Space Flight Mechanics

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Module No. # 01

Lecture No. # 40

Attitude Dynamics (Contd.)

Searching about the attitude dynamic of a torque free rotating body so, we will continue with that and more over if time permits then will start with the a professional topic.

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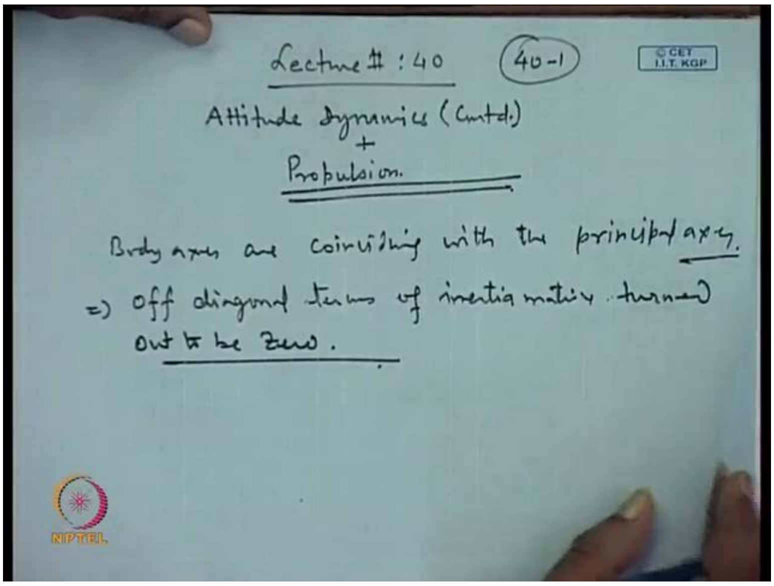

So, what last time we worked out that a rigid body is given and the body axis are coinciding with the principle axis. So, our assumption was body axis are coinciding with the principal axis. So therefore, the off diagonal terms, off diagonal terms of inertia matrix turned out to be zero. And then, we worked out the equation of motion using the Euler's dynamical equation by setting the torque in that equation to zero.

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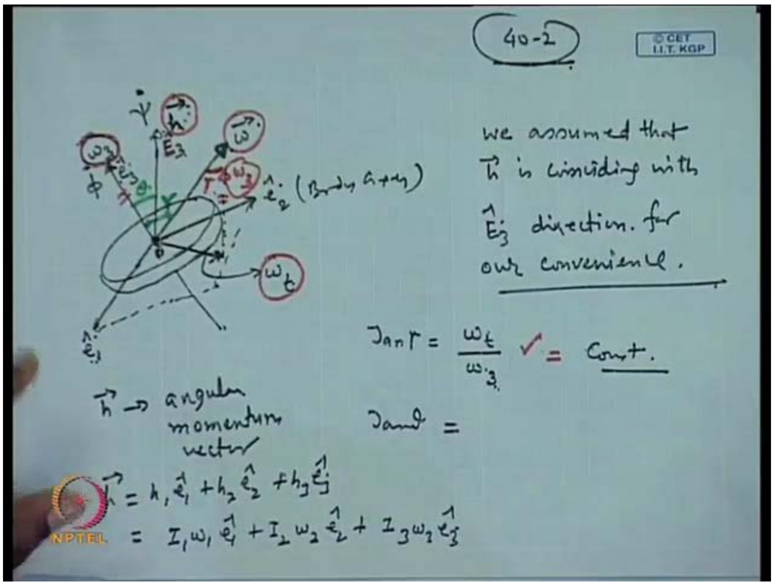

So, next we have what we are looking at? That a disc is given which is rotating on its axis, omega 3 is in this direction, phi dot is in this direction. And, we assume here that the angular vector h is given to us, a angular vector h is shown along this direction and then psi dot also we showed along this direction. And this direction we wrote as E 3 and the final body axis because phi dot is along the body x. So, this axis was shown as E 3 cap. E 3 or E 3 cap, E 3 cap is the unit vector in this direction. So similarly, you can put here E 3 cap. So, we have e 1 vector in this direction and let us say that the e 2 vector is pointing like this.

So, these are the body axis coordinates forming the body x coordinate system. While this is part of the inertia reference system and then we of course, omega vector we showed as and this angle first taken as theta and this angle was taken as gamma. So, omega was broken in the e 1 e 2 plane. So, let us consider that this e 1 e 2 plane. So, we broke omega in this plane and say it is lying somewhere here. So, this was termed as omega t and it is a perpendicular omega t. And because this vector is perpendicular the omega t that is omega three, e 3 vector is perpendicular to the e 1 and e 2. So, the omega 3 is lying here in this place. This is our omega t and this is omega 3 and here our omega lying and h vector is coinciding along this direction.

So, we assume that h is coinciding with E 3 direction for our convenience. This is the point to the center of mass of the body. So, there after we proved that tan gamma, this is nothing but omega t divided by omega 3. So, this is your vector omega and here in this figure is which, be careful here that this vector this is the vector here e 3. And, this is the actual component omega 3 which is being shown here. So, they are not appearing parallel it is a just because of this figure, but this line and this dotted line both are parallel to each other. So, omega 3 here if say appearing to go in this direction, while this omega is appearing in this direction. This is not true; this should be exactly parallel to this. But once we make it parallel, so the figure does not look very good. So, I have kept it like this.

