Space Flight Mechanics

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Module No. 01

Lecture No. 04

Conic Section

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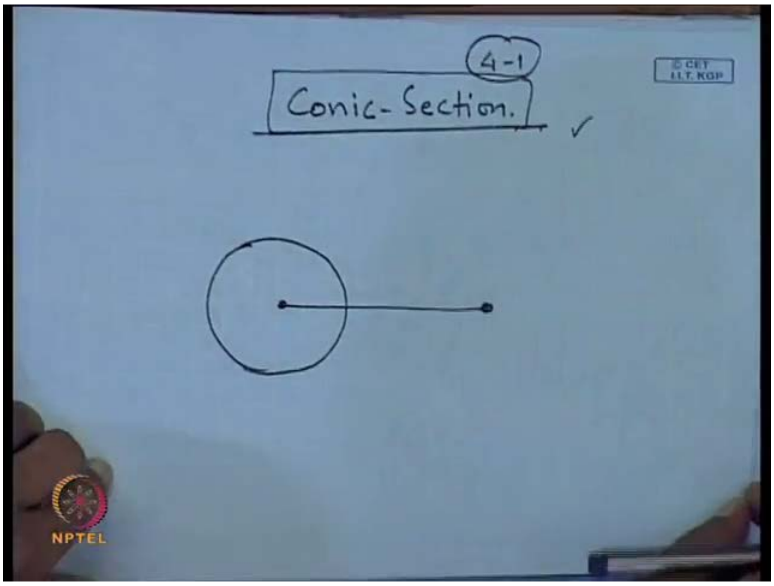

In this lecture, we are going to discuss about the conic section. So, the conic section, it is important to us, because later on, we will see that, once a particle is moving about a big body, or either the two particles are moving under the mutual gravitational forces, so, their motion, their trajectory can be reduced to a conic section. So, in this lecture, we are going to study about the conic section. So, conic section, it can be defined aslocus of point, locus of points, whose distance, whose distance is, from a fixed point and a fixed line, are in constant ratio.

So, the fixed line be, let us represent like this. Another, the conic section, if we represent by this curve, and if we take any point on this conic section,so, the distance from here to here, this is the point B and the fixed point, what we are talking about, this is our focus. So, this is your fixed point, and this is the fixed line. This fixed line is called Directrix and this fixed point, this is called focus. This distance, we can show this as r and here, let us say, this point, we are writing as C. So, from this fixed line, and from this fixed point, the locus of the points, whose distances are in a constant ratio. So, this implies that, BF by BC, this will be a constant and this constant, we write as e.

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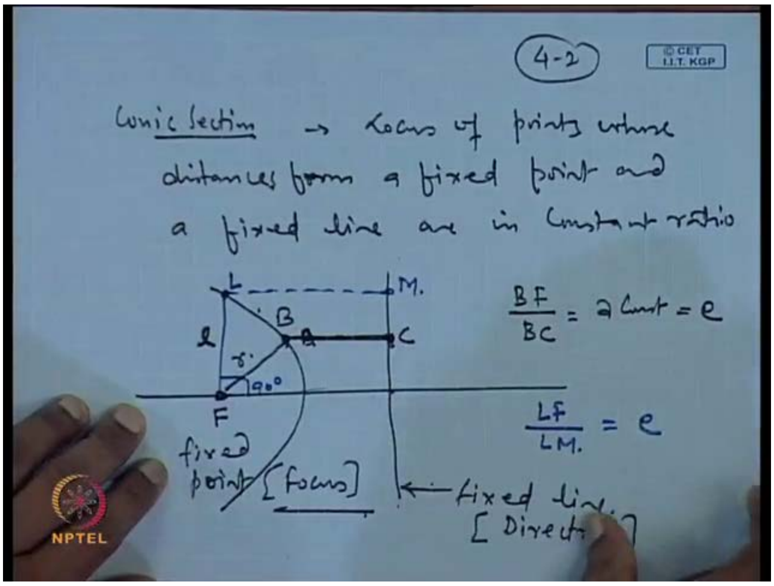

So, take another point. Now, by taking this angle to be 90 degree,sorry, we are choosing another point, and let us name this point as L and,and here, we indicate this point as M, which is parallel to this line. Now, because this point is also lying on that trajectory, and therefore, LF divided by LM, this must also be in constant ratio. So, this LF, we will write this as the l, this distance from here to here, this is l. And, this is called the Semilatus rectum. So, putting LF is equal to l, therefore, l by LM, this becomes equal to e, and this implies, LM is equal to l by e.

Similarly, from here, we can see, this BF, this is the distance which we are writing as r. So, BF is r, and this is the BC,BC is this distance. So, this is BC, and this is in the constant ratio, so, which is equal to e, and therefore, we can write BC equal to r by e. So, this distance, then, it becomes l by e and this distance becomes r by e. Now, once we have done this, s So, it is a, many important properties of this conic section, we can derive;one by one, we are going to work out all of them.

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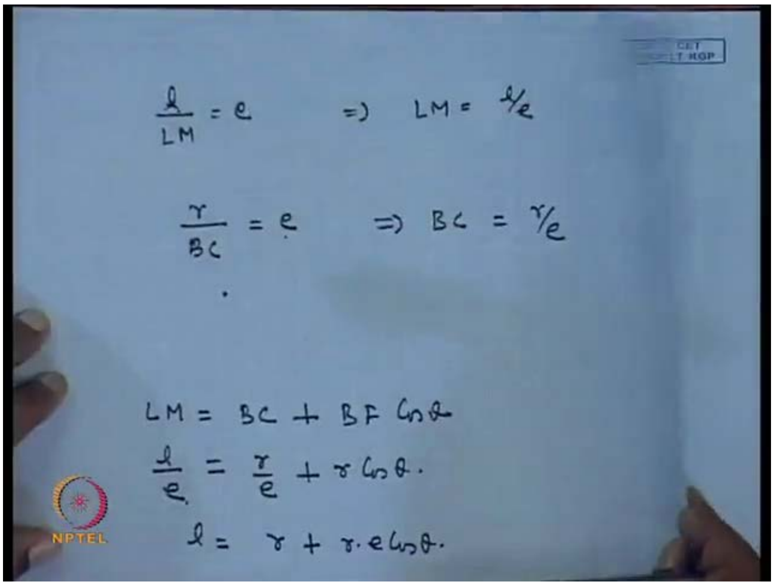

Now, from here, from the previous figure, in this figure, we can see that, LM, we can write as, this LM can be written as BC, plus the projection of this distance BF cos theta, where theta is this angle. It has the angle from here to here; this is angle theta. The blue line we have shown here, this is 90 degree; this angle is, which is the vertical one. And, BC, we know from here that, this is r by e and BF, we have written as r and l by m, we know this is equal to l by e, as derived here, and therefore, we can write from this place, l is equal to r plus r times e cos theta.

