Space Flight Mechanics

Prof. M. Sinha

Department of Aerospace Engineering

Indian Institute of Technology, IIT Kharagur

Lecture No. # 06

Two Body Problems (Contd.)

So far in the last lecture, we discussed about the central force motion. So, today we will continue with that and some for remaining formulation, we will complete and thereafter we will look into the problem of a in a motion of 2 particles or 2 bodies under the mutual gravitational attraction.

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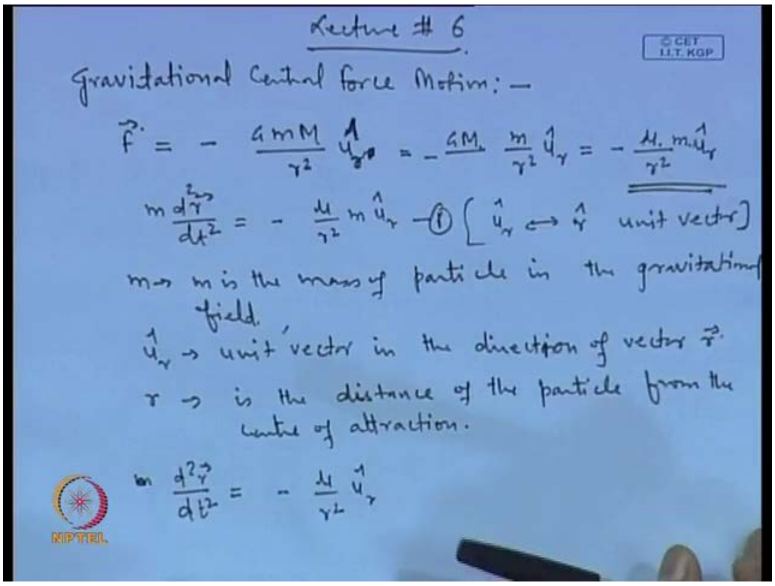

So far, now for the gravitational central force motion in whatever the last time we have discussed. So, last time we did not assume anything about the gravitational force. How its varying, but today we will assume that the gravitational force it varies as the inverse s square of the distance between the two particles or in this case, because the two only two particle is present and there is a center of attraction.

So, we assume that this gravitational speed or the gravitational force due to the centre of attraction. It falls of as a 1 by r s square or it varies as 1 by r s square. So, we have the equation for the motion can be written as f is equal to, where f is indicating the force acting on the particle equal to minus G m M by r s square u r, in this case we have this is that this part it indicates the Newtons gravitational law, where m is the mass of 1 particle and capital M is a mass of another particle and the distance between them is r s square, then it is a this is the force acting and here u under s square r u cap r, this is the unit vector in the direction of r.

So, because we are discussing with the central force motion, so here we assume that the only the center of the force is available to us. So, it may be given in terms of a just like a writing g times m as mu and m by r s square u r. So, this can be represented as mu by r s square m times u r u r cap. So, where mu is a constant of propos tonality, because here we are just assuming let us say in central force motion. So, we may not be aware of actually what is the mass of the central force of attraction, but of a here given in this format where this is the propos tonality constant. So, we can infer from this that there is some mass involved in that, let us this is the best way of this equation is the best way of writing the force of motion in this case.

So, therefore, for left hand side we can expand it, and we can write it as d s square write for dt s square this where 2 minus mu by r s square m times u r cap, where u r cap or if sometimes it say u r cap is a also written as just r cap to indicate. This is the unit vector. So, here m is the mass of particle in the gravitational field u r cap unit vector in the direction of vector r r is the distance of the particle from the centre of attraction. So, therefore, equation one; therefore, equation 1 can be written as d s square r dt s square this is equal to minus mu by r s square u r cap.

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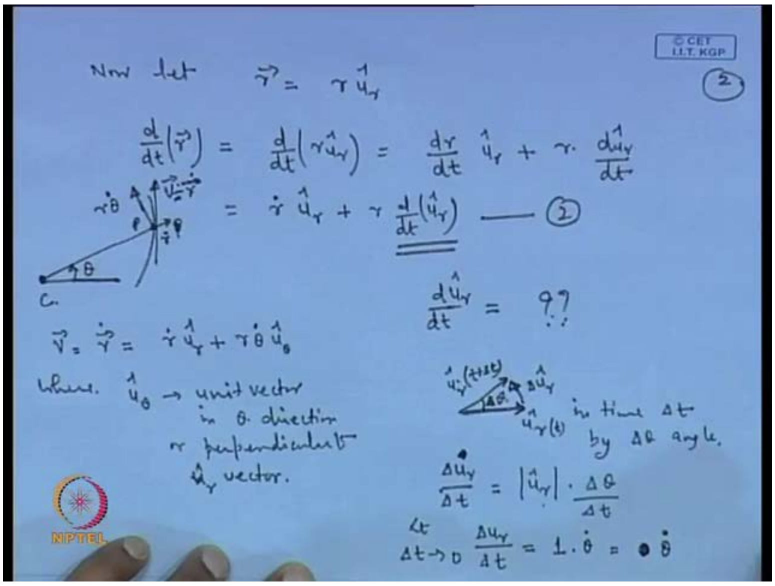