So, from here we can see that tan gamma this angle is from here to here gamma. Because this and this line are parallel, so this angle is your gamma. So, immediately from this place you can see this value is here omega t and this is omega 3. So, tan gamma is omega t by omega 3 and we already prove that omega t is a constant, omega 3 is a constant. So, this turns out to be a constant. Similarly, the tan theta we had the, Now, h is the vector the which is the angular momentum vector. This is the angular momentum vector and this we can, we have broken and written as h 1 e 1 cap plus h 2 e 2 cap plus h 3 e 3 cap.And this was finally, written as I 1 omega 1 and e 1 cap plus I 2 omega 2 e 2 cap, just I am repeating from the last lecture and I 3 omega 3 times e 3 cap.

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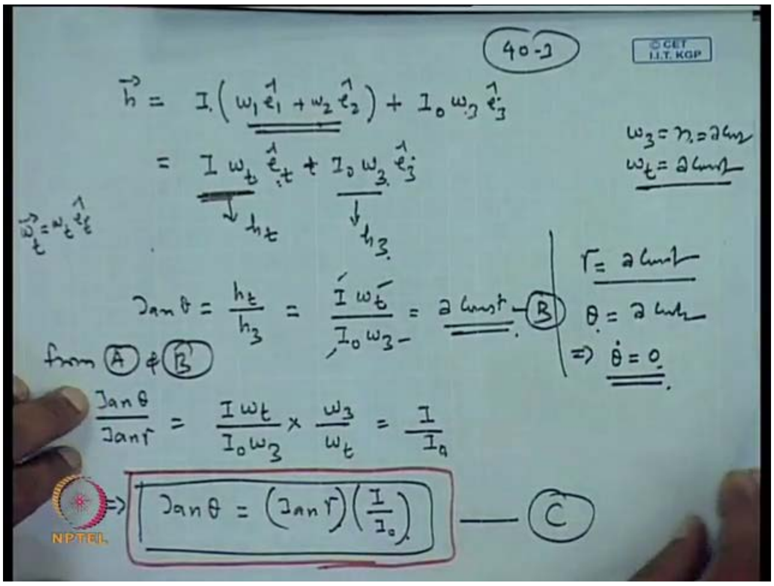

So, h becomes now I 1 and I 2 they are equal 2 each other and this was written as I. So, we wrote it as omega 1 e 1 cap plus omega 2 e 2 cap and I 3 was written as I 0. And the finally, this vector is nothing but omega t. So, we wrote this as the omega t times, we can write it as e t cap this is unit vector in the omega this is a unit vector. So, omega t basically we are writing as omega t times e t cap. This is I 0 and omega 3 is nothing but your n which is a constant. So, omega 3 equal to n, this is a constant and omega t also we proved this is a constant so, e 3 cap.

Now, tan theta. So, this is your basically h 3 and this is the tangential component and also we last time proved that the vector omega 3 h and omega, they are lying in the same plane. Also therefore, the vector omega t, because omega t is a combination which is making here omega t and omega they together compose omega. And omega and we have the omega t here and this is the vector omega whose projection is we are taken on the even 1 e 2 plane. Now, omega this tan theta, again we wrote as see from this place this is the h vector and we can decompose this is vector, we can break it into 2 qwords. So, this component will be h 3. So, here you have h 3 and this component this is your h t. So, from here tan theta is obviously, h t by h 3.

So, we can write here h t by h 3 and this is nothing but your h t here. So, this h t we wrote as I times omega t and this quantity is I 0 and omega 3 and you can see that I, I 0 these are constants omega t omega 3 are constants. So therefore, this is also a constant. So, thus we have proved that, now at gamma is a constant because tan gamma turns out to be a constant and similarly, theta is a constant. So, this theta is a constant implies that theta dot equal to 0.

Now, we can write tan theta by tan gamma from these two equations. This is let us say, this is equation number B and this is our equation number A. So, from A and B tan theta by tan gamma becomes I omega t divided by I 0 omega 3 times divided by gamma. So, this is omega t and omega 3 this becomes I by I 0. So, this implies tan theta is equal to tan gamma times I by I 0 and this a let us say this is our equation number C. This equation will give you very important conclusions.

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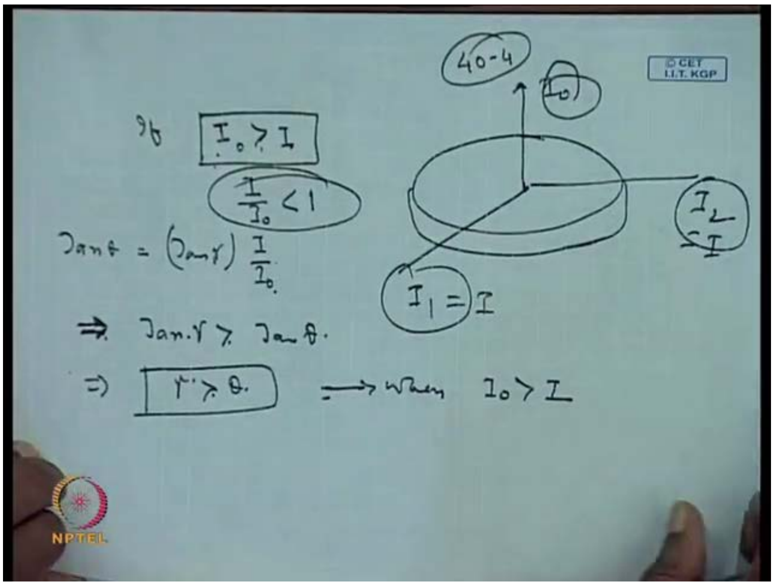

Now, taking this equation number C, if I 0 is greater than I. If I 0 grater then I, which is the case we have been considering here. We have suppose a big disc, we are taking a big disc and it is rotating on as shown x, this is your phi dot shown here in this direction. So, phi dot this is the basically the Euler rate while omega 3 is the body rate, what we have shown. So, if a big disc is rotating like this. So, we can consider that this is a big disc and it is a rotating. So, this is your I 0 components we are showing in this direction, while I 1 and I 2 will be perpendicular to each other in this plane. So obviously, in this case I 0 is greater than I.