Now, this particular equation, this can be reduced and it can be written in, by taking r common out of this, you can write l by 1 plus e Cos theta and this constitutes equation of the conic section. Now, if in this equation, we put e equal to 0…So, simply this integrates l is equal to r. Later on, we will see that, this, the l that we are writing here, in general, we write this l as, a times 1 minus e square. So, for e equal to 0, this gets reduced to r equal to a; means, this is describing the position of the particle, where the, this point, as it is moving. So, this r is remaining constant, from the focus; this is what it indicates. Therefore, this implies a circle. So, this is the equation of a circle.

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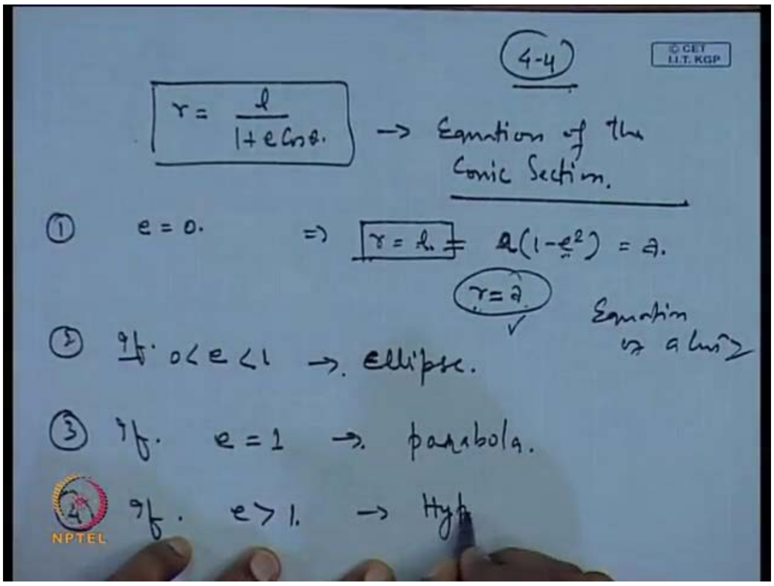

If e is greater than 0, but less than 1, so, this becomes equation of an ellipse. So, this indicates ellipse. If e equal to 1, this is the equation of a parabola; and, if e is greater than 1, this results in a equation for a hyperbola. So, what this hyperbola, ellipse and other things are? Just now, we will have a look of this. Now, the conic section that we call, this derives a terminologies from the sections of a right circular cone. We have, this is a right circular cone. So, if we see, this right circular cone in the front view, so, this is your right circular cone, and we are looking at the front view. If I take a section here, in this place, so, here, I take a section like this, and then, see from the top. So, if you see from the top, it will give you a circle. So, the cut area on the lower portion, or either on the upper portion…So, the bottom of the upper portion, or the top of the lower portion, it will show you as a circle.

Now, if you cut the same thing by an inclined plane like this, so, this will appear as an ellipse. On the other hand, if you cut it now, say, the generators of this cone, these are the lines drawn on the surface of the cone at equal distances. So, if you have generators… So, I draw the generator from here to here, and on this space, from here to here. Now, say this is the point B, and this is the point B prime. So, I take a plane, and say, this is the point B and B prime, in the front view; if I take a plane, and cut it from here to here, by a plane, so, the plane which will appear, here in this place, so, this plane will give you a parabola.

So, this dotted line, this is showing you the parabola. And here, this is your parabola. This is a circle, we have drawn at first, and, this one was your ellipse. On the other hand, if you extend this cone upwards, and take a section like this, take a section from here;so, as you cut it, so, you will see that, the section is getting generated here and similarly, on this side, a similar section will be generated. So, the lower section will, if you look into the side view, so, it will appear as this;this is the line here. So, it will appear in this way.

So, here again, this lines are the Directrix. The Directrix, we will define as the line, from which the distance of, the constant ratio of the distances are being maintained. So, if, say, these are the focus here, for the hyperbola, so, again, the same rule applies that, the distance from here to here, this is r and if my Directrix is located, say somewhere, Directrix will be located, this is the center; this is not Directrix; this is the center line; we call it the center line; this is the center line and Directrix, we have, here, in this place and similarly, we will have Directrix on this side.

So, from this Directrix, the ratio of thistwo distances will be a constant. And, it may happen that, say around the earth, some satellite, or some object is moving in a hyperbolic orbit, which is being represented like this. So, this will be called, because the earth is present here, so, this will be called the occupied focus, and this will be called vacant focus. So, motion along this trajectory, if your heavenly body is here…

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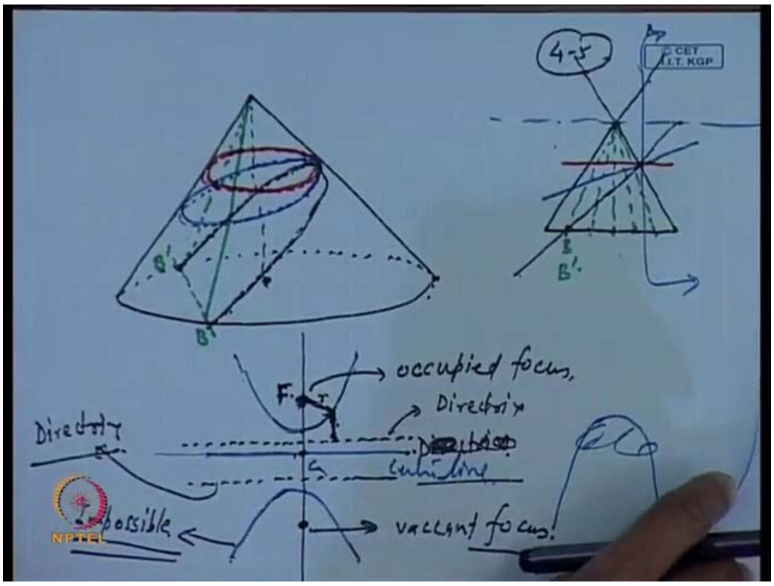

