Space Flight Mechanics

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Lecture No. 07

Two Body Problem (Contd.)

So, in in the lecture number 7 we are continuing with the gravitational central force motion and thereafter we will go into the two body problem.

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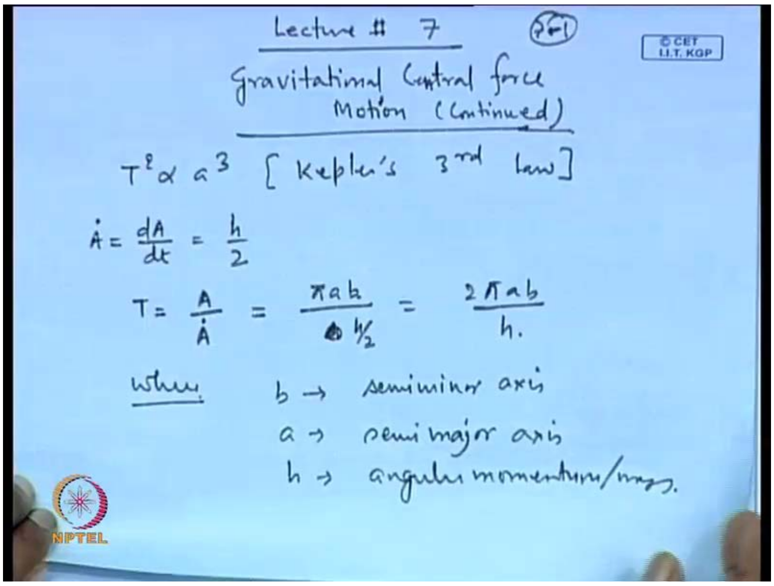

See in the gravitational central force motion the last thing was to be proved that T square is proportional to a to the power cube which is Kepler's, Kepler's third law. So, we know that d A by d t is equal to A dot the rate of ship of the area, this is nothing but h by 2 where h is the angular momentum per unit mass. Therefore, from this place the time period can be written as the A by A dot, that is the the A dot which is the rate of shift of the area. So, total area to to be shifted is the ellipse area. So, the ellipse area can be written as pi a b divided by A dot. So, A dot we replace as h by 2. So, this becomes 2 pi a b divided by h where b is the semi minor axis, semi minor axis and a is the semi major axis and h is the angular momentum per unit mass.

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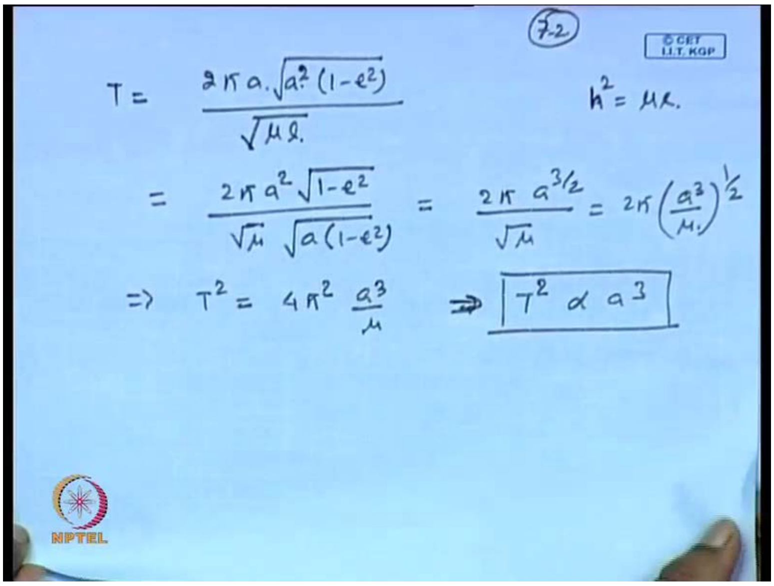

So, time period then will become 2 pi a and b can be written as a square times 1 minus e square under root and h obviously can also be replaced. So, h is nothing but mu times l under root, we have seen this early earlier that h square is equal to mu times l. So, if we insert here into this place, so this becomes 2 pi a times this becomes a square 1 minus e square under root divided by mu under root and l is nothing but a times 1 minus e square. So, then this under root. So, together this gives 2 pi a to the power 3 by 2 and 1 minus e square under root will cancel out. So, leaving out mu under root. So, this is 2 pi a by mu cube to the power 1 by 2.

This implies T square is equal to 4 pi square a cube by mu and this implies T square is proportional to a cube. So, this is what we wanted to prove. That square of the period of the body in the orbit is proportional to the cube of the semi major axis or the also this is sometimes called the average distance from the sun in the case of earth. So, till now what we did using the central force motion for inverse square law of the gravitation.

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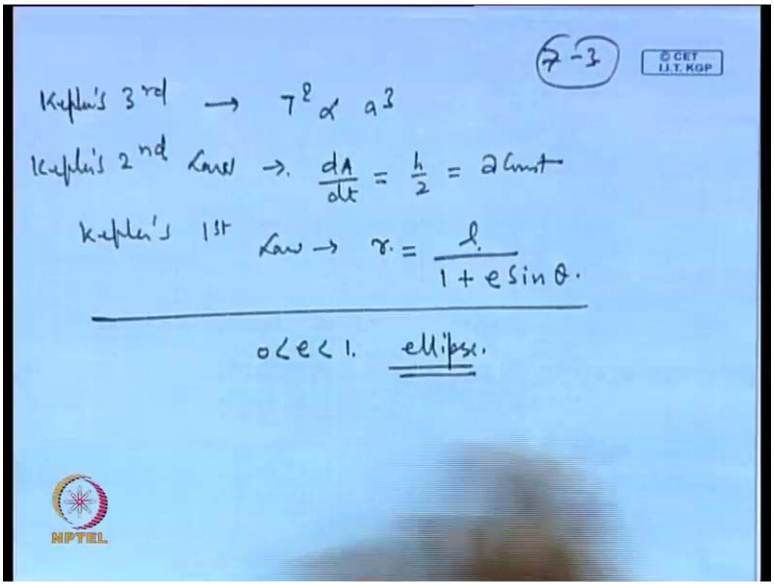

We proved the third law. So, third law has been proved as T square is proportional to a cube. We proved the second law, Kepler's third law, Kepler's second law is nothing but d A by d t, h by 2 this is a constant. This is what we have stated in the beginning. And Kepler's first law that every heavenly body moves or the every planet moves around the earth in an elliptical orbit with sun at once, one of its focus and the corresponding the equation for the conic section this ellipse, this can be written as l by e sin theta. So, this is what we have derived till now. So, for ellipse we have written e to be between 0 and 1 for ellipse. Now, we go into the two body problem.