So, for the disc we have, this is the I 0 in this direction and I 1 will lie in this direction I 2 will lie in this direction and this is equal to I, this equal to I. So, if I 0 is greater than I, then this equation see what does it say? The tan theta equal to tan gamma times I by I 0. So, this implies that I by I 0 this is less than 1. So, this simply implies that, tan gamma will be greater than tan theta and this implies gamma will be greater than theta. So, this is the case when I 0 is greater than I. Now, take the case of a cylinder, we can discuss little bit more here before taking the case of a cylinder to, We will make a phase figure of this.

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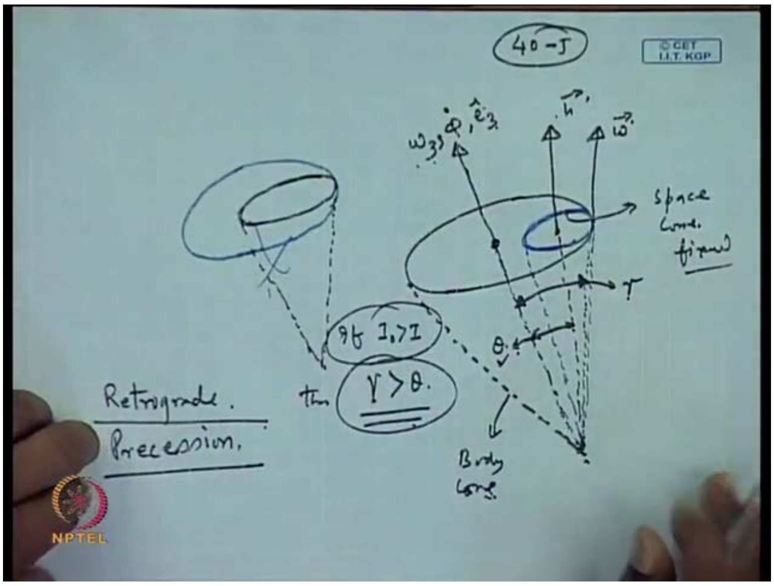

So, what we are showing here? Just now we have computed, this gamma is greater than theta. Gamma is the angle between, as we have seen in here. In this figure gamma is angle between omega 3 and omega. So, this is the omega vector and this is the omega 3 here. So, this is your angle from here to here we can show as, this is angle gamma and while your theta angle is between h vector and omega 3. So, this is the h vector and this is omega 3. So, this is this angle is your theta so obviously, you can see that here theta is smaller than gamma, that is gamma is greater than theta. This kind of rotation, this is call retrograde precession.

What is the exact meaning of this we will discuss it little later, once we further develop the, because right now whatever the equations we have develop it is in the terms of the body rates. And to see them, so body rates until unless you are a riding over the body itself you want be able to see anything. So, form the as an observer we want to see it from the an inertial x. So, we have to get back in to the equation and look at the Euler rates. So, we will try to find out the Euler rates equation and from there will conclude the how the motion it looks like.

But right know the picture is giving like this that, if I 0 is greater then I then gamma will be greater than theta and this is what it implies. That omega vector will be directed like this, h vector will be directed like this, and phi vector is directed like this and this is called body cone. And this one is called space cone, this is fixed. So, you can see that here as earlier, we are discussing that the omega h and omega 3, omega vector is vector and omega 3 they always lie in the same plane. What is exactly happening here? This bigger cone this will be rolling over this cone. So therefore, it will be such that omega vector points initially in same direction, while omega and omega 3 they will be rotating. And rotating such that h always lie in the plane containing omega 3 and omega.

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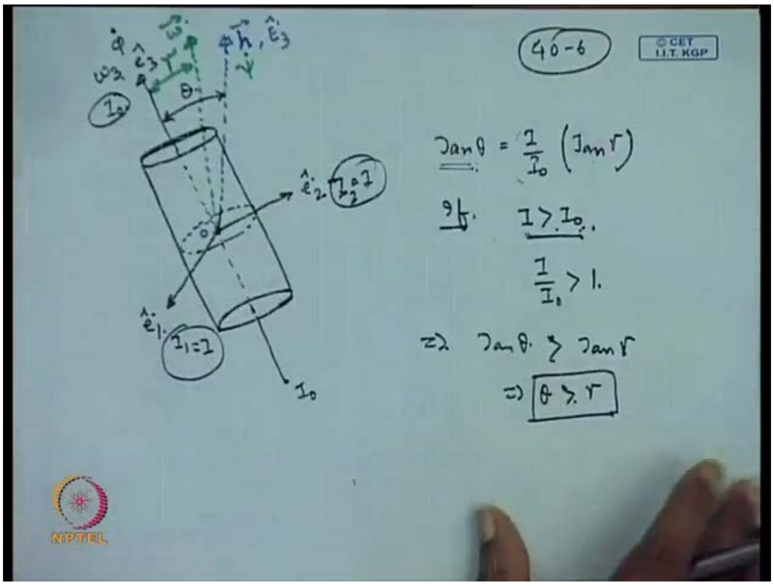

So, next we take the case of the cylinder. So, the same logic applies here also, what we have shown here in this figure the only thing the dimension changes. This is your omega 3; this is the center of mass of the body.