Now, let r is equal to r times u r, so basically r is here the magnitude of the vector r and they u r is the unit vector in the direction of r. Therefore, d y dt r this becomes. So, we can differentiate and write it in this way now here dr y dt this is nothing but r dot which is the component of the velocity vector in the direction of r. So, we can show it here, this is the center of attraction the particle p is moving here. So, this is v v is equal to nothing but r dot, and this component then becomes r dot and this is this is the vector. So, this is the velocity vector, this is the r dot and perpendicular to this we can break it.

So, this is the radial component and similarly the tangential component can be written here it will be r time’s theta dot where theta is that it is suppose we are measuring angle from some reference. So that this is the angle indicated here theta. So, the tangential component can be shown as r times theta dot, now basically here we are indicating v is equal to r dot is equal to r dot times u r cap plus r turns theta dot times u theta cap.

Now, here u theta cap unit vector in theta direction or perpendicular to u r vector therefore, dr by dt this becomes r dot times u r cap plus r times d by dt u r cap, now if i this is our equation number two issue is to determine this particular value this particular term. So, this quantity is what we will the how to indicate this quantity how to remove the differential of the u r cap and indicate in terms of the other known parameters.

So, here we have basically u r we can indicate as suppose this is a u r vector. So, because this u r vector is indicated in the direction of r, so after some time delta t this will rotate by delta theta angle. So, it rotates in time delta t by delta theta angle. So, the new u r then is indicated by. So, this u r is a time t. So, another u r at time t plus delta t can be indicated by this vector. So, this is the net change in u r. So, this we can write as delta u r cap.

So, therefore delta u r cap by delta t or let us take for the magnitude of delta u r, so without taking the direction of this. So, this can be approximately written as u r cap magnitude times delta theta into divided by delta t and in the limit delta t tends to 0 delta u r by delta t, then it becomes this quantity is equal to 1, then there because this is unit vector and this becomes theta dot. So, this quantity basically equals to theta dot.

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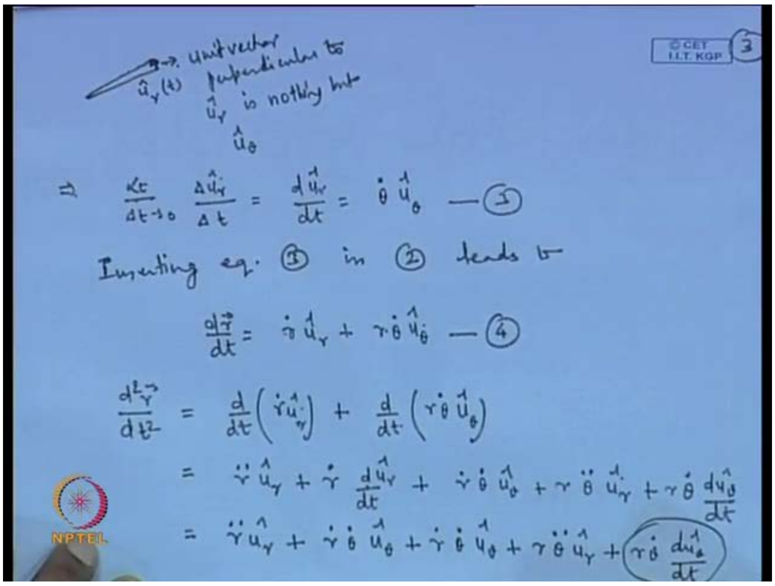

Now, this change as we are talking this u r here u r at t. So, as t becomes delta t becomes a smaller and a smaller. So, the change can be indicated by an arrow which is perpendicular to u r vector. So, this is perpendicular to u r vector. So, this vector is nothing but the unit vector in this direction can be indicated by unit vector perpendicular to u r is nothing but u cap theta this implies that limit delta t tends to 0 delta u r by delta t cap this can be indicated by dr by dt equal to nothing but theta dot times a theta cap therefore, in equation 2.