So, motion along this trajectory is not possible, because this kind of trajectory is possible, only under repulsive forces. While the gravitational forces, it is a attractive force, therefore, this trajectory will be followed, and this trajectory becomes impossible; but this is just a mirror image of this trajectory, in the center line; this is the center line. So, in this,this whole thing is reflected; here, this is your Directrix; this is also Directrix.

So, thus, we have seen that, how the hyperbola, parabola, ellipse and the circles are generated. Now, let us find out, what this l will be, for an ellipse. This distance, for an ellipse, we have shown as l. This ellipse equation, we can write as r equal to 1 by 1 plus e cos theta. So, this is your r, and the equation for r is given here, where theta is this angle. So, theta here, this is written as true anomaly. Now, using this, you can see that, this point, we,we can write this as, say A, and this point as B. So, this point will be called the pericenter and this, we can call as the apocenter, or a, this also called as apoapsis, apoapsis. So, the distance from here to here, this distance is written as r p and distance from this point to this point, this is written as r a.

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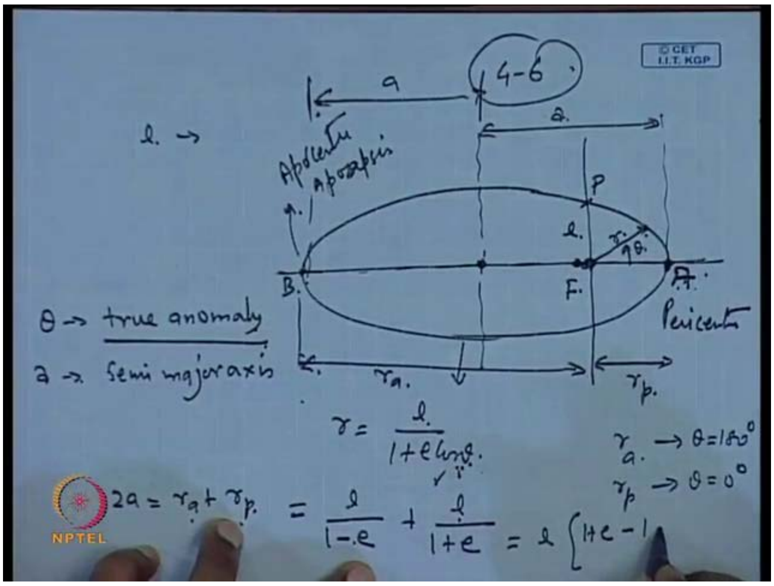

This is the center line here; this is also a center line, for this ellipse. The distance from here to here, let us say, this distance, we write as a. This a, we say this as the semi major axis. Therefore, you can see that, this distance also, this will be, from here to here, this distance will be also a. So, you can see that, 2 a will be equal to r a plus r p. Now, r a, this distance r F, from here to here, this will correspond to theta equal to 180 degree. So, ra corresponds to theta equal to 180 degree and r p corresponds to theta equals to 0 degree. Now, put in this r a. So, r a will be l by 1 minus e, because cos theta is equal to 180 degree and r p correspondingly, it will be 1 by 1 plus e, because theta equal to now, 0 degree.

So, you can write this as, l times 1 plus e minus 1 plus e divided by 1 minus e square. So, this gives you,2a equal to l times 2 eby 1 minus e square; this 2, 2, you can cancel out, and you can write, l equal to a times 1 minus e square. So, your semilatus rectum,this becomes l equal to a times 1 minus e square. Now, next, we can look into the equations that we have developed earlier. So, what we did, we developed the equation for r equal to l by 1 plus e cos theta as the conic section equation; in originality, we got this as l is equal to r plus r e cos theta. So, this, often it is written as, r dot e, or e dot r, where e, this is an eccentricity vector.

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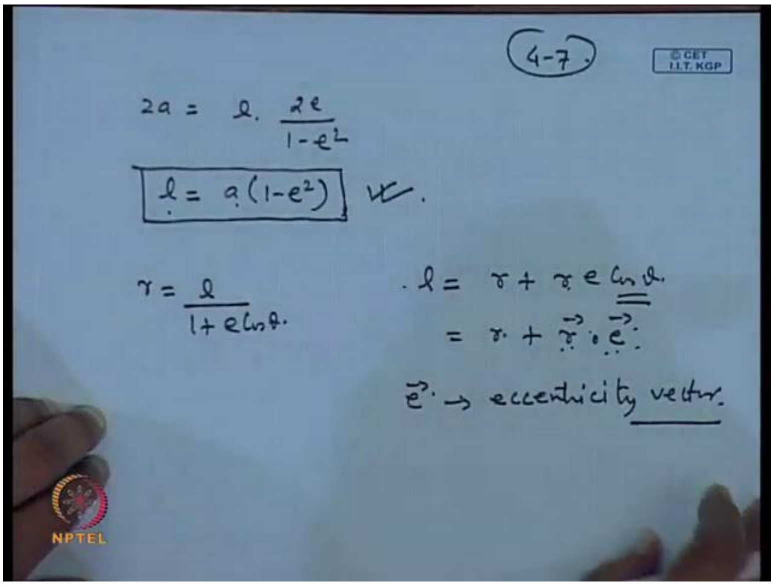

So, if you break it…so, the angle between the r vector and e vector, this angle gives you cos theta. So, your eccentricity vector, this is directed in this direction. And later on, we will see that, this eccentricity vector, it appears in the equation of motion of the heavenly bodies. So, automatically, once we are dealing with the equation of motion of the heavenly bodies, so, this terms will appear. So, another part, you can see here that, r cos theta is nothing, but projection of this r on this axis. So, if we take the projection of this, so, this distance is rcos theta. And, if you are measuring distances from this point, say, the distance you are measuring from here to here as x, so, this distance will become r cos theta equal to x;x is equal to rcos theta. While on the other hand, if you are measuring the distances from this point, if you are taking this as the origin, so, you will have a different set of equation in x and y. So, this coordinate, we can write here this, coordinate of this point as x, y. Before we do derive anything, let us find out how much this distance will be, from here to here.