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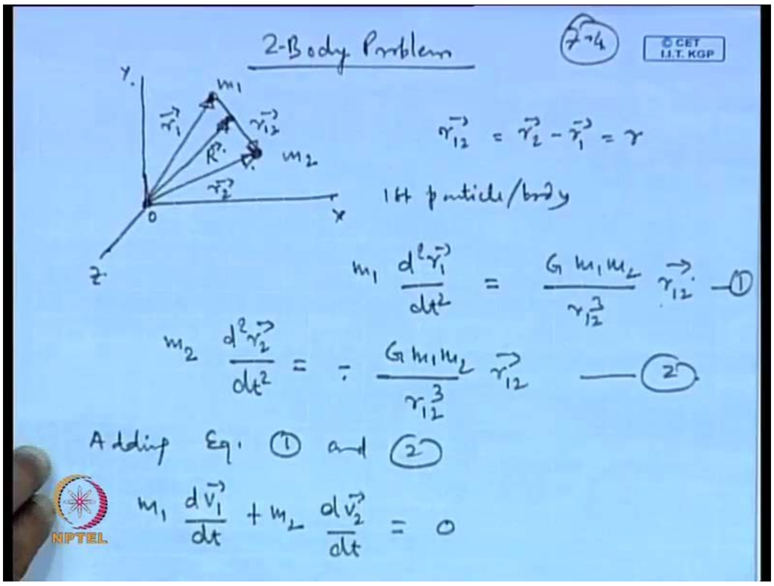

So, in the two body problem we have two body problem, we have two masses m 1 and m 2 whose radius vectors are given as r 1 and r 2. A vector from m 1 to m 2 is written as r 1 2, the center of the mass of the particle m 1 and m 2 this is written as R. So, we can write the equation of the motion of each of the particle in this inertial frame. Let us say this is x, x y and z is in this direction, this is o. Therefore, we will get for each of the particle a second order differential equation and first let us develop the equation.

Thereafter, we will see what insight it gives into this problem. So, we are, we write r 1 2 is equal to r 2 minus r 1 is equal to r. So, for the first particle body we will have m 1 times d t square. This is equal to G times m 1 m 2 divided by r 1 2 whole cube r 1 2. So, motion of the particle m 1, this is getting is attracted by the mass m 2. The the force on this will be m 1 times m 2 multiplied by the universal gravitational constant divided by the distance between this cube of the distance between this two particles and the unit vector in the direction of the force. So, this is what we have written. So, this is our equation number one.

Similarly, for particle two we can write r 2 d t square, this is equal to minus G times m 1 m 2 r 1 2 whole cube r 1 2. Now, in the case of particle two the force is applied opposite to the r 1 2 direction. So, we put a minus r here in this place. So, this is our equation number two. Adding equation one and two, so what we do? We add this two equation. So, we get from here m 1 times and this we can write as in terms of v 1; velocity of the particle so we can also write this as m 1 d 1 d v 1 by d t plus m 2 d v 2 by d t. And we see on the right hand side that it cancels out is equal to 0.

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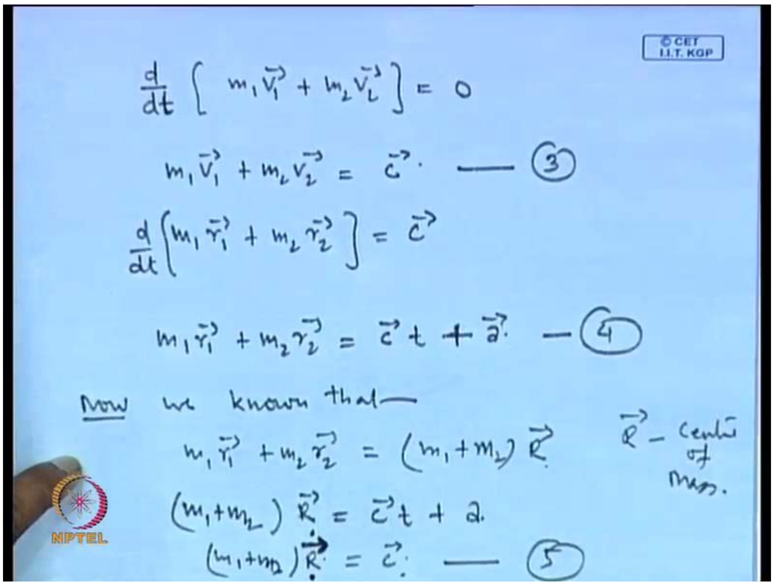

So, we can if the mass is constant the same thing we can write as d by d t m 1 v 1 plus m 2 v 2 is equal to 0. Now, the quantity inside the bracket, this is nothing but the linear momentum. So, we can write here now m 1 v 1 plus m 2 v 2 is equal to a constant. Let us write this constant as some constant let us say this is c. Now, again v can be written in terms of r. So, if we write the same thing. So, in terms of r, so this becomes m 1 r 1 plus m 2 r 2 is equal to c then this equation we will write as third equation.

So, next we have if we integrate it further we get m 2 r 2 is equal to the c t first let us see another constant a. So, this is equation number four. Now, we know that Now, we know that m 1 r 1 plus m 2 r 2 is equal equal to m 1 plus m 2 R where R is the center of mass. Here R is the center of mass. Therefore, equation four can be written as m 1 plus m 2 times r is equal to c t plus a or equivalently we can write m 1 plus m 2 if we differentiate with respect to t. So, this becomes R dot is equal to c.

So, this equation, the equation number five implies from here we can see this is the R is, R dot is the, R dot is the velocity of the center of mass and m 1 plus m 2 is the total mass of the two particle. So, from here we and the right hand side is the constant vector. So, from here we can conclude that the velocity of the center of the mass it continues to move move with a constant velocity.

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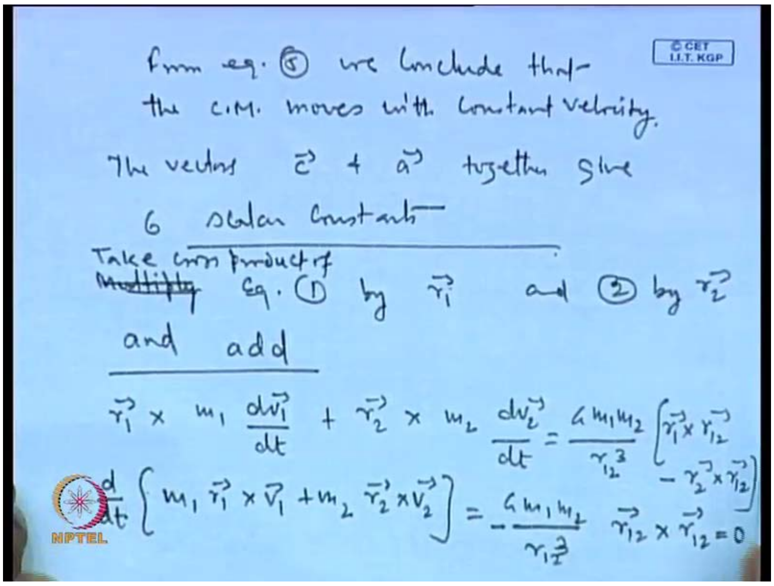

So, from equation five, from equation five we conclude that the centre of mass moves with constant velocity. So, in this equation we have been able to identify a is a constant vector here. So, we have been able to identify two constant vectors c and a. c and a, this two constant vectors together, the vectors c and a together give 6 scalar constants. So, for this motion of the two particle in the inertial reference frame we have been able to identify right till now 6 scalar constants. So, let us see if we can identify any further more scalar constant.