So, inserting or inserting equation 3 in 2 leads to dr by dt is equal to r dot times u r cap plus r times theta dot turns u theta then next step is to point the second differential of this. So, now we can determine d s square r and dt s square. So, having differentiating in this equation number four now this can be expanded. So, as usual d u r by dt here it can replaced by r times theta dot times u theta cap, that is another theta dot turn u theta cap then r times theta double dot time u r cap and r time theta dot times d u theta cap by dt.

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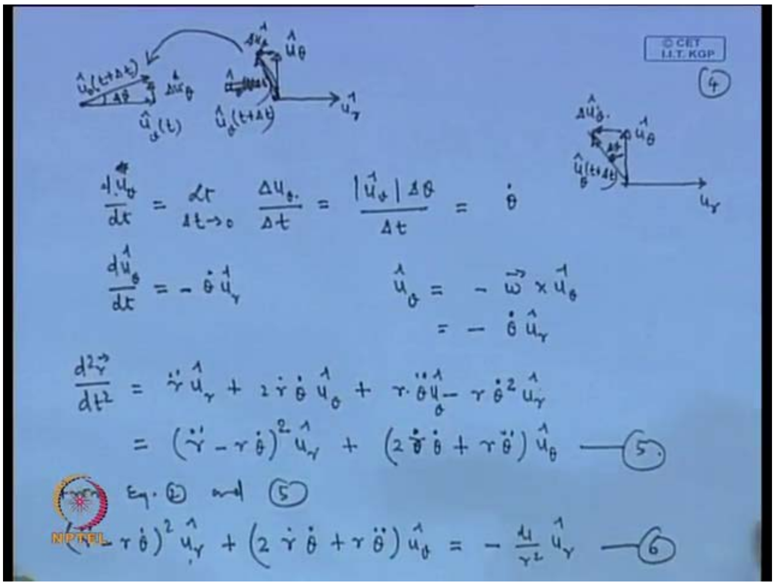

Now, this is the term that we have to evaluate as usual. We have done now, we can indicate vector unit vector of y u theta. So, we find delta t this u vector will undergo a change of u theta t plus delta t the next changes here delta u cap theta and this vector rotates by an amount delta theta see the vectors are like this is the u r vector, and perpendicular to this is the u theta vector. So, both of them there rotate at the same rate.

So, here from this 1 we can write again that d u theta cap by dt or we can remove the cap find here and just we can have a look of this that limit delta t turns to 0 or this is delta u theta divided by delta t and delta u theta almost this is going to be. So, delta u theta we can write from this place u theta cap magnitude times delta theta divided by delta t. So, this becomes theta dot and a, we can see from this place that if we have any change in this vector u theta. So, your delta u theta will be indicated delta u theta. So, this is your vector theta cap t plus delta t this is your vector u cap theta t plus delta t.

So, this is the change which has been indicted here in this figure. So, as the time delta t becomes very small. So, the change delta u theta cap this will be just opposite to the vector u r cap. So, we can write here d u theta cap by dt, this is equal to theta dot times u r cap within negative sign, because this is indicated in this negative direction of u r just opposite direction of u r. So, I will make the figure over here this is u r this is u theta u theta cap t plus delta t and this is the change here delta u theta.

So, this delta u theta as delta theta becomes various small or delta t is become various small this is delta theta. So, this vector just becomes opposite to the u r vector. So, we have written with minus sine here in this place.

So, in general we can write u theta is equal to minus omega cross u theta cap. So, omega is nothing but here theta dot and a corresponding vector which will be perpendicular to both theta and u r. So, this will give us theta dot times u r cap this can be easily using from here. So, now completing we can write d s square r by dt s square is equal to r double dot u r cap plus 2 r dot theta dot u theta cap plus r times theta double dot minus r times theta dot s square times u r cap.

Now, collecting similar terms to whether this gives us u r cap and here we have missed out u theta. So, we can insert u cap theta plus 2 r dot time theta dot plus r time theta double dot u theta cap this is our equation number five. So, from equation number 2 from equation number 2 and 5 we can write r double dot minus r theta dot s square u r cap plus 2 r dot theta dot plus r theta double dot u theta cap this is equal to minus mu by r s square u r cap.

