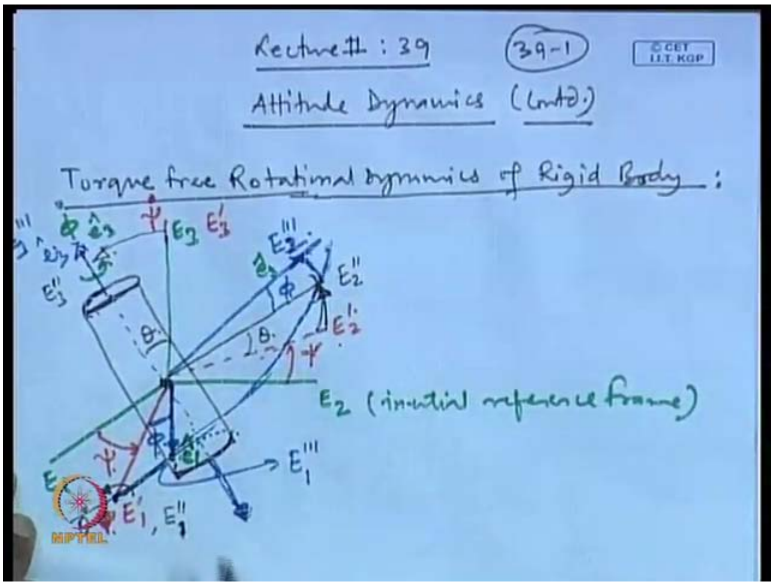
Space Flight Mechanics

Prof. M. Sinha
Department Of Aerospace Engineering
Indian Institute Of Technology, Kharagpur

Model No. 01
Lecture No. 39
Attitude Dynamics (Contd.)

We have been discussing about the attitude dynamics. So, in that context we are stared with the torque free rotation of rigid body.

(Refer Slide Time: 00:29)



So, let us consider the case where a cylinder is given and this E 1, E 2 and E 3the inertial reference.

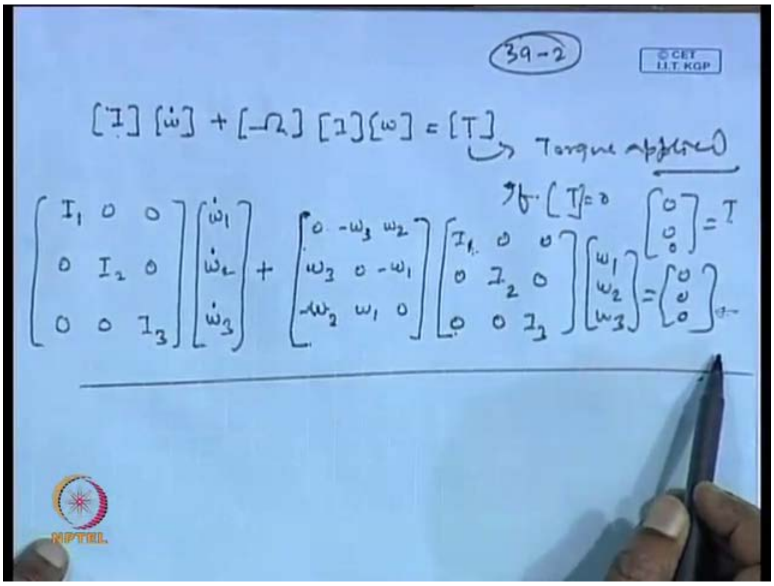
So, as we have done the in the last class this is the inertial reference frame and then the body axis is to be oriented with respect to this. So, the fine last time we saw that this can be oriented by first rotating this body can be oriented with respect to this inertial reference frame in the way it has been write now put by first rotating y angle psi. So, psidot is in this direction and then this also rotate from this place to this place. So, this we can show by dotted line, this is the angle psi, this is your E 2 prime and E 3 prime is here in this place itself then we rotated about this axis by angle theta.