Now, the from the equation one and two if we multiply them by so by r 1 and r 2 vector respectively and add so multiply multiply equation. This is equation one and two, 1 by r 1 tip cross product, take cross product of equation 1 by r 1 and 2 by r 2 and add. So, if we do this what we get r 1 cross d v 1 by d t plus r 2 cross m 2 times d v 2 by d t. This will be equal to G m 1 m 2 by r 1 2 whole cube times r 1 cross r 1 2 minus r 2 cross r 1 2. So, this is what we get. So, the left hand side can be written as d by d t m 1 times r 1 v 1 plus m 2 times r 2 cross v 2, while on the right hand can see that this gets reduced to 0 because r 1 minus r 2 is nothing but minus r 2 minus r r 1 2. So, the right hand side is G m 1 m 2 by r 1 2 whole cube with minus sign here. r 1 2 cross r 1 2. So, this becomes equal to 0.

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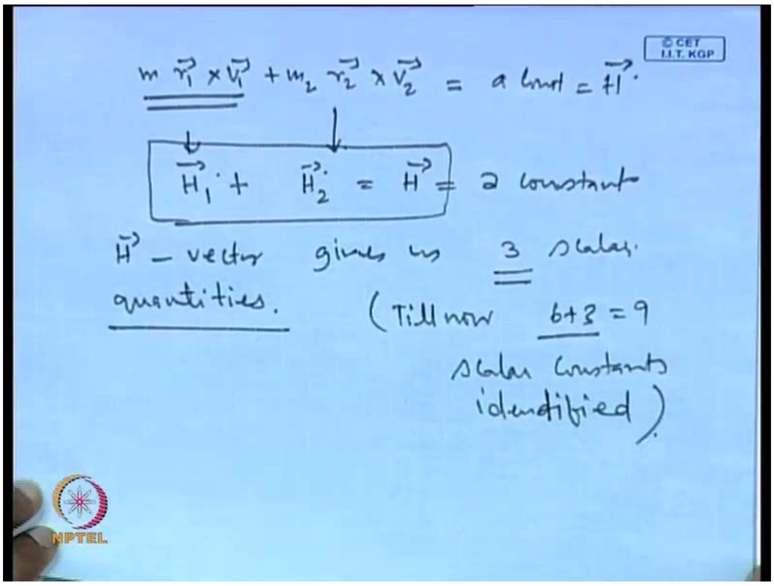

So, from this equation we can conclude that m r 1 cross v 1 plus m 2 r 2 cross v 2. This is nothing but a constant and we write this as is equal to H. So, if we look on the left hand side this quantity is nothing but H 1 the angular momentum of vector of the particle one and this is nothing but the angular momentum of the particle two. So, together this is a constant. So, H is a constant, H is a vector, H vector gives us three scalar quantities. So, what we have been able to do 6 scalars we identified with for the center of related to the motion of center of mass and here for the angular momentum of the two particles that the total, the angular momenta is remaining constant.

So, here we are getting a total of 3 scalar constants. So, total of 9 scalar constants. So, till now, till now 6 plus 3 is equal to 9 scalar constants identified. Further, we will be able to identify only 1 scalar constant which is the total energy of the system which we are going to compute now. So, what we do?

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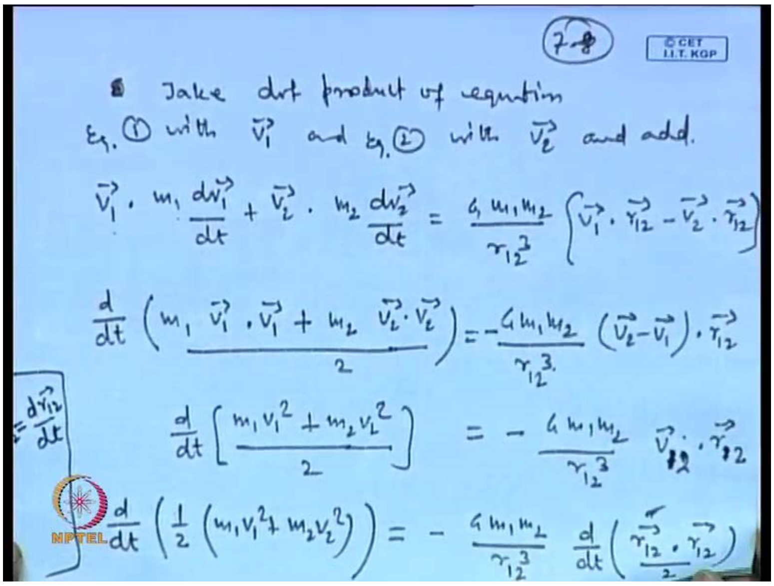

We take the dot product of equation one and two, take dot product of equation one with v 1 and two with, equation two, two with v 2 and add. So, doing this, this gives us v 1 dot m 1 times d v 1 by d t plus v 2 dot m 2 times d v 2 by d t. This is equal to G m 1 m 2 by r 1 2 whole cube v 1 dot r 1 1 minus v 2 dot r 1 2. Now, this quantity we can write as d by d t m 1 v 1 dot v 1 plus m 2 v 2 dot v 2 divided by 2 while the right hand side we can write as r 1 2 whole cube. Now, here r 1 2 we can take it outside and minus sign we can put here and v 2 minus v 1 we can write as v 2 minus v 1.

So, this is v 2 minus v 1 dot r 1 2. So, this quantity we can further write as 1 2 whole cube v, v 2 minus 1 1, let us say that we write it as v 2 1 dot v 2 1 2 times r 1 2. So, the left hand side this is the dot product here. So, left hand side will get reduced to this is m 1 v 1 square and this gets reduced to m 2 v 2 square divided by 2. This is very easy to see it if you differentiate this, this is going to yield this respective quantities here. So, you can do it yourself.