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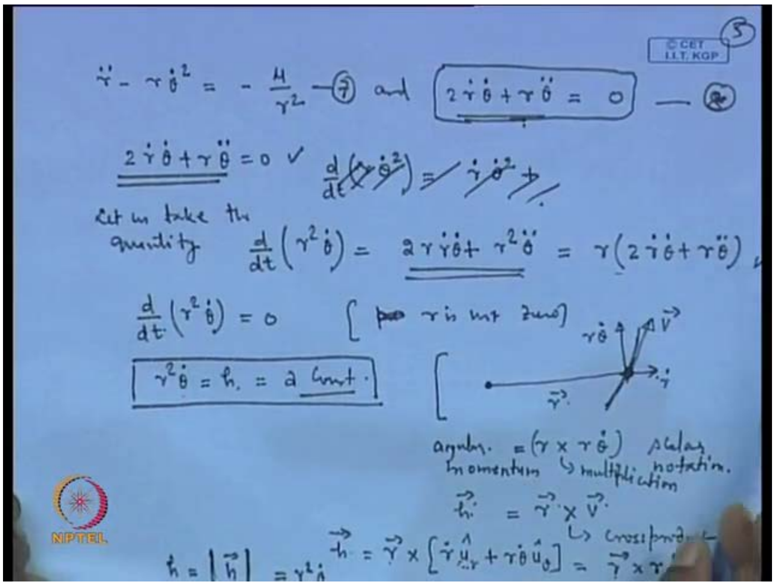

Now, comparing the coefficient of the unit vector u r, and theta on both sides in this equation number 6, comparing the coefficients of u r and u theta in both sides, we have r double dot minus r theta dot one quotation is there r theta, where this square is not outside this is inside. So, there is square is inside, so comparing this coefficient. So, this gives us r r double dot minus r time theta dot s square. This is equal to minus mu by r s square and 2 r dot theta dot plus r theta double dot this is equal to 0, because there is no term of u cap theta on that I consider in equation number 6. So, this is equation number 7, and this equation number eight.

So, now after this simplification we can further explore this equation. So, let us take this equation 2 r dot theta dot plus r times theta double dot equal to 0 if we can represent in a with a better way. So, we can get more information or we can understand the system in much better way. So, if we write r time's theta dot s square we differentiate this quantity suppose. So, what we get here r dot r dot times theta dot s square plus mu, let us take the quantity d y dt r s square theta dot not r times theta dot s square. So, if we differentiate this.

So, this quantity then it will become two times-two times R and R dot plus r s square times theta dot theta double dot. So, now we can see that this quantity and, but even we have the quantity theta dot is missing where we could hear theta dot also. So, this quantity and this quantity they differ by 1 term which is r. So, we can take r outside. So, this becomes two times r dot theta dot plus r times theta double dot. So, because r is not 0, so we can easily write d by dt r s square theta dot, you this quantity is equal to this quantity equal to 0.

Here r is not 0. So, if we integrated. So, r dot r s square times theta dot this becomes equal to h equal to a constant. So, what this quantity is this quantity is nothing but the angular momentum vector per unit mass, we can shift from here this is the r vectors this is the center of attraction particle is moving along this path. So, this is the velocity vector this is the r r dot component and this component perpendicular to this is r times theta dot.

So, the angular movement term, this is the given by r times angular momentum r times r times theta dot this. We are writing in as scalar notation scalar notation, if we want to write the same thing in vector notation then we have write here r cross v. So, this is the angular momentum vector h here, and if we expanded we get the same quantity. So, here this is simple multiplication while here this is the cross product, and we can check from this place h is equal to r cross.

Now, v becomes r dot times u r cap plus r times theta dot time u theta cap. So, unit vector in the direction of r is nothing but u r. So, this cross product will vanish leaving us with r cross r times theta dot u theta cap. So, the angular momentum vector basically perpendicular to both the r vector, and the u theta vector and the magnitude of this angular momentum vector which can be written as r s square times theta dot, because the magnitude of r vector will be r. So, this term make it r s square and times theta dot and the unit cross product of unit vector will give a unit vector, and easily write here h is equal to s magnitude is equal to r s square theta dot.