The next rotation we are giving this is about this axis itself and this n rotation is by theta. So, theta dot will be in this place and this will rotate from this place to it will rotate something like this by angle theta this is coming out of the plane. So, this will also move. So, this is moving in one plane, E 2 prime and E 3 prime it will move out and it will go from here to here. So, this is your angle from here to here this is your theta.Then, the final rotation we give about this itself phi angle phi. So, phi dot is in this direction and once we are rotating by angle phi. So, we have to constitute one plane now. So, let us say the plane we indicate.

So, this is this line is showing one plane in which this was your E 1 prime. So, in this plane this point will move from this place to somewhere else. So, here we can name this point as E 2 double prime and here we will have itself E 2, E 1 double prime because the rotation is given in this place itself.And E 3 double prime can be indicated here in this place now, once we next rotate about this axis we are giving a rotation like this. So, if you rotate by phi so, it will this line will move from somewhere here to let us say it is moving to this place.This is the angle phi and this will move away from this place to here and this we can show by angle phi.

And the final values are here in this the notations are E 3 triple prime and E 3 triple prime and this E 2 triple prime, E 3 triple prime and this point will designate as E 1 triple prime.So, the notation that we have used it is a very important so, this is our final body orientation. So, this is basically your E 1 cap this is your E 2 cap and finally, in this direction itself you are getting E 3 cap. So, we are using now a body of revolution to ensure that the axis that we are choosing those are the axis of thin, those are the principal axes. So that the half diagonal terms will be 0 in the equation of motion.

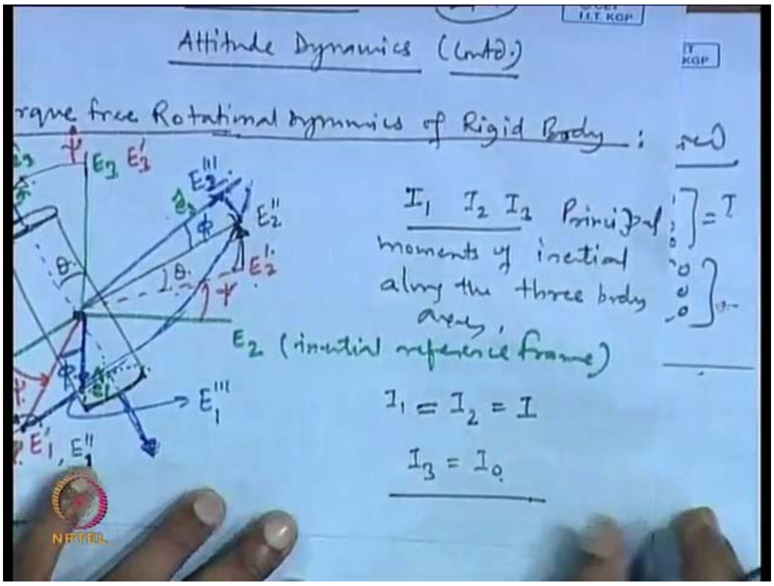
(Refer Slide Time: 06:31)



So, finally, for this body the same Euler equation we can use which we have developed earlier using Eulers dynamical equation. So, we wrote the equation in this way, this is the inertia matrix I times omega dot plus and we use the term capital omega and this we wrote as the T this is the torque applied, if T equal to 0 means all the elements in this are 0. So therefore, writing in an expanded notation I 1, I 2, I 3.