So, this distance, here to here, you can calculate. This distance is known to you; this is r p and therefore, this distance can be calculated. So, r perigee distance, which is the distance F a, we have written as l by 1 plus e cos theta equal to 0 cos 0 degree. So, this resulted in 1 by 1 plus e, l by 1 plus e, and just now, we have proved that, l equal to a times 1 minus e square. So, this results in, this implies r p will be equal to a times 1 minus e. So, this is the distance from here to here, which is equal to r p, you have written. So, this is 1 minus e. And therefore, this distance, we can calculate from this distance; it is easy to work now. So, the total distance is here, a,so, we subtract from this distance. So, for your distance, let us say, this is the center,C here. So, this implies,CF will be equal to C A minus F A and C A is nothing, but your equal to a and FA is nothing, but equal to r p, which you have written as a times 1 minus e.

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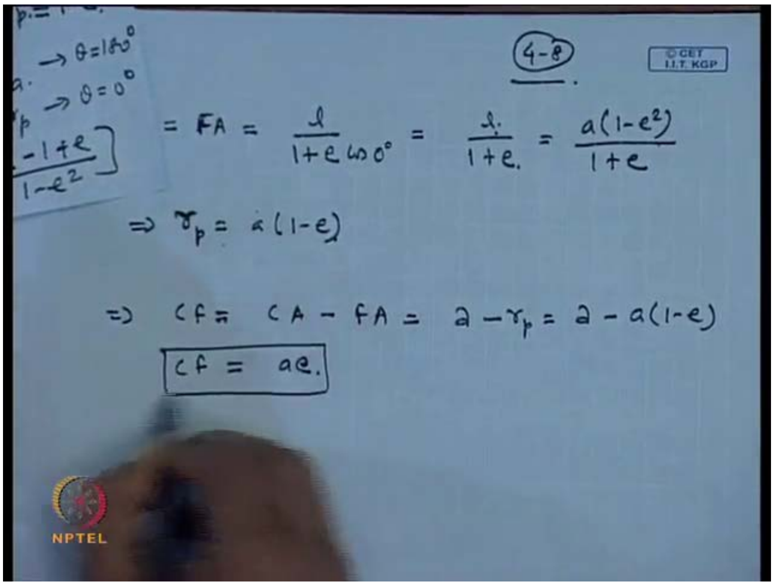

So, this gives you, CF is equal to a e. So, this distance is your a e. And, the distance from here to here, this we have written as 1 minus e. So, if you want to describe the trajectory for this equation of this ellipse with this as the origin, so, it is easy doing. You need to write here, supposing that, the origin is at the focus. So, we take a situation. Let us do this, we find the, with the origin at the center, and then, find the equation of the ellipse, rather than working out that part. So, origin lying at center C; find the equation of ellipse. Now, you can refer the points, x from this place. So, here, we have to draw a separate figure for this;in the same figure, they get confused.

So, distance from here to here, this is x; this distance is y; this distance is theta; this is r. So, obviously, r square will be this distance here, given as a e. So, r square you can write as, x minus a e square plus y square. From this distance, we have subtracted this ae and for this y square, this equal to this r square. Now, after certain substitution and l you, or you know that, r can be written as l by 1 plus e cos theta.

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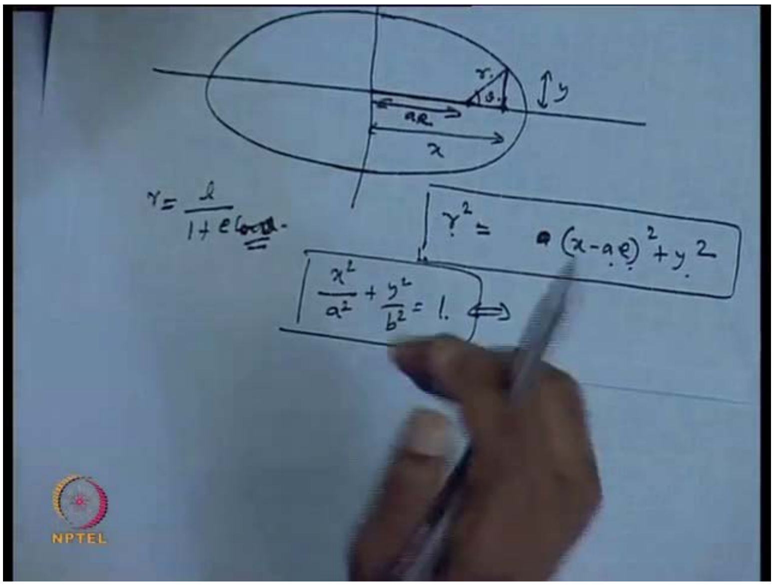

So, you need to replace this r by l by 1 plus e cos theta and this cos theta also, you later on, you need to eliminate. So, if you eliminate, your equation of ellipse will result, so, which you can write as,x bar, x square by a square plus y square by b square; this is equal to 1. This is what the equation will result. So, we do not have, there is only one lecture is devoted for this particular conic section. So, we are not going into details of this and where b, we have already seen that, this can be written as a times 1 minus e square under root.

Now, we can go into the next stage, where we have equations for, we have already done. Now, we can look into the parabola. So, ellipse, already it is over. So, let us look into the parabola, how does it look like. If this is the directrix, just like the ellipse case, the, we have the focus here. Any point lying on this parabola, say this point is B, this distance is r; this angle is theta. As earlier we have shown that, from the directrix, and from a fixed point, the distance will remain a constant. So, that is remain going to remain a constant, but here, in this case, e turns out to be 1. So, this you can write as C here. So, what does it imply that,once e equal to 1, the ratio of this and this equal to 1 means, both these distances are constant.