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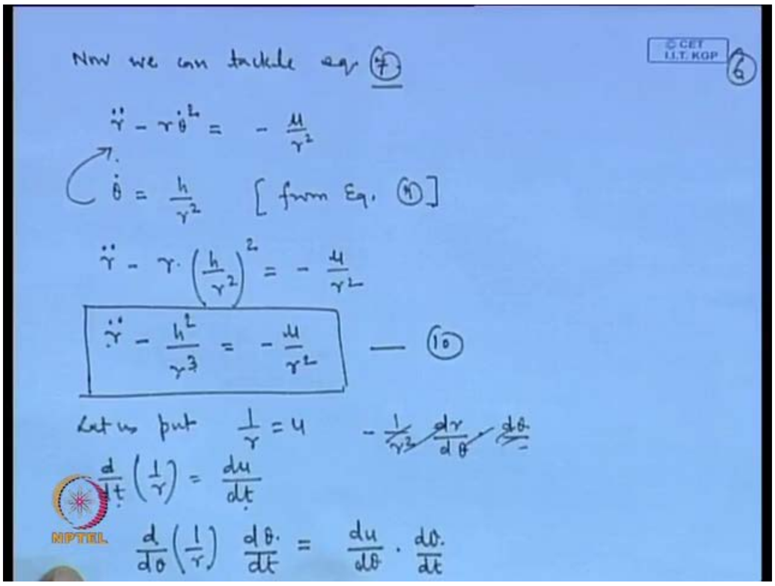

So, this is the information we get from the quantity, which is indicated here now after getting this information we can take the equation number 7. Now, we can taken equation 7. So, we have from equation 7 r double dot minus r times theta dot s square this is equal to minus mu by r s square, and here we can name this as equation number nine. So, theta dot will be equal to h by r s square, and this we get from equation number nine.

So, insert this into this equation. So, this gives you r double dot times r times theta dot. So, insert it now this will become r s square, and s by r s square whole s square is equal to mu by r s square r cube, now this equation is you can see it is in this usual way we cannot tackle it. We cannot solve it, but if we do some manipulation it will become easier to solve it. So, this we write as a equation number ten, now let us put 1 by r is equal to u.

Therefore 1 by r s square minus sine dr by d theta or we will do like this, what we want we look for this quantity r double dot. So, if we differentiate with the first with respect to t 1 by r. So, this will be du by dt. What we intend to do here we want to replace the variable independent variable t by theta. So, we will be using independent variable theta instead of t and then this equation, equation will get simplified, and then we will be able to solve it. So far doing this 1 second we can write it in this way.

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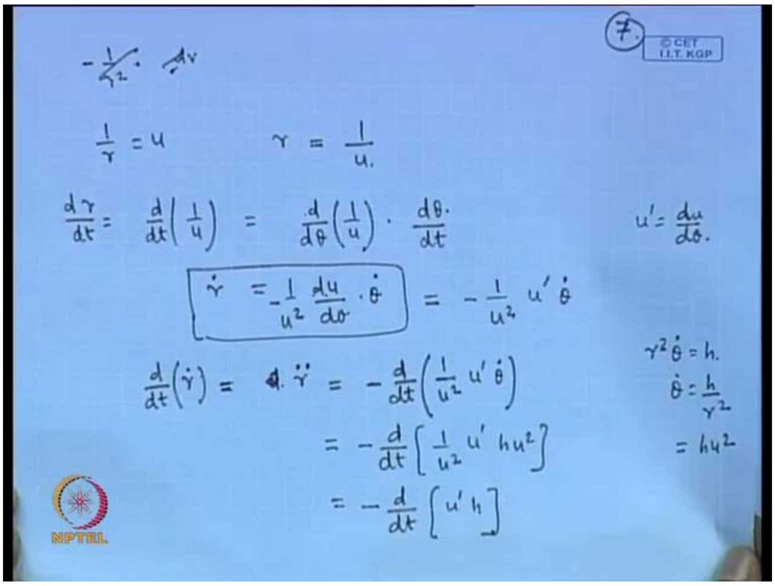

So, the quantity now this will become t square what we need to do here dr is equal to dr by dt, let us write 1 by r is equal to 1 by u we have taken. So, better to express it as r is equal to 1 by u then it will be easier to work it out. So, therefore, we can write dr by dt this is equal to d by dt 1 by u. So, this way it is easier to write. So, therefore, this quantity here becomes d u by d theta minus 1 by u s square and d theta by dt is nothing but theta dot and on the left hand side we have r dot.

What we need r double dot therefore; we need to differentiate this once more. So, u theta d u by d theta we will write as u prime u prime is nothing but d u by d theta. So, next time once we differentiate this further this becomes r double dot equal to minus d by dt 1 by u s square u time theta dot. Now, we need one further substitution for theta dot here, because we did we are not going to put here theta double dot we want to replace this. So, we will do it further here in terms of 1 by u s square u prime, and theta dot we know that r s square theta dot is equal to h; therefore, theta dot we can replace by h by r s square and this is nothing but 6 times u s square. So, theta dot then becomes h times u s square. So, the whole thing then goes like u prime h where h is a constant.