So, we have the left hand side. This is the, if we look into this quantity. This is nothing but the total kinetic energy of the two particles in the inertial reference frame. Right hand side we need to compute further. So, d by d t 1 by 2 m 1 v 1 square plus m 2 v 2 square this is equal to minus G m 1 m 2 r 1 2 whole cube and you can check again that this quantity is nothing but r 1 2 is r 1 2 dot r 1 2 divided by 2. Here we are using the fact that v 1 is nothing but d r 1 2 by d t. So, the right hand side can also be further reduced.

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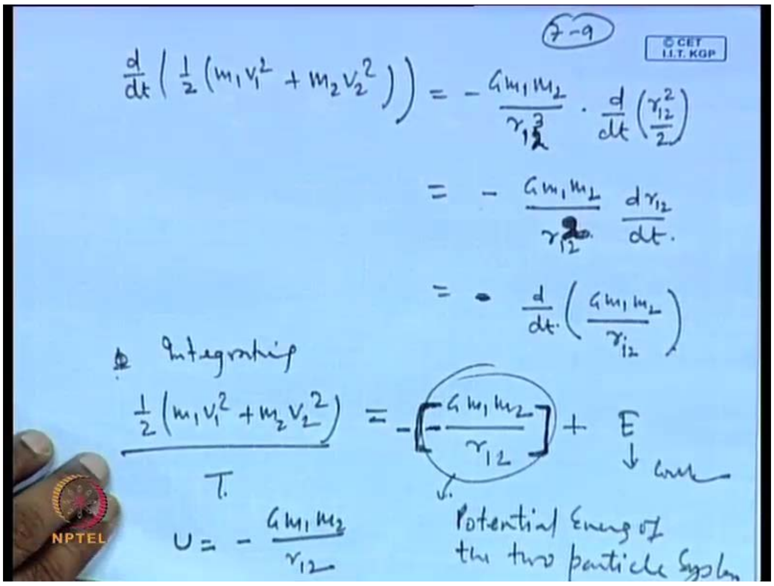

We write it as 1 by 2 m 1 v 1 square plus v 2 square is equal to minus G m 1 m 2 r 1 2 whole cube times d by d t and r 1 2 whole square, m 1 m 2. This will give divided by 2. So, 2 2 will cancel out and this will us r 1 2 whole cube with a minus sign, this d r 1 2 by d t. This is what we get. So, now if you look into this quantity you can write this quantity as d by d t G m 1 m 2, this is r square here, this becomes r square, r 1 2. If you differentiate this, so r 1, r 1 2 will get differentiated and this r square will r 1 2 will appear in the denominator with a minus sign and d r 1 by d t.

Now finally, integrating it, so what this we get integrating, integrating we get 1 by 2 m 1 v 1 square plus m 2 v 2 square, this is equal to G m 1 m 2 divided by r 1 2 plus say constant E. This is a constant. Now, we can look into this quantity. This is nothing but the potential energy. This is nothing but potential energy, potential energy of the two particle system and this is nothing but the kinetic energy. So, we can write it. So, this is the potential energy with what we write with a potential energy, negative of the potential energy we can put here minus sign inside and minus sign like this.

So, with minus sign this quantity appears as a potential energy. So, with minus sign, so we have here rather I will put a bracket like this to make it more convenient. So, minus sign is taken outside. So, we have, we can write U is equal to minus G m 1 m 2 by r 1 2.

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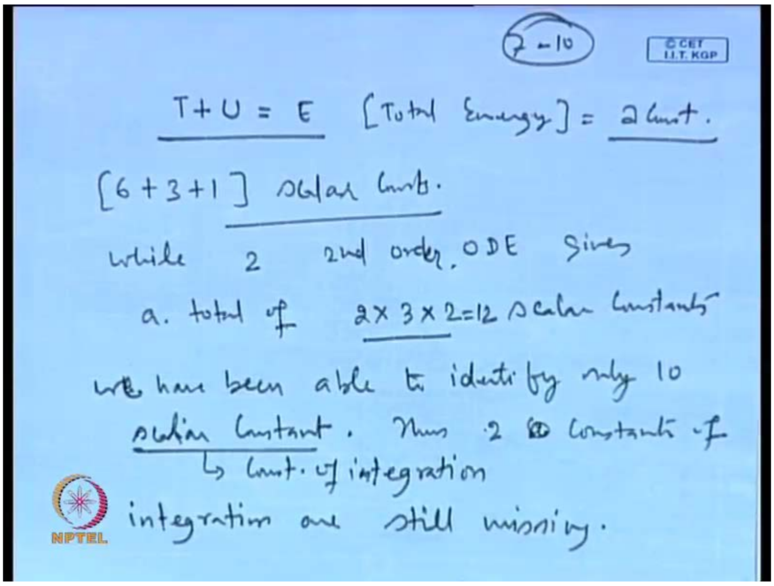

Therefore, this equation gets reduced to this T is, T is the kinetic energy. So, we can write T plus U is equal to E and which is the total energy is equal to a constant. Thus, we see that the total energy of the two particle system remains constant. So, altogether we have been able to identify 6 plus 3 plus 1 scalar constants while the 2 second order ordinary differential equation gives a total of 2 into 3 into 2 scalar constants because each equation is a vector equation. So, it has 3 scalar parts. So, 3 second order ordinary differential equation and 3 second order ordinary differential equation will have 6 number of constants of integration.

So, 2 second order ordinary differential vector ordinary differential equation it will have total 12 scalar constants. So, total 12 scalar constants. So, we have been able to identify only 10 scalar constants thus 2, 2 scalar constants which are the constants of integration, constant of integration. So, thus 2 constants of integration are still missing and it is not possible in fact to identify any any other more than 10 scalar constants the, which are the constant of integration.

Therefore, this two particle system cannot be solved completely if we take the absolute motion of the particles. We are, because we are not able to identify all the constants of integration while on the other hand it is possible to solve the relative motion of the two particle system and it can be completely, all the parameters related to that can be worked out.