So, once you come to this point, and the distance from here to here, then, we can calculate and write it. So, again, looking into the ellipse part, in the ellipse part, we have seen that, the distance from here to here, this we can write as l and this distance was then written as l by e, and obviously, this distance was written as r by e. So, the same thing applies here; only thing there, here l, e equal to 1 and therefore, this distance and this distance equal; this distance and this distance equal. Now, if this distance is l, and accordingly, the distance, the perigee position for here, in this case, this distance you can write as r perigee equal to l by 1 plus e. Now, in this case, this distance, you, as you see that, if we take this point…So, this point and this point, this also maintain the same ratio. Therefore, this distance will be equal to l by 2, and this distance is also going to be equal to l by 2.

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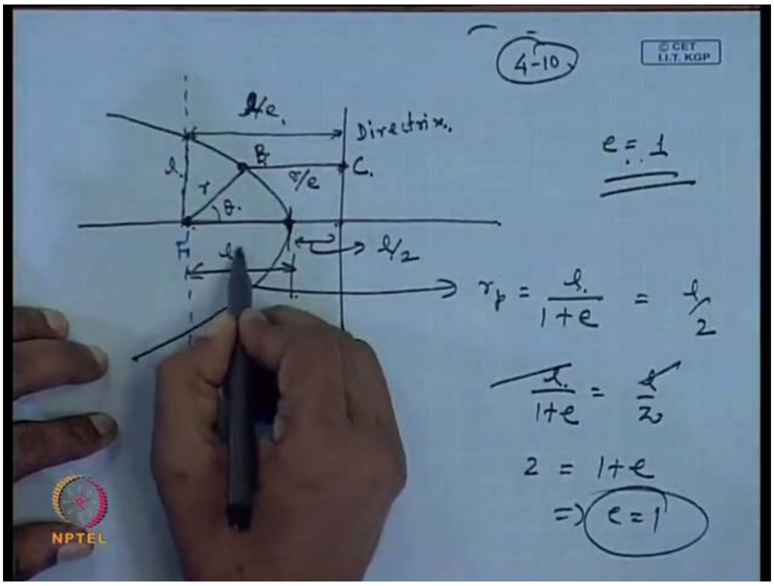

So, can we do anything from this place? Let us look into this, whether this satisfies this relationship or not. So, l by 1 plus e is equal to l, l by 2. So, this cancels out, and we see 2 equal to 1 plus e and this implies, e is equal to 1. So, the things are satisfied; is this distance here, we have shown this is the larger distance it is appearing, but really, this one is equal to this one. So, this is l by 2 and this distance is also l by 2.

While in the case of hyperbola, so, hyperbola, we had the center line here. So, the lines which are shown here, they are asymptotes, asymptotes to hyperbola. Say, this is the focus. So, on the right hand side also, you can make the same kind of the figure. As usual, this is point F and this is point F star. So, this is vacant focus and this, we call as the occupied focus. The distance from hereto here, as usual, this distance will be written as r p, which is the distance of the pericenter. You can take any point on this trajectory and this angle we can show as theta. So, this is your theta. Somewhere here, our directrix lies. So, this is your directrix and from this directrix, as usual, you will find that, the ratio of the distances from any point on this trajectory to this line and from this focus, it will maintain a constant ratio. So, if we write here, this point, we take it as B and this point as C. So, already we have seen that, and this is the distance r. So, r by BC, this will be equal to e and this implies, BC equal to r by e. For this case, e is greater than 1;for parabola e was equal to 1, and ellipse, e was less than 1. So, already we have proved that, l by e equal to r by e plus x, or this we have written as r cos theta. So, this was the first property that we proved. This is also called the focus directrix property.

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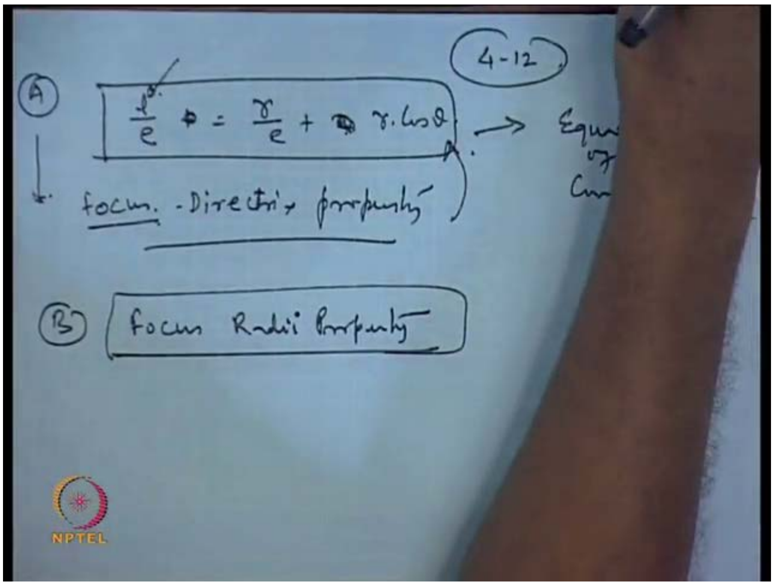

So, right now, the relationship that we were proving that, the distance from this directrix and this one, this is remaining a constant. And therefore, we wrote in terms, the whole thing in terms of this semilatus rectum. So, this relationship, particularly this is called the focus directrix property. So, from here, we have derived this and this is the equation of the conic section. This is the equation of conic section, but the property that we have mentioned, this property, we will call as the Focus Directrix property. Then, the, before we do any other thing, the next property, what we call as Focus Radii property, this we are going to do little later, depending on the availability of the time. In the meanwhile, we will try to find the distance b; b we have not proved.

b, we have still not proved. So, will first do for the b, which is the semi minor axis. So, given an ellipse…Few more things here,which are remaining. So, let us cover this first. So, in this, few more distances that we can list; the distance from here to here, this is a; and, the distance from here to here, this is ae. So, this is a center line and center line to the focus; just you can see the, in the case of ellipse, we measured the distance from the center line to the focus, and that distance turned out to be a,a e and the same distance is also appearing here, in this place.