Let us say this is the e 1 cap and this is e 2 cap and here you have e 3 cap, phi dot is in the same direction. As usually we make h 2 coincide with the e 3, the h vector and the E 3 cap it is in the same direction this is point O let us see. Now, in this case, if we get back to equation tan theta is equal to I by I 0 times tan gamma. Now if I is greater than I 0 which is the case here, this is the case of a cylinder. So, I 0 you we are taking along this axis, while I 1 and I 1 is equal to I and here I 2 is equal to I we have taken. So, these are along, these are the principle axis of the body, this is the body of revaluation. So, this easer thing in the principle act in this case and this is the third principle axis.

So, I 0 we have chosen along this direction so therefore, we can see that I 0 is less then I. Therefore, I by I 0 this will be greater than 1 and this implies that, tan theta. This will be you can see from this place that this quantity is turning out to be greater than 1 and this is to be made to equal to this quantity. So, what will happen in this case? If this quantity is less than only to (( )) by a larger quantity this will be equal to that and that you can see just by dividing. So, in this case tan theta will be greater than tan gamma and this implies theta is greater than gamma and we need to fix the angles. So, the gamma we have chosen as the angle between omega vector and the omega 3. So, we need to choose all this since now carefully and theta is the angle between the h and omega 3.

So, as usual this is our theta angle and gamma angle is now a small. So, gamma angle let us say in this case, this is your omega vector. So, angle between this and this is your gamma. So, you can see the difference here in this omega was a located here and omega 3 here while the h vector was located here in this place. Obviously, psi dot we are pointing in the same direction again. So, the this vector as moved away from this position to this position, to the intermediate position between e 3 small e 3 and capital E 3.

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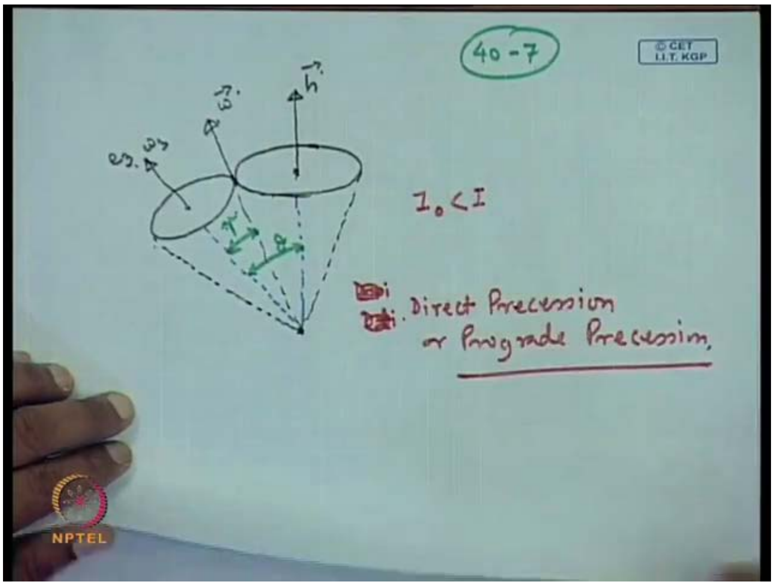

This kind of case we can show by, our h vector is point in direction. This is omega 3 e 3 as usual and here we can show this is the omega vector. So, we will have. So, the angle between omega 3 and omega is our gamma, this is the angle gamma and the angle between h and omega 3 this is theta. So, this is your angle from here to here, this is theta. So, the motion that we have just now depicted here this is called direct precession or also called pro grade and this refers to the case when I 0 is less than I.

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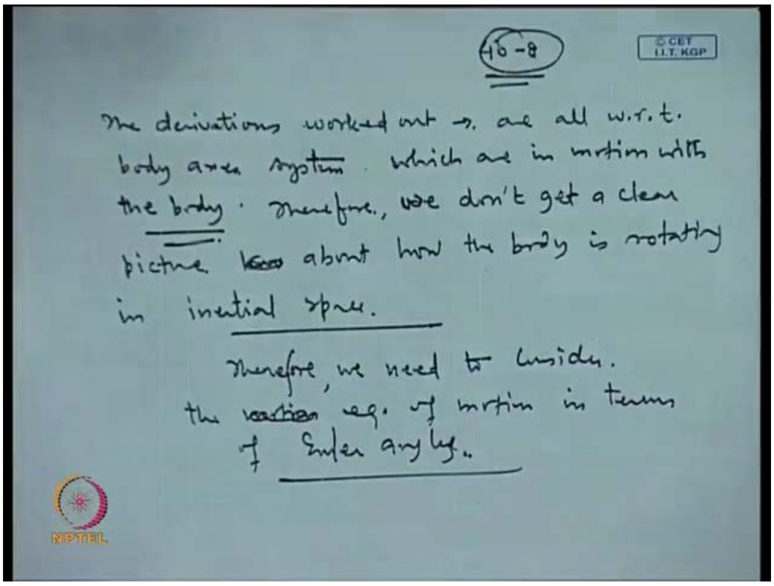

So, derivation till now we have done, derivations worked out are all with respect to the body axes system which are in motion with the body. So therefore, we do not get a clear picture how the about, how the body is rotating in inertial space? Therefore, we need to consider the motion equation of motion in terms of Euler angles. So, this is what we are going to do. It is very clear from here that this is the angular velocity the equation that we wrote in terms of the body components of the angular velocity. This is not giving a clear picture, but it has given us some very interesting thing that these angles are constant in the case of the torque free rotation.