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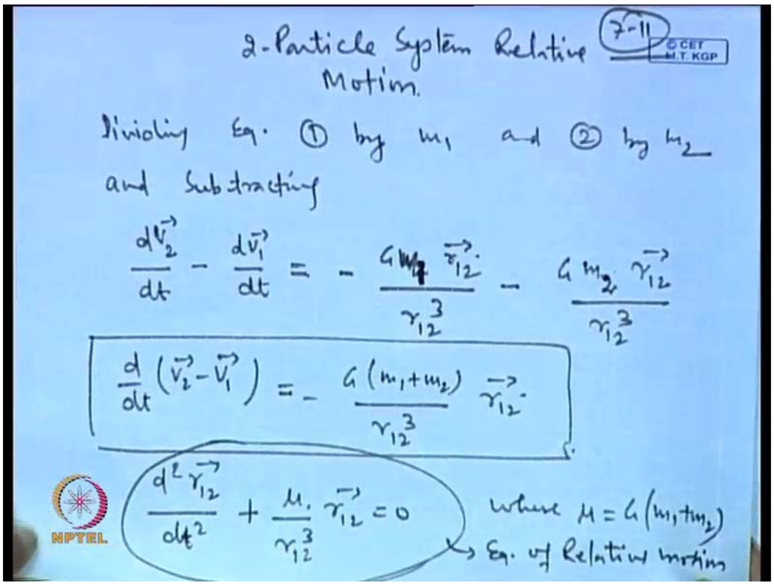

So, we next we go for the 2 particle system relative motion. 2 particle system relative motion. So, again we can utilize equation number one and two, so dividing equation one by m 1 and two by m 2 and subtracting this will give us d v 2 by d t minus d v 1 by d t, this is equal to minus G m 2 r 1 2 this is m 2, r 1 2 by r 1 2 whole cube minus G. We think to we have the negative sign and for this we get G, the first one we take, this here we get 1 if we divide by m 2 and here we get m 2 if we divide by m 1. So, this is G m 2 r 1 2 by r 1 2 whole cube.

So, the left hand side it is nothing but v 2 minus v 1, while on the right hand side we can write G m 1 plus m 2 minus with minus sign here r 1 2 whole cube times r 1 2. So, this is nothing but d square r 1 2 by d t square and if we put G times m 1 plus m 2 is equal to mu. So, this will look like in this format into 0 where mu is equal to G times m 1 plus m 2. Now, this equation is the equation of relative motion, equation of relative motion and it is a possible to solve this, solve this equation.

And so this is a second order differential equation ordinary differential equation, but it is a vector. So, again this will have a total of 6 constants of integration and it is possible to identify those 6 constants of integration. But for the 2 particle system for their absolute motion or the, their motion in the inertial reference frame not the relative motion. So, their absolute motion cannot be worked out totally while the relative motion can be worked out and the, we will see that later on, we can develop this into a, in the equation of an ellipse for the s as the Earth satellite or the motion of the Earth around the Sun.

And its orientation, its various parameters. So, total of 6 parameters can be identified in that context. So, for now solving this equation. So, let us write this equation as equation number A. So, for solving this equation A we try the earlier tricks applied.

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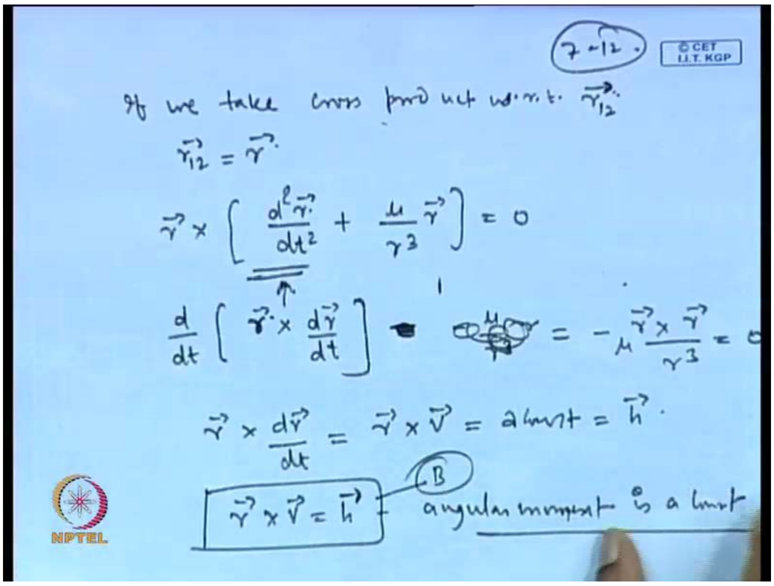

So, if we take, if we take cross product with respect to r 1 2. So, what we are going to get from this place, this will give us the, as earlier we have done this will give us the angular momentum of the, for this relative motion. So, for if we take the cross product with respect to r 1 2. So, r 1 2 here we write as first r 1 2 is equal to r. So, the notation gets simplified. So, we write r cross d square r by d t square plus mu by r cube r equal to 0. Now, as earlier we have done the first term this can be written as d.

Let us write in this way, r cross d r by d t and this will be equal to minus mu r cube, here also we have to take the cross product. So, we take it on the right hand side and simply take the cross product r cross r by r cube times mu and this is nothing but 0. So, if you differentiate this you will get this quantity here. So, this will give you this quantity. Now, integrating so this gives us d r by d t. This is nothing but r cross v is equal to a constant. We shall, we can write as h. So, what we have found that r cross v this is equal to h, this is the, this gives the angular momentum is a constant. So, for this relative motion angular momentum is a constant. So, this we write as equation number B. Now, if we take the dot product of equation number A with respect to v or r dot.

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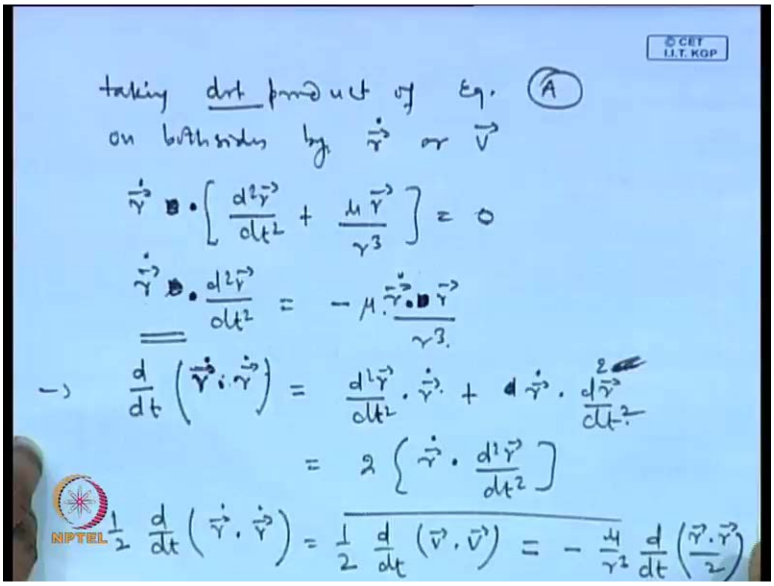

So, taking taking dot product equation A on both sides by r dot or v, this results in r dot cross d square r by d t square plus mu r by r cube is equal to 0 and the quantity which is present here we can break it, take it outside d t square. This we can write as minus r dot cross r mu times r cube. So, let us look into this quantity. This quantity we can surely we can write it as d by d t. This quantity is nothing but r dot is v and this is d b by d t. So, we can write as d b by d t r dot. Here we are taking the dot product, not the cross product. This is the dot product, dot product here.