Now, the distance from here to here, this is the asymptote… So, if we draw a vertical from perigee position…So, again, you look into this distance, this is the center line, here. So, from the center line, the distance to the perigee position, as in the case of an ellipse…In the case of ellipse, this is the center line and from this is the perigee, here. So, this distance is nothing, but a, and also to the apogee, this distance is also a. So, here the apogee, obviously, in this case, it is going to infinity;this trajectories are going to infinity. So, apogee, we do not worry about; only the perigee position is available here.

So, the distance from the center line, this is equal to a; it is a same as in the case of, equivalent to a distance from the center line for the ellipse to the perigee position. So, this two are resolved here. Now, if this is the asymptote, so, the asymptote, if you draw a vertical from the, this perigee position, this is your P. So, from here, if you draw the vertical and the line where it cuts, so, this distance from here to here, we write this as b. And, distance between the Directrix and the center line, this is a by e. Now, we can go into the ellipse, and few more things are still remaining; but we do not have much time left for this particular lecture.

Later on, we will come to the hyperbola again. Let us first finish the ellipse case. So, what we are interested in finding, this distance b. This distance, we have written as a e; this is our r and this angle, we have shown as theta. So, this angle is 180 minus theta. And, we need to prove that, b equal to a times 1 minus e square under root. So, this r, we can write as l by 1 plus e cos theta. b square can be written as r square minus a square e square; and r is nothing, but l by 1 plus e cos theta from this place; this whole square minusa square e square.

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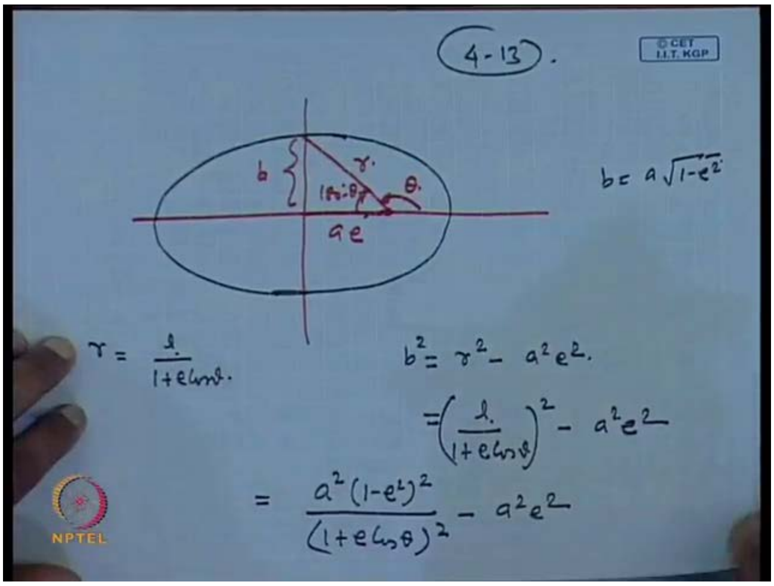

Replacing l in terms of a times 1 minus e square, so, this becomes 1 minus e square whole square divided by 1 plus e cos theta whole square minus a square e square. Also, we considered r cos 180 minus theta. rcos 180 minus theta, this is nothing, but equal to a e. So, this implies, minus r cos theta equal to a e, and therefore, cos theta, we can write as minus a e by r. We need to eliminate this cos theta from here, and therefore, the relation that you will be getting, it will be in terms of b,a and e.

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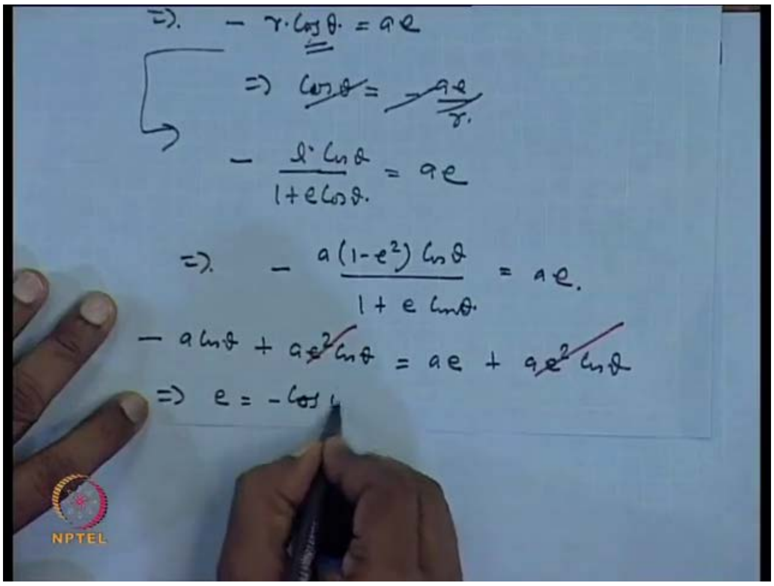

So, this equations, let us write this as, r is equal to l by 1 plus e cos theta. This gets multiplied by now, cos theta. So, this becomes equal to a e. Therefore, this implies, putting l equal to a times 1 minus e square…ecos theta is equal to a e. And, if we solve it further,so, this will give us, minus a cos theta plus a times e square cos theta equal to a e plus a times e square cos theta. So, this two terms will cancel out, leaving you with e is equal to minus cos theta.

Now, we can utilize this relationship here, in this place. So, cos theta, we are going to replace by minus e. So, this implies b square is equal to a square times 1 minus e square whole square, divided by 1 minus e square. This one cancels out and this implies,b equal to a times 1 minus e square. So, this is what we wanted to prove. So, this is your semi minor axis, and this is your semi major axis. For an ellipse, earlier, as we have stated that, the ellipse equation can be written in this way, if origin is taken at center of the ellipse. So, if we take the origin at the center of the ellipse...