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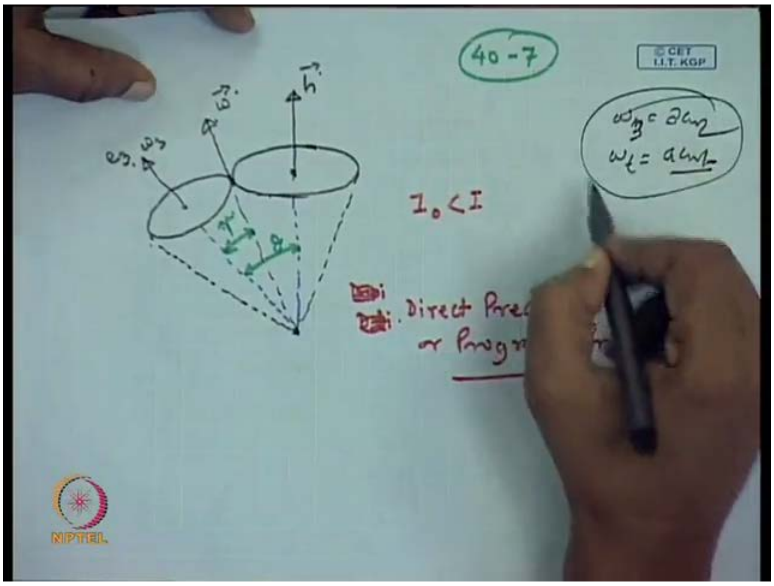

And more over the and this we are derive from the fact that omega 3 is a constant and similarly, we also got that omega t is a constant. So, all these are important conclusions what we need further in site in to the motion and therefore, we they sort to the Euler equations.

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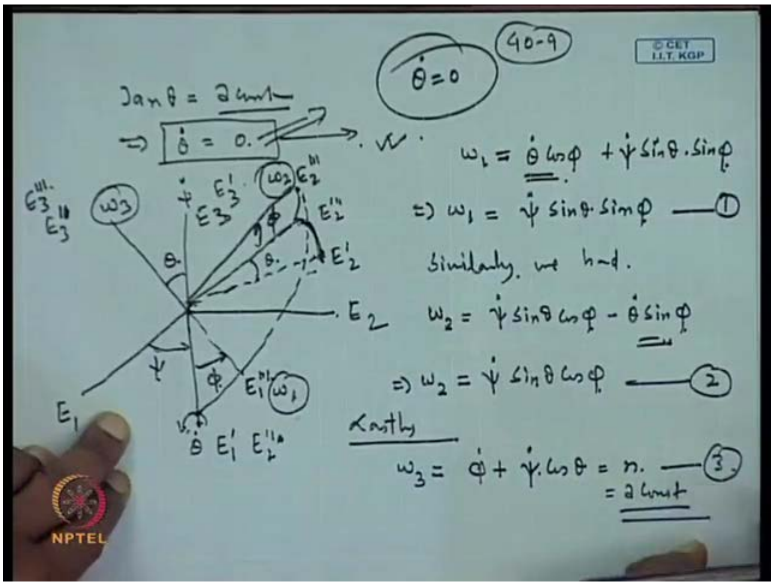

So, what we just now worked out that tan theta this is a constant. This is very obvious and therefore, this implies theta dot equal to 0, earlier also we have written. We will utilize this result for working out the Euler's equation.

So, this was our angle psi we rotated about this was angle theta we have to theta dot is there psi dot E 1 capital E 2 capital an E 3 capital in this direction. This move the, once we rotate it from here to here this moved in this direction and there after once we are given in the theta rotation. So, this moves up and the E 2 comes here and this is E 2 prime it came to E 2 double prime from this position to this position. And, here this is E 1 prime and E 2 double prime, E 3 prime was in the same place and E 3 this is double prime.

Now, next we once we are rotating it from to here to here. So, it moves from this space to this space and this rotation was given by angle theta, there after we further moved it away from here to here and this moved in the same plane as E 2 double prime will move. So, it was move from here to here by phi and here to here from phi. So, phi angle and phi angle here both, so E this is E 2 triple prime E 1 triple prime and this E 3 triple prime in the same direction. So, with this figure we worked out that omega 1, omega 3 is in this direction and omega 1 is in this direction, this is omega 3 omega 1 finally, omega 2 is here. So, equation for omega 1 we wrote as, theta dot cos phi plus psi dot sine theta times sine phi.

Now, just now we have seen here theta dot equal to 0 this implies omega 1 this cos term, this term will become 0. Therefore, omega 1 becomes psi dot sine theta times sine phi. This is let us say this is our equation number 1.Similarly, we had omega 2 equal to psi dot sine theta times cos phi minus theta dot sine phi. So, this term will become 0 here. So, this implies omega 2 is equal to psi dot sine theta times Cos phi if they, we are utilizing the same result theta dot equal to 0.

So, this is our equation number 2 lastly we have omega 3 this is equal to, this was a constant basically. So, phi dot plus psi dot Cos theta is equal to n, this is a constant. This is our equation number 3, now we differentiate equation 1 2 3. So, while differentiating and there after we have to do some insertions here and here, after some manipulation we will be able to get to the results for theta dot psi dot and phi dot. So, already what we have seen from this places that theta it dot equal to 0. So, this conclusion is already made, just we will have to work out for psi dot and phi dot.