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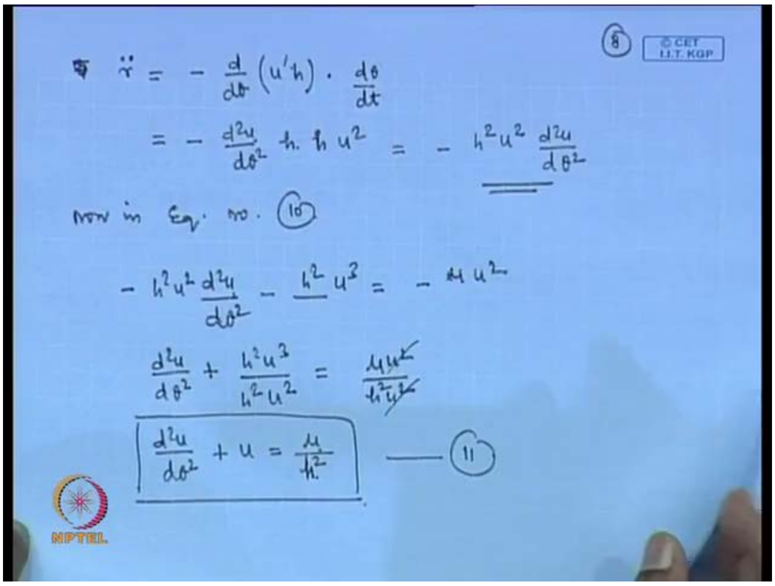

So, therefore r double dot this becomes minus d by d theta times u prime h times d theta by dt again we need to replace this d theta by dt in terms of h and u. So, now, this becomes d s square u by d theta s square times h times theta dot, now theta dot is h by r s square is equal to nothing but h times u square.

Now, the whole thing then turns out to be minus h s square e s square d s square u by d theta s square. So, if we go back to our equation number ten this is our equation number ten. So, now in equation number ten we have r double dot, we can insert from here minus h s square u s square ds square u d theta s square and minus h s square by r cube. So, 1 by r is nothing but u. So, we can simply replace this as u cube have to minus mu times u s square 1 by r s square becomes u s square.

Cancelling the minus sign on both sides, so this can be simplified as d s square u by d theta s square plus h s square u cube by h s square u s square is equal to mu s square by h s square u s square. So, ultimately what we are going to get I am checking it out this is mu u s square u s square cancel it out mu by h s square, and this is our equation number. So, now this equation we can look into the equation number eleven this is a linear second order differential equation, and right hand side is mu is a constant and h is a constant because this is the magnitude of the angular momentum vector.

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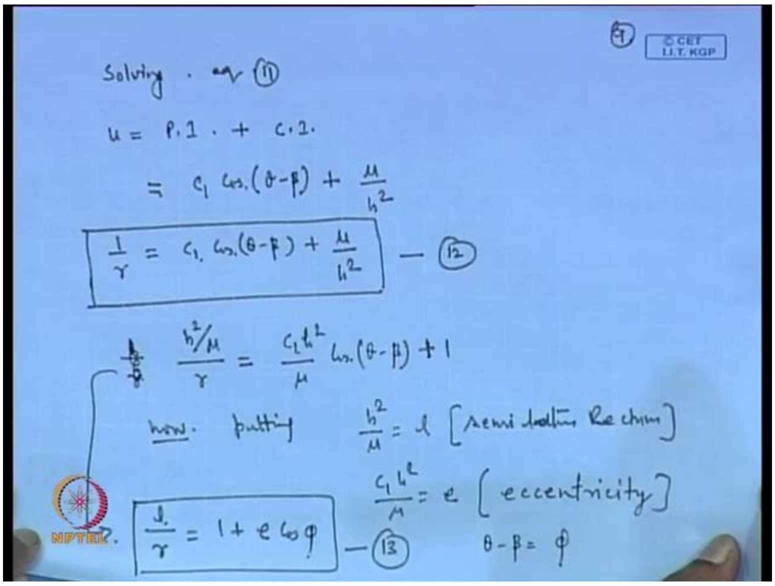

Therefore, it is easier to solve it and the standard solution to this equation is available solving equation eleven. So, the total solution u will consist of the particular integral and plus the complementary integral or complementary part and the particular integral. So, the complementary part this can be written as C 1 times cos theta minus beta and the particular integral, it is very easy to see in the equation number eleven if we put u is equal to mu by h s square. So, because this term is a constant. So, this will become automatically this term will become 0 and both side will then become equal therefore, particular integral mu by h s square u is equal to mu by h s square it satisfies this equation. So, particular integral we can write as mu by h s square and thereafter we can put u is equal to that. So, 1 by r. So, this becomes C 1 cos theta minus beta plus mu by h s square.