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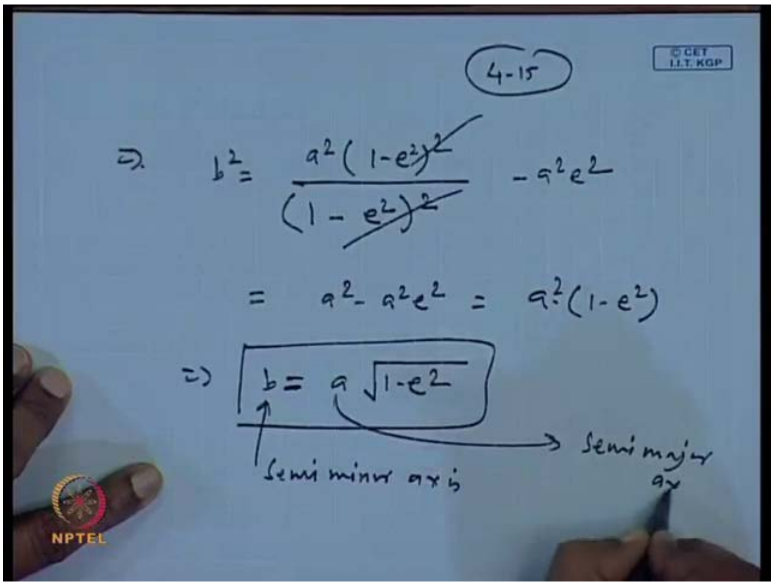

Now, in this equation, we can write here, b, we can write as a times 1 minus e square. So, this whole square, it will come here, is equal to 1 and you can see for this place, x square plus y square divided by 1 minus e square; then, it becomes equal to a square. In general, if your e is, what we have seen that, if e is less than 1, greater than 0, so, this constitutes the case of an ellipse.

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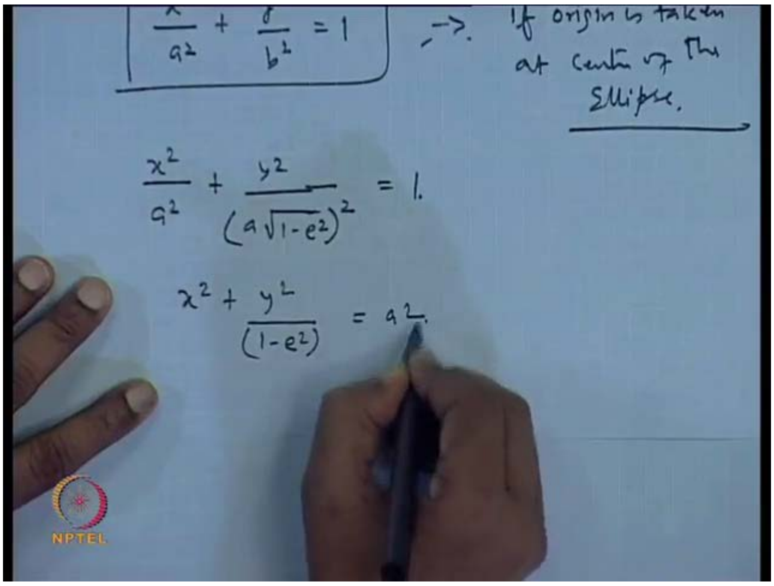

If e becomes greater than 1, so, this equation, rather than writing in this way, we will write this as, y square by e square minus 1 equal to a square, where e is greater than 1.

So, nothing we have done here, just taking minus sign outside, and taken the, here in this place.

So, now this quantity is positive, and this becomes the equation of hyperbola. So, here, what we want to show that, there are few very interesting properties. So, taking the equation to be x square by a square plus, or let us write, x square plus y square by 1 minus e square equal to a square, for an ellipse. So, say, this is the major axis and from here to here, this is your semi major axis. So, holding this a as constant, hold this as constant and change e. So, if you do that, then, what will happen? Keeping a as constant, change e. So, the relationship for b, you know that, we have written as 1 minus e square.

So, as you are increasing e, and it is, e is going towards, proceeding towards 1, say, approaching 1,sSo, you can see that, b will approach 0; and in the limit e tends to 1, we will see that, the b has reduced. So, this is constituting your major axis, while the b vanishes; the same, the minor axis vanishes; only the major axis remains. So, here, an ellipse, in this case,it degenerates into a, into a line. And, in this kind of motion, so, the, your focus, which are lying here in this place, these are the two focuses which are lying here. So, as you increase the value of e, so, the eccentricity is increasing. So, you remember, this was your r perigee position from this focus; this distance, you took as r p, and this distance r p, you wrote as a times 1 minus e.

So, as you are making e approach 1, therefore, r p is vanishing; means, the focus is moving towards the, this perigee position, which we,we are writing here as P; if you write this as P, so, this focus is moving here.

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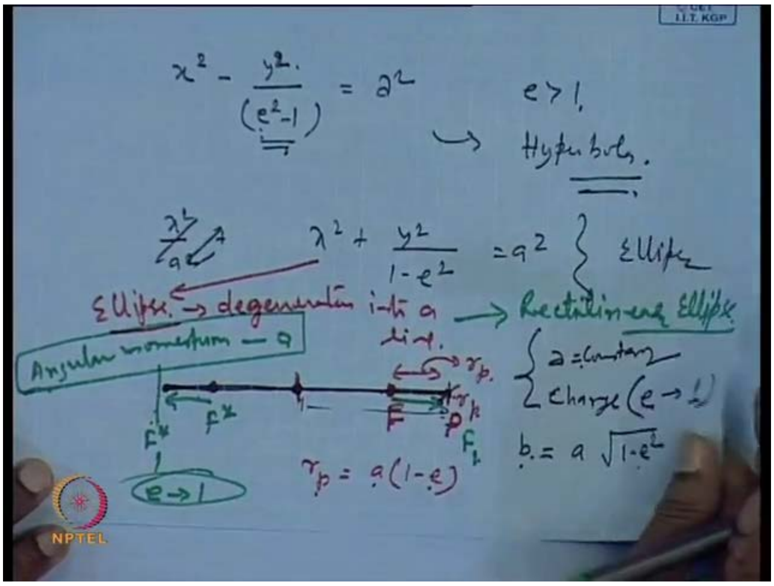

So, in the moved position, this is moving from here to here; and, this will move from here to this position; this focus F star, it will move and come here in, next this is the position and this, you have the focus position here, in the limit e tends to 1. So, in this case of, in this kind of motion, your angular momentum is 0 and this is called a rectilinear ellipse. So, this case of, this is called a rectilinear. So, this kind of motion will be defined by where the angular momentum is 0.