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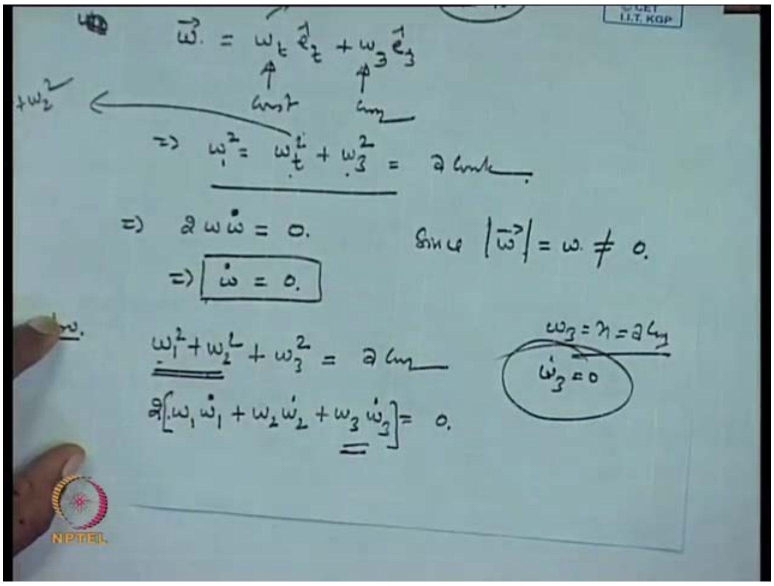

So we have, now let us get back into the omega vector which we wrote as omega t times e t cap plus omega 3 or n times e 3 cap, where omega t is a constant omega 3 is constant .

So, from here certainly we can write omega square is equal to omega t square plus omega 3 square. This 2 are constants, already proved in the last lecture, so this implies omega square will be a constant. So, if we differentiate this so, this will be 2 times omega times, omega dot this is equal to 0. And since omega the magnitude of this vector which is omega this is not equal to 0 therefore, this implies omega dot will be equal to 0.

Also we have, so by the same logic we have omega 1 square omega 2 square, this is omega basically your omega t square which we have written here. These quantities are here; we write this is nothing but your omega 1 square omega 2 square. So, we add the finally this term omega 3 square, this is basically a constant and if you differentiate this. So, this simply indicates omega 1 times omega 1 dot omega 2 times omega 2 dot plus omega 3 times omega 3 dot this is equal to 0.

So, in this equation omega 3 is a constant, already we have seen omega 3 is equal to n this is a constant and therefore, omega 3 dot this will be equal to 0. So, this term will vanish. We have to put 2; we can be taken outside the bracket. So, as a whole what we are getting.

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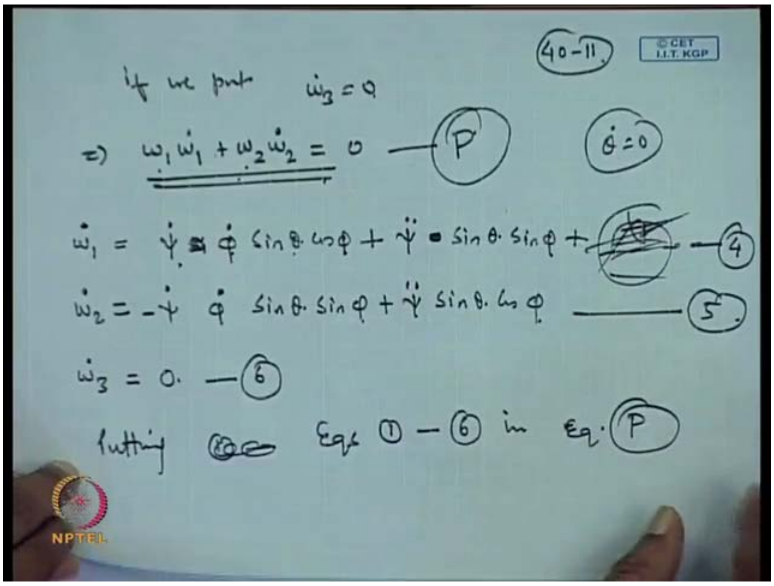

So, if we put omega 3 dot equal to 0, so this implies omega 1 times omega 1 dot plus omega 2 times omega 2 dot this is equal to 0.And just now we worked out the relationship for omega 1 omega 2 and omega 3 so these are available to us. And, what we need further, this is omega 1 dot omega 2 dot. So, omega 1 dot we can use the equation number 1 here, omega 1 dot can be written as psi dot times, let us say the phi dot sine theta times Cos phi plus psi double dot. We are differentiating this equation here, psi double dot times sine theta times sine phi and plus the third term we have to differentiate sine theta.

So, if we differentiate this dot term will be 0 because theta dot equal to 0. So, that term we are just eliminating this will not be required here in this place. Similarly, we will have omega 2 dot, we use this equation again the same operation we have to do here. First we have psi dot times phi dot sine theta and this is Cos phi. So, this will be sine phi here now and minus sign we put here in this place and plus psi double dot sine theta times Cos phi. This is our equation number 4, this is equation number 5 and omega 3 dot already we have proved this quantity equal to 0. Omega 3 is a constant and omega 3 dot, this turned out to be 0 this is from the last lecture itself.

So, now we have omega 1 dot and omega 2 dot available to us, so omega 1 omega 2 also available. So, insert in this equation, now let us write this as equation number we put here 4 5 6. So, will term this as equation p, putting 1 3 equations 1 to 6 in equation p.