So, this gives us the solution to our second order diffrential equation this is equation number eleven we can further simplify it and put in a standard format. So, which this format is known as the equation of the conic section. So, let us modify this little bit or we can do we will write here as h s square by mu brideging the whole thing from this side to this side. So, this will become h s square by mu divided by r, and this terms will become C 1 times h s square by mu cos theta minus beta plus 1. Now, putting h s square by mu is equal to l, and this quantity is known as m I semi lotus rectum and we put C 1 times h s square by mu is equal to e this term is called eccentric c 2.

So, this equation then gets reduced to l y r is equal to 1 plus e and also put theta minus beta is equal to phi. So, they become. So, this is the equation of a standard conic section. So, here 2 parameters are 3 parameters are. In fact, are involved one is the two parameters are there one and e is the eccentricity, and one is the semi major axis and we have the two variables involved here which are the radius vector and this is the phi which is called the true anomaly. So, we will look into this pictorially later on we will leave this at this stage.

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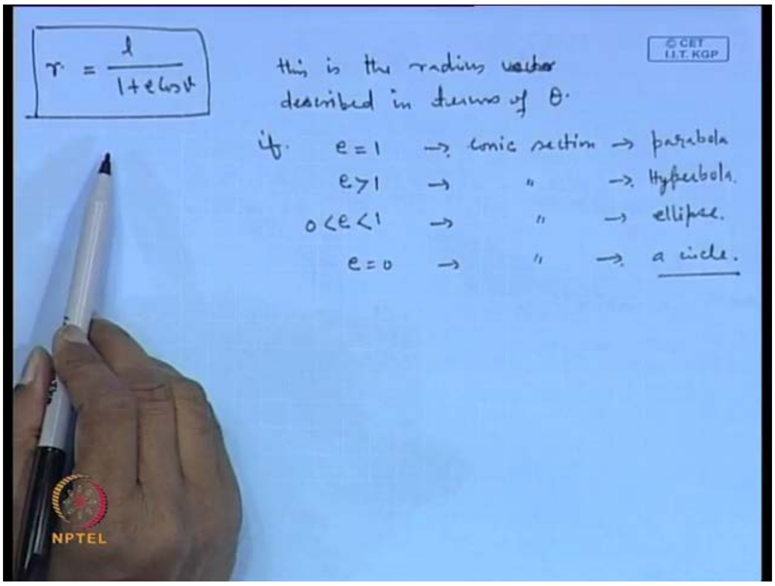

So, here we can write this as for the l y 1 plus e cos theta is equal to r. So, this is the radius described in terms of theta. Now, if e is equal to 1 this conic section becomes a parabola if e is greater than 1 then the conic section is hyperbola if e is less than 1, but greater than 0 and this conic section is ellipse with t equal to 0. So, this conic section gets reduced to s circle besides other cases are there will consider it later on. So, we stop at with this solution at this point. So, our working with the central force field motion it is a complete right now.

So, now we are going to get into the two particle motion let see there are two particles, and they are moving under the mutual gravitational attraction of each other, and then we try to look into the whether this problem is solvable or not or if solvable, then what are the junctions involved in for solving this problem. So, here the problem was very plane that we have a center of attraction, and then we want to describe the motion of a particle. So, using very simple mathematics, we have been able to describe the motion of the particle in the central force field as a conic section.

Now, depending on the value of the eccentricity here this turns out to be either an ellipse or a circle or a hyperbola or a parabola. So, these are very relevant in the case of satellite, because if we consider from the point of view of a satellite. So, once we inject the satellite into the orbit. So, at that time of injection or at the time of injection into the orbit what is value of the velocity r p m means in which direction the velocity vector is there what is? It is magnitude. So, that will decide what kind of orbit this is going to be...

So, accordingly the e will be fused. So, the velocity vector and the radius vector together with both of them are having three components total of a six components. So, easily we know these six components of a they are known to us from the onward measurement units are there this internal measurement units are there. So, from there we have the velocity component three components of velocity and three components of the radius vector they are available. So, they help in determining all this parameters which are involved here. So, if a let us look into the case of n an ellipse.