Now, another case you can take, let e tends to infinity. So, if e tends to infinity, you can see here, in this place, that this quantity will vanish, as e tends to infinity. Therefore, you can write x square is equal a square and this implies, x is equal to plus minus a. So, this is again the equation of lines. So, here, what the situation is, this is the center line, and, you had the hyperbola on the two sides of this. So, as you are changing the eccentricity of the, making the eccentricity tend to infinity, so, this hyperbola, constantly it will, it is becoming more eccentric, and later on, it is getting reduced to a straight line; and here also, this will gets reduced to a straight line. So, this eccentricity is increasing and finally, it is, for e equal to infinity, this is equal to infinity, this turns into a straight line.

Now, you can look into the distances, what we have written here, x is equal to plus minus a; this is what your distance is. So, here, your hyperbola, which was shown here, this is getting generated, degenerating into a line and from the center line, this distance is shown here to be a; and, the same distance is here; you can see here, this is the, distance a is manifesting here, in this place. So, from the center line, this equation for the, if you are writing the equation for the hyperbola, so, obviously, in the case of hyperbola, you will write it as, x square minus y square e square minus 1 times a square is equal to 1; and, this is being referred to this center, this equation. And therefore, the distances are measured as x and y, positive and negative in the either direction, depending on how you choose the coordinate x. So, either way, you can say, this distance as negative, or either you can take this distance as negative.

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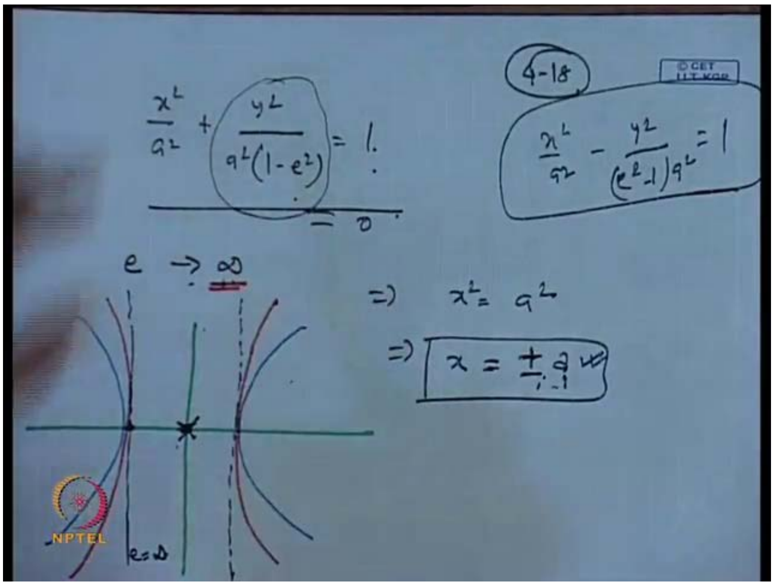

So, this distance, it tells from here that, this is the distance a, which also we have written here, in this place, and this we interpreted, again, from the polar equation, we interpreted here; while here, the same thing, we are interpreting from the Cartesian equation. Now, for hyperbola, once e equal to greater than 1,r equal to l by 1 plus e cos theta, and if we choose cos theta equal to minus 1 by e,so, then, r becomes equal to infinity. So, cos theta equal to minus 1 by e. So, here, theta, we can write as cos inverse minus 1 by e, and this defines your asymptote.

So, this is that asymptote being defined. So, theta, we are measuring from this place. So, you can consider that, the, if we draw a line, as r tends to infinity, so, you will see that, you can draw a line parallel to this line.

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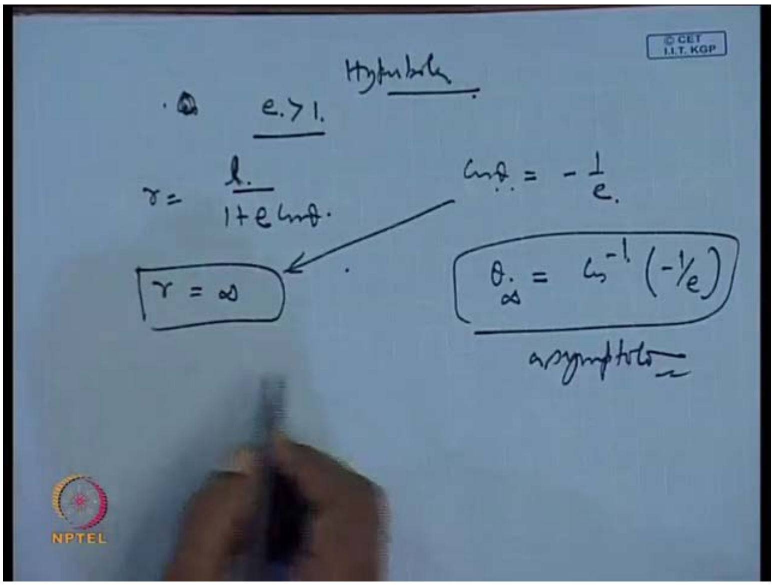

So, this is defining here; from here to here, this is your theta infinity. And, this will be parallel to this asymptote. So, this is basically describing the asymptote equation. And, if you write this angle as beta, so, beta will be equal 180 minus theta infinity. And, using this, similarly, using this, you can define many interesting properties. So, our time is getting over. So, we stop here, in this place, but it is a possible to discuss something more about this; but we remain content with this, for this particular lecture. Thank you very much.