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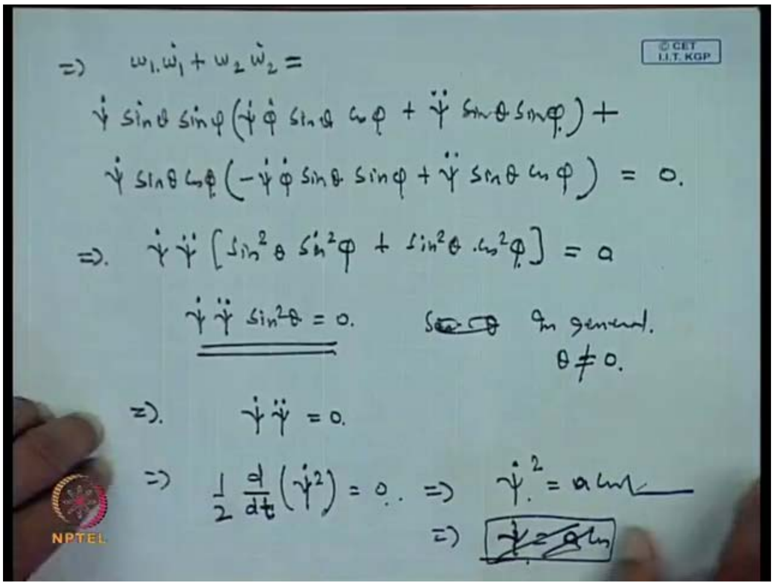

So omega 1 times omega 1 dot. So, we have omega 1 psi dot times sine theta and then we put the omega 1 dot value, just now we have expression we have derived. Psi psi dot times phi dot sine theta Cos phi plus psi double dot sine theta and sine phi and plus omega 2.We have written as psi dot sine theta times Cos phi and this to be multiplied by omega 2 omega 2 dot is minus psi dot phi dot sine theta and sine phi plus psi double dot sine theta Cos phi. And this is equal to 0, so this implies omega 1 plus omega 1 dot plus omega plus omega 2 dot equal to this equal to 0.

So therefore, simplifying this just you have to multiply and see most of the terms, they cancel out and what you get is psi times psi double dot sine square sine square theta timessine square phi plus sine square theta times cos square phi. This is equal to 0 and this will give you psi dot times psi double dot sine square theta equal to 0.Since, in general theta this is not equal to 0. So, this implies psi dot times psi double dot this is equal to 0.

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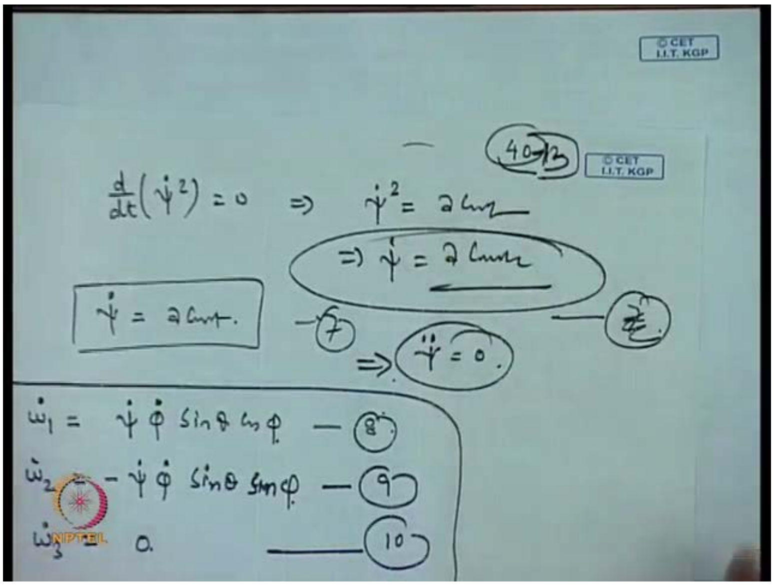

So, this we can write as d by dt psi dot square 1 by 2 equal to 0 and this implies psi dot square equal to 0 or ultimately we have psi dot equal to 0. Psi dot square this is equal, this will term to be constant once we take the integrate it. So, after integrating we are writing as constant so, this psi dot becomes a constant not 0. We write it on the next page. So, d by dt psi dot square equal to 0 implies psi dot square is equal to a constant and this implies psi dot is a constant. So, this is our equation number let us say 7. So, psi dot equal to a constant, this we are writing as equation number 7.

So, once we have got omega 1 dot omega 2 dot omega 3 dot so, we got all these results. So, what ultimately we see that the equation for omega 1 omega 2 omega 1 dot omega 2 dot omega 3 dot, they will get reduce to now omega 1 dot. We can write as psi dot is a constant and therefore, psi double dot naturally this will be equal to 0. So, this implies psi double dot equal to 0. So, we have omega 1 dot is equal to psi dot phi dot sine theta Cos phi, this is our equation number say 8.Similarly, omega 2 dot this becomes minus psi dot and theta times sine phi and this is equation number 9 and omega 3 dot we have got as 0 and this is equation number 10. So, we can see that this term and this term gets eliminated because psi double dot equal to 0.

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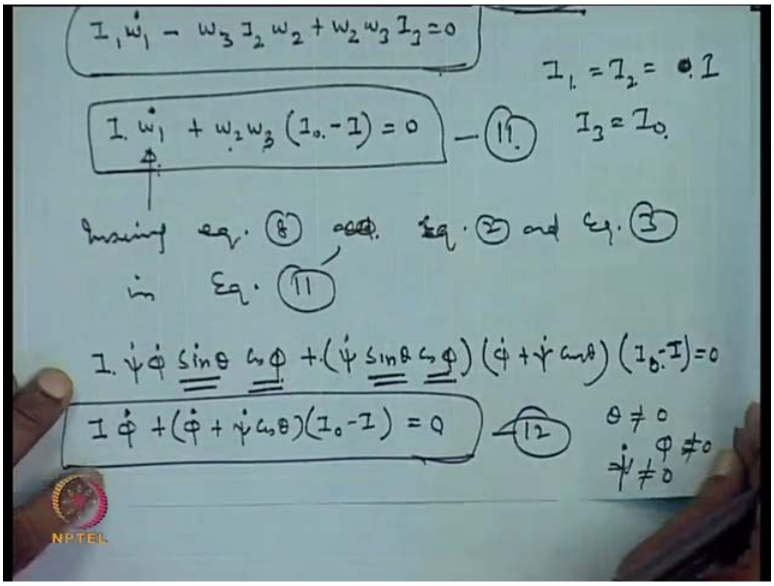