So, r dot times r dot, let us check this. d d t, d by d t r dot times r dot how much this quantity is? This becomes d square r by d t square dot r dot plus r dot times dot d r dot by d t. This this is we can write here by moving this dot, d square r by d t square. So, this is nothing but 2 times r dot d square r by d t square. So, here this quantity we can write as 1 by 2 d by d t times r dot times r dot is equal to this also we can write as d by d t v dot v dot. This becomes on the right hand side we have minus mu by r cube and similarly, this can be written as following the same way r dot r dot divided by 2. See this is the earlier technique we have used, the same technique it is exactly, only thing the, if you now we are dealing with the equation of relative motion.

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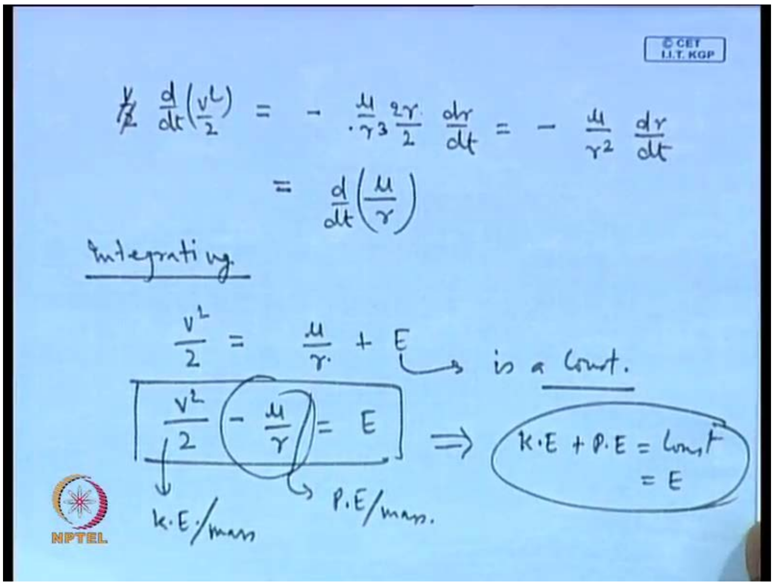

So, we get 1 by 2 d by d t v square or we take the 2 inside. This is equal to minus mu by r cube and this becomes, this quantity is nothing but r square. So, if we differentiate this we get r here and so, 2 r divided by 2 times d r by d t. So, this is minus mu by r square d r by d t and the right hand side then this is nothing but equal to mu by r d by d t. This can checked if we differentiate this, this becomes minus mu by r square d r by d t. So, integrating we get v square by 2, this is equal to mu by r plus E where E is a constant.

Thus, v square by 2 minus mu by r this is equal to E. Now, we can identify this is kinetic energy per unit mass and this is, this whole thing is potential energy per unit mass. So, we can write this equation that kinetic energy plus potential energy. This is total constant into a, total energy of the system remains conserved. So, the total angular momentum remains constant means it is conserved total energy of the system remains conserved. So, now what do we conclude from all these results that for 2 particle system if we are looking for the absolute motion of the particle, then it is not possible to completely solve it.

But if we are looking for the 2 particle system or the 2 body system wherein we, wherein we want to describe the motion of one body with respect to the other body. Then it is possible to describe the motion and in this case what are the constants we have identified? We have identified that the total energy of the particle it remains constant and total angular momentum of the particle is say, angular momentum of the particle for the relative motion also that is constant. So, now for 2 particle system we want to solve the equation of motion.

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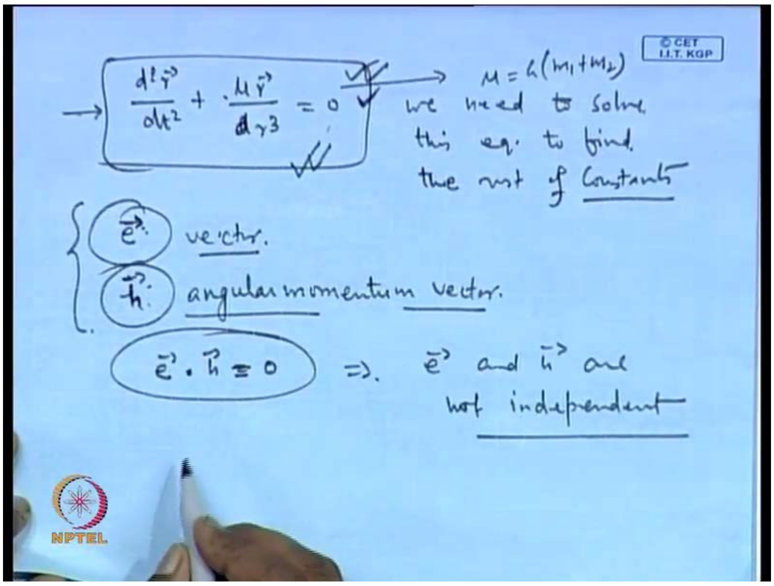

Then we have this d square r by d t square plus mu r by r cube is equal is to 0. We need to solve this equation to find the rest of constants. So, in the next lecture we will see that we will be able to identify the eccentricity vector e. So, eccentricity this is a vector and similarly, we will be able to identify, we have already identified, we have already identified the h which is the angular momentum vector. So, we using these two it will be possible to find out the constants of integration of the motion for this particular equation. So, if we integrate it we get 6 constants of integration, but these two are not independent. Later on we see that in the next lecture that e dot h will be equal to 0.

Therefore, this implies that the e vector and h are not independent. So, we need to apply or we need to work out further to identify all the 6 constants involved in this equation, involved in this equation of motion for the relative motion of the two particles. Equation which is written here if we look into this equation, this equation is quite similar to the central force equation. It is exactly the same appearing in the same way as the central force motion except the difference here arises for mu. So, in this case, in this case the mu is nothing but G times m 1 plus m 2 while for the central force motion only the quantity G times m 1 appeared, m 2 appeared, G times m 2 appeared because there the, if say let me repeat it.