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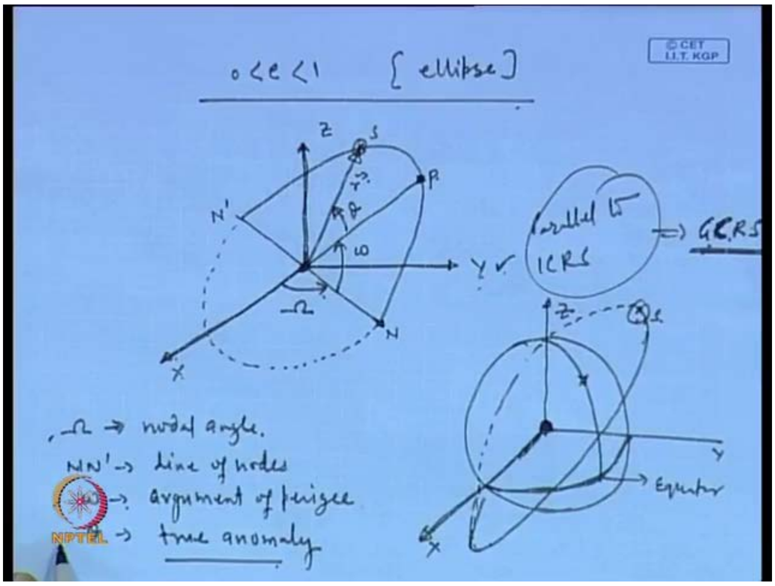

So, once e is greater than 0, but less than 1. So, this is the case of an ellipse, so how does this conic section look like and how this can manifest in the form of an orbit. So, let us consider that we have the inertial axis here x y and z. So, any satellite what we are showing here, this is the inertial axis x y and z inertial x means its orientation remain same and it is a non accelerating reference frame. In fact, non accelerating reference frame it is shape physically, now a days it is a fixed in the ICRS system this is call a international celestial reference system with a once we work with because we do all the measurement from the earth. So, we fix a an axis parallel to that and a fixed at the center of the earth, but it is an non rotator means its orientation remains parallel to the g c r s, and this it is the orientation of this reference frame it will remain parallel to the ICRS.

So, this is parallel to ICRS and this we will turn this whole thing as the geocentric celestial reference frame or the reference system. So, if, so in the g c r s because x y z here this indicates and the reference frame g c r s, so in fact this g c r s will have because suppose we have earth here. So, at the center of the earth we are fixing this mutual reference frame g c r s. So, I am this earth will be going around the orbit around this one under satellite is moving around the earth.

So, orientation of this reference frame while the earth goes around the sun it remain same, it does not change means it is and a snap shot at this insistent and another a snap shot at another insistent they will look exactly parallel to each other, but because it is a going in a electrical orbit. So, it as certain amount of acceleration, so in true sense this is not a exactly an inertial reference frame, but for our earth purpose or for many of our measurement purpose. This will serve as the inertial reference frame, because this acceleration is quite a small.

And therefore, if we look into this case, so here we have the frame eversible and on the earth we have this is the equator of the earth. So, this is the equator and here somewhere say our ground station is there. So, from the ground station we are looking at some satellite in the sky here. So, the same satellite is suppose indicated here by yes, and the orbit of the satellite around the earth may look like. So, this orbit is indicated here by this particular curve and we can extend below. So, the solid line is given up the x y plane and the dotted line lies below the x y plane.

Now, in this plane the satellite is lying here and this is the radius vector of the satellite this angle from here to here this is capital omega this is called nodal angle the line l n prime this is known as the line of nodes. So, satellite position is measured from certain position which is called perigee. So, say this is the perigee of the satellite, this is the nearest position of the satellite from the center of the attraction or in this case from the center of the earth.

So, the angle from here to here this angle is written as small omega and this is a small omega this is nothing but argument of perigee, and the angel from here to here up to the satellite this is written as theta and theta this is call the true anomaly. So, in this equation that we have referred this theta is nothing but the true anomaly. So, here we have theta true anomaly besides.

So, we have covered here capital omega a small omega and theta these are the three parameters beside this to determine or to describe an orbit we have another three parameters. So, total of a six parameters. So, will describe this in the next lecture and then we will continue with the motion of two particles system and we will further develop that into the three particles system. So, thank you.