Now, earlier if you remember in the last lecture we derive few results, which we are written as I times or I 1 times omega 1 dot which was basically the reduction of the Euler's dynamical equation for the torque free case. This is what the equation we wrote the last time. So, here I 1 in our case I 1 and I 2 these are taken as equal to I and I 3 is written as I 0. So, this we wrote as I times omega 1 dot plus omega 2 times omega 3 times I 0 minus I equal to 0. This is the last time development, now omega 1 dot is available to us, so this we name as equation number 11. So, omega 1 dot is available to us omega 2 is available, omega 3 is available. So, we insert in this equation.

So inserting equation, omega 1 dot from 8 equation 8 and omega 2 we have written as equation number, omega 1 and omega 2 omega 2 and omega 3. So, this is from equation number 2 and 3, inserting equation 8, equation 2 and equation 3 in equation 11.So this will, I times omega 1 dot or the omega 1 dot is psi dot times phi dot sine theta times Cos phi and plus omega 2. Now omega 2, we have to take again from this equation this is psi dot sine theta cos phi and omega 3 this is a constant.

So, we simply write this as and constant and this can be written as phi dot plus psi dot cos theta and then we have I 0 minus I. So, from this equation let us see what we can throw out. So, sine theta Cos phi sine theta Cos phi these are on both sides and theta this is not equal to 0 and phi also not equal to 0. So, under that circumstances what we get and psi dot also psi dot is also not equal to 0. So therefore, what we get here I times phi dot plus phi dot plus psi dot cos theta times I 0 minus I, this is equal to 0and this is equation number 12.

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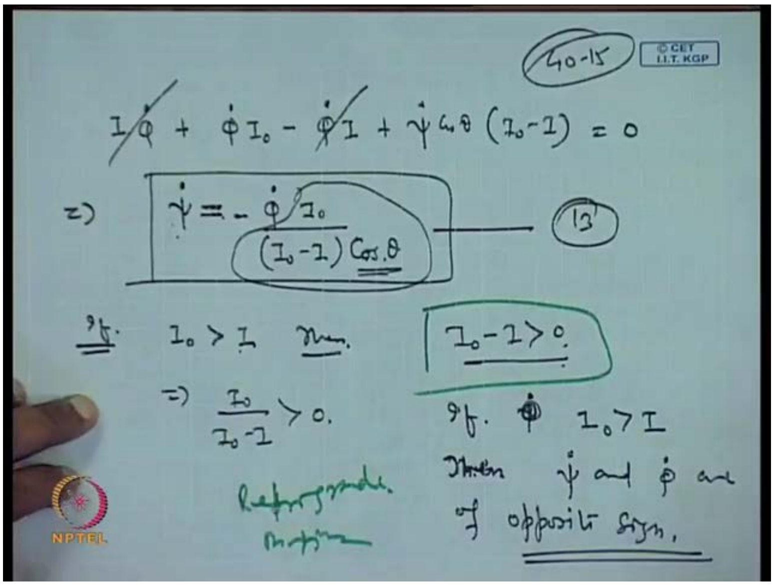

We can break the bracket, this is I phi dot plus phi dot I 0 minus phi dot I plus psi dot Cos theta times I 0 minus I this is equal to 0. So, this implies this we cancel out, leaving us with psi dot equal to phi dot minus sign here, phi dot I 0 divided by I 0 minus I Cos theta. You have seen how the equation for the Euler rates it looks like. So, this is our equation number 13, now this equation will give you a lot of insight, how the body is rotating. So, we see that if I 0 is greater than I, then this quantity will be positive I 0 minus I. So, I 0 minus I this will be greater than 0, if this is greater than 0 then simply it implies that the I 0 by quantity. So, this implies that I 0 by I 0 minus I this quantity is basically greater than 0.

And therefore, you can see if Cos theta is positive. Therefore, this term turn sort to be positive it says that, if I 0 is greater than I then psi dot and phi dot there are of opposite sign. So, if psi dot if I 0 is greater than I, then psi dot and phi dot are of opposite sign. So, you can guess what is happening here in this case. So, psi dot is basically the nodal, is the rate at which the line of node it processes. So, you have here the case that phi dot is the on the axis on which the body is rotating. So, you are body is rotating like this and psi dot we have taken along this line. So, psi dot is here and phi dot what we have shown here in that case so, phi dot will just get reversed.

As the case in the figure we have shown, so phi will come here in this picture. So, what it shows now here and this is what we have termed as the retrograde motion. So, this was our retrograde motion. So, psi dot is positive it is up. So, it is a rotating like this, the precession is taking place like this, while the body rotation is like this. It is just opposite of that it is a body is not rotating in the same direction. But it is rotating in the opposite direction, so basically this vector is in this direction. So, this give gave us a lot of very good conclusions.

So, we are almost done with the attitude or torque free rotation of a rigid body, where the body was taken as there case of body of it was basically body of revolution. So, in the next class we will take the stability of the torque free body rotation. So, this is the last part remaining I expected to complete it today, but is not possible. So, next time we will finish the stability part a stability of this rotating body and then we go into the propulsion. Thank you very much.