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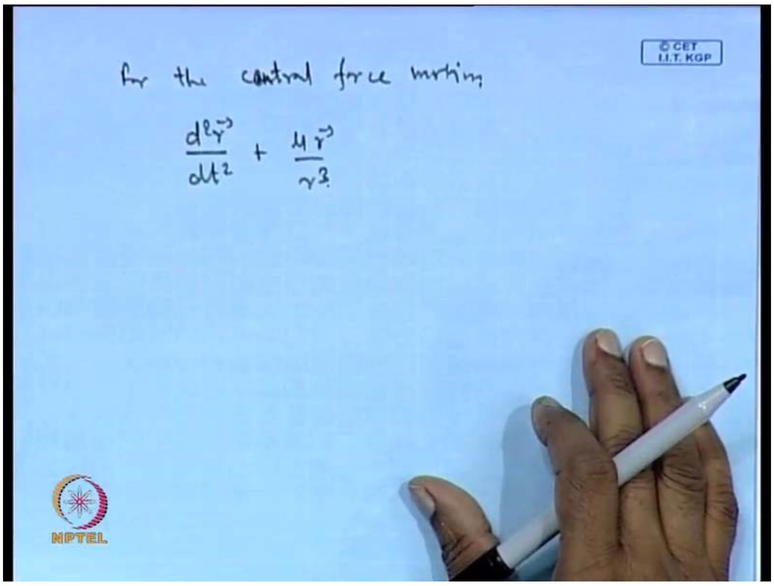

So, for the central force motion, for the central force motion we got exactly the same kind of equation mu r by r cube. So, here if we assume that the mass m 1 m 2 here in this case we have taken the motion of the mass, let me check it. So, m plus m 2 while writing the relative equation of motion. So, this we have written as here d v 2 minus d v 1. So, what we are doing?

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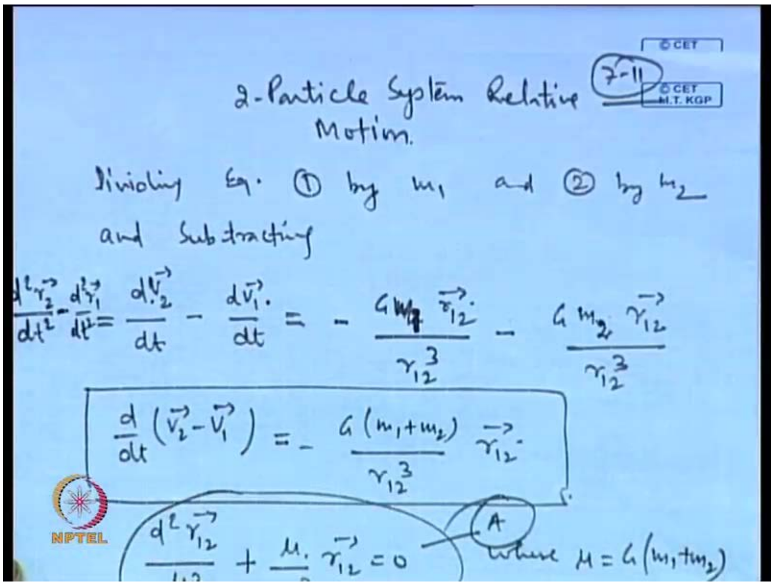

We are writing the equation of motion of the particle 2 with respect to particle 1. That is this quantity is here d square r 2 by d t square minus d square r 1 by d t square is equal to d v 2 by d t and d minus d v 1 by d t. So, this is the motion being described with respect to the particle number 1. The motion of the particle 2 with respect to particle number 1.

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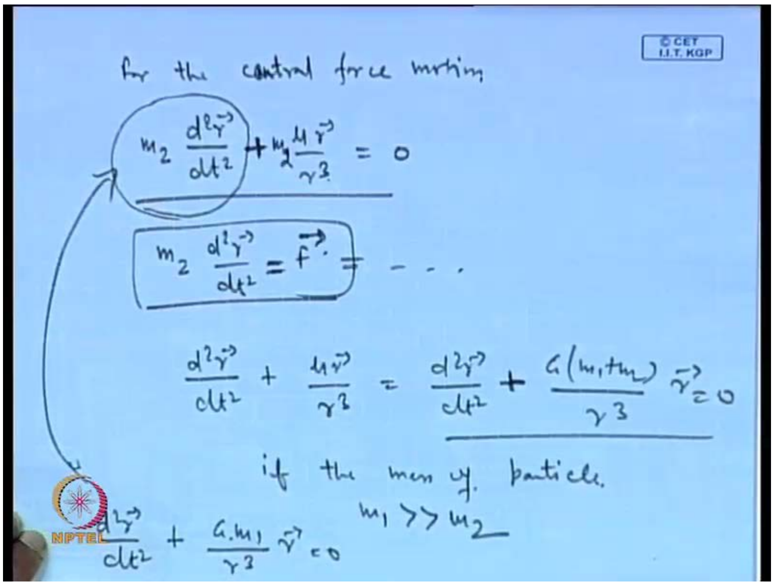

So, in that situation if we multiply both side by, both side of this by m 2. So, we get the motion of the particle 2 with respect to the motion of the particle 1. If this let us multiply like this. So, this is m 2 multiplied by m 2. Now, the left hand side this appears as the according to Newton’s law if the force acting is f, so m 2 times d square r by d t square, this can be written as f. And whatever the magnitude of the, this f is the force vector is we can write it on the right hand side.

So, the difference that I want to make it here that the quantity d square r by d t square plus mu r by r cube wherein this is nothing but d square r by d t square plus G times m 1 plus m 2 by r cube r is equal to 0. Look into this equation. So, if the mass of particle m 1 is much larger than m 2 then the, this equation can be written as d square r by d t square plus G m 1 by r cube r is equal to 0. Now, compare this. If we multiply both side of this by m 2. So, what this will look like?

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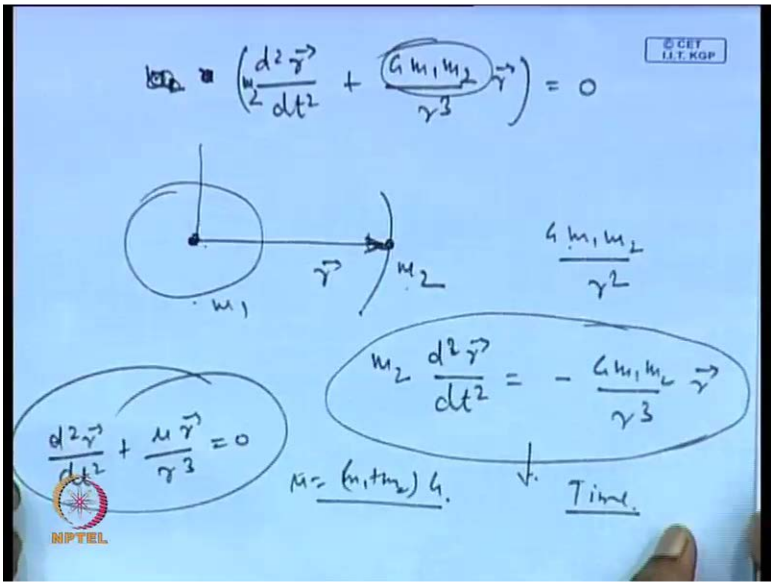

This will appear as if, so if we multiply this equation by m 2. So, multiplying it by m 2 d square r by d t square plus G m 1 m 2 r cube r is equal to here m 2 we will take it inside. So, the quantity which appears here it tells that there is a mass m 1 fixed in this space and about which a mass m 2 is moving. Therefore, the force of attraction between this mass, masses are G times m 1 m 2 by r square and therefore, the equation of the motion can be written for the mass m 2 is equal to m 2 times d square r, this is the radius vector r here, d t square is equal to minus G times m 1 m 2 by r cube r.

