v square by 2 minus mu by r. So, we can write this as v square by 2 minus mu by r equal to 2 h square or e square minus 1 times mu a square divided by 2 h square it is a very easy to see from this place h square is nothing but mu time one. So, e square minus 1 mu square times h square is mu time one. So, this gets reduced to mu times a square minus 1 divided by two one and one we know what this one is… So, if we put if we have an elliptical orbit. So, one we can replace as a times 1 minus a square this becomes minus mu by 2 a for elliptical orbit.

So, the same result we get in multiple ways, but this is very important and this is basically our b square or from here we can solve this and we can write. So, this implies b is equal to mu times two by r minus 1 by a under root the earlier relation that we have got and here this gives the energy is constant the total energy is constant. So, this total energy e prime is equal to minus mu by 2 what we have written this is also called this is this is a constant of integration, and is called this viva integral this viva integral this viva integral.

So, we see that the same relationship can be derived in multiple ways, but this is also gets into the insight how the how to work out the problems in what the eccentricity is exactly. So, earlier we got the eccentricity to be just in a scalar quantity where here it turned out to be a vector which is nothing but the lap lace vector divided by mu. So, after going into this we can find out the further results, but I would like to restrict today in this lecture for till this extent only discuss further what are its implication.

(Refer Slide Time: 51:42)



So, what we have seen till now that we have found the e vector which was nothing but e vector was r dot cross h by mu minus r dot r. Now, let us take the dot product of e. So, this page number is its eighty seven this is 80 this is 89. So, take the dot product of this with h and let us see what happens. So, this we can write as by mu minus r dot and h is nothing but r cross r dot or r cross v divided by r. So, if we look into this quantity is cross h is 0. So, this becomes 0 and here we can write this in similar way r dot by r. So, this quantity is also equal to 0 therefore, this gets is gets reduced to 0. So, this is what we have got.

So, while working with the equation d square r y d t square plus mu r by r q is equal to 0 we found out in the way two integration constant h and e. So, these two integration constants both are constant and they together contain the information about the orbit shape size and orientation, but here h and e dot they are not independent as we can see from here they are not independent e and h. So, alone these two or not able to completely describe the orbit, so what we need further this is the true anomaly true anomaly which is a quantity which defines the position of the satellite-satellite in the orbit.

So, we need to define the true anomaly. So, next lecture what we will see that we want to define the orbit completely. So far defining the orbit completely we need those 6 parameters we have discussed earlier that is the small omega-omega the argument of perigee capital omega the nodal angle I the angle of inclination then a the semi major axis and then e the eccentricity, and the last one the true anomaly. So, these are the six constants of the orbit or the six constant of the motion, if we integrate this we get six constants of integration constant six constant of integrations. So, those integration constant this will be defined as the six orbital parameters.

So, next time we start with given the initial position and the velocity of the particular or the body then find this six orbital parameter, So, overall the conclusion is that given this problem of two body the absolute motion we cannot describe, because as of total we can get only ten constants of integration we are able to identify only ten constants of integration what it involves twelve constants of integration.

But if we are working for the relative motion of one body with respect to other then; of course, we can solve it and describe the motion of the second body about the first body and the motion is given by this equation, and this is a second order ordinary differential equation. So, it will have six constants of integration and those six constants of integration are the six parameters of the orbit which are called the orbital parameters. So, thank you very much we continue next.