So, what it tells that? That if the mass of one of the particle is very large, if the mass of one of the particle which is the primary particle m 1 is very large as compared to the m 2. So, it is as good as the mass 1 will appear to be fixed in a particular position and with respect to that then the motion of the m 2 can be described. So, this is exactly the inverse inverse square central force period motion and we have already solved this motion. So, we have already determined for this kind of motion what will be the constants of integration, some of the constants of integration.

So, one thing is very easy to find out from this place. The time period can be easily estimated from this equation because this equation we have already worked out. So, what we do? We write this equation, but we go back to the equation again d t square mu times r by r cube is equal to 0 wherein mu is equal to m 1 plus m 2 times G. And find out the time period for this particular motion.

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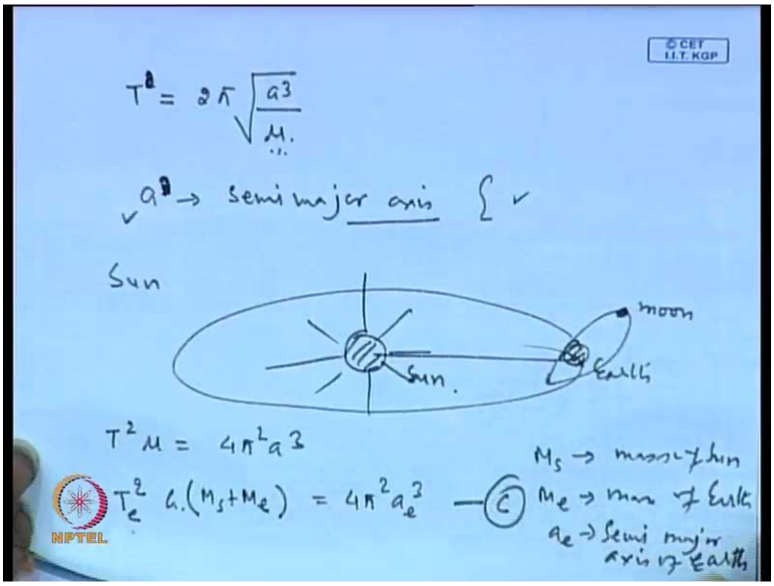

So, time period will be given by T square is equal to or T is equal to 2 pi a cube by mu under root where a cube is the, a is the semi major axis. And we will be doing all this things in next lecture. So, how to determine this a we will do it in the next lecture. Now, suppose a is known, mu is here modified mu because mu in the mu term, m 1 plus m 2 both are appearing. So, this is not exactly the m 1 times G as this, this is not exactly this quantity.

Now, under this modified quantity we can have certain insight into the planetary motion. Suppose, Sun is there, so we have the Sun here and somewhere the Earth is in its orbit. So, it is moving around the Sun and around the Earth, there is the planet Moon. So, this is Moon, this is Earth and this is Sun. So, from here we can write few things. So, let us expand it and write it T square times mu, this we can write as 4 pi square a cube. Let us say the period of the Earth around the Sun is written as T e and then mu will be G times M Sun plus M Earth into 4 pi square a Earth cube. So, M s is mass of Sun and M e is mass of Earth and a e is semi major axis of Earth. So, this is our, let us say this is equation number C.

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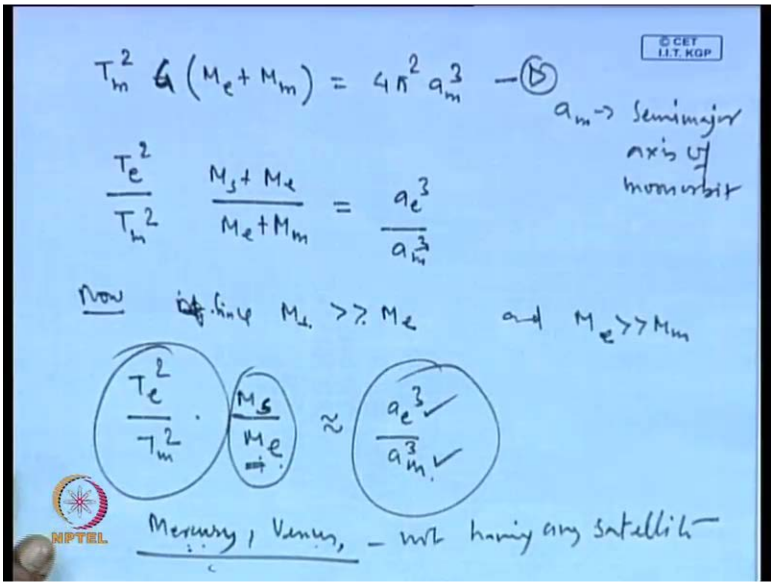

Similarly, for the motion of the Moon around Earth we can write as T square m times G times M Earth plus M Moon this will be equal to 4 pi square times a moon whole cube where a m is the semi major axis of moon orbit. Here the axis of earth orbit. So, we can see that from equation C and D, if we divide equation C by D. So, we get T e square divided by T m square G G cancels out and we have M s by M e plus M e plus M m. On the right hand side we will have a e cube by a m cube.

Now, if M s is much greater than M e and now since M s is much greater than M e and M e is much greater than M m we can write T Earth square divided by T m square is equal to M s by M e. This is a cube by a m. So, if we know the a m whole cube, if we know the semi major axis of the Earth, if we know the semi major of the Moons orbit and we know the period then it is possible to get the ratio of the, ratio of the mass of Sun and the Earth.

So, from here it is very clear that if we use this equation, it is possible to find the ratio of the mass of Sun to the mass of the Earth. Similarly, suppose if any planet here in this case Earth was having of satellite Moon. If any planet is not having any satellite then it is not possible to find this ratio. So, in few cases we do not have this satellite and therefore, in that those cases we are not able to find the ratio of the masses Sun to the mass of that planet for example, Mercury, Mercury, Venus these two are not having not having any satellite.

So, it would not be possible to find the mass of the Sun to the mass of the corresponding planet. So, for this type of problem then we need to apply some different method. So, this is the limitation of this method here, but obviously this result have been very useful that we can determine the mass of the Sun to the mass of that planet provided the time period is known and the semi major axis, these are known. Thank you very much.