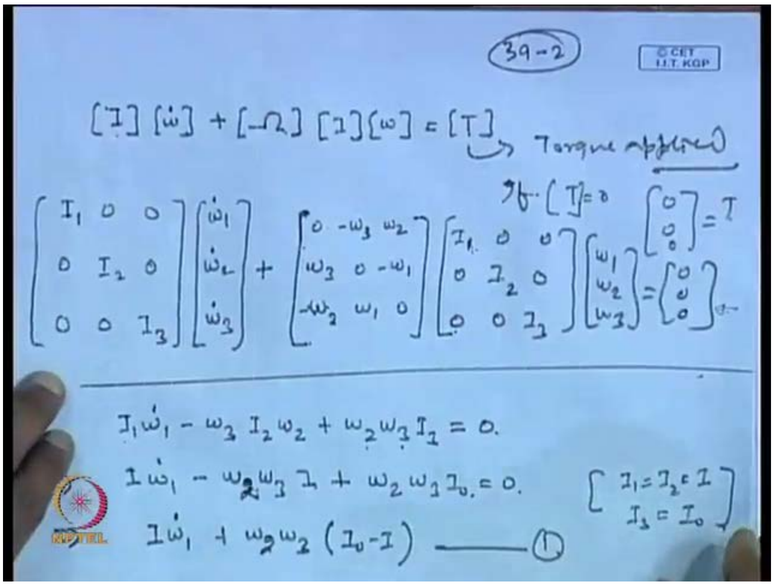
This as I 2, I 3, 0, 0 and finally here omega 1, omega 2 omega 3. So, this is equal to 0 0 0. So, this is our Eulers dynamical equation in which the torque applied is set to 0 because now we are considering the torque free rotation of a rigid body. So, now for the case of a cylinder here we are taking the I 1 in this so, final body axisis this one E 1 cap,E 2 cap and E 3 cap.

(Refer Slide Time: 09:10)



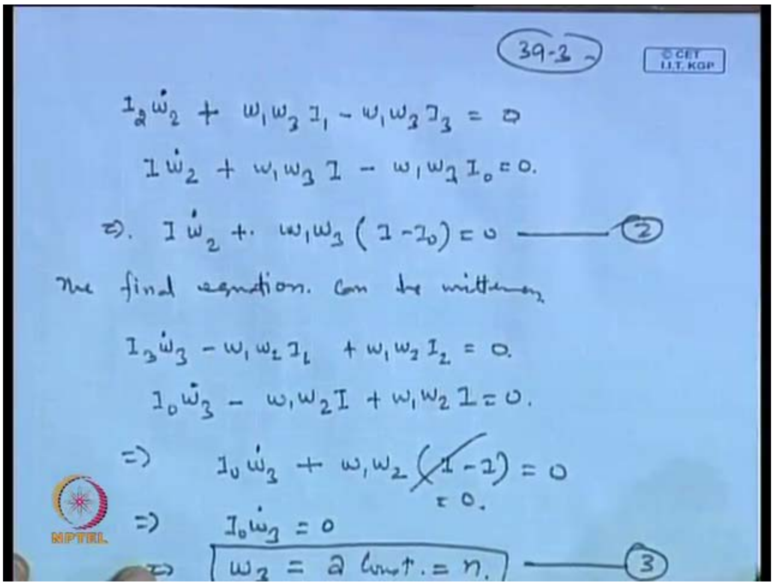
And the principal movement of inertia axis, these are I 1, I 2 and I 3 these are the principal movement of inertiaalong the three body axis. So, I 1 and I 2 because they are equal to each other, this we set as I and I 3 we can write as I 0.So, with this notation from here this we can break it down and write separate equations.

(Refer Slide Time: 10:04)



So, the first one will give I 1 times omega 1 dot and from here if you break it out. So, this will give you omega 3, times I 2, times omega 2, plus omega 2, times I 3, times omega 2, times omega 3, times I 3 this is equal to 0. So, we will put here I 1 and I 2 we are already setting it as I. So, I times omega 1 dot omega 2, times omega 3. So, omega 2, times omega 3, times I plus omega 2, times omega 3, times I 0 equal to 0.We are using the fab dot we have set I 1, I 2 equal to I and I 3 equal to I 0.And little bit rearrangement we can do here omega 2, omega 3, I 0 minus I and this we let us write this as the equation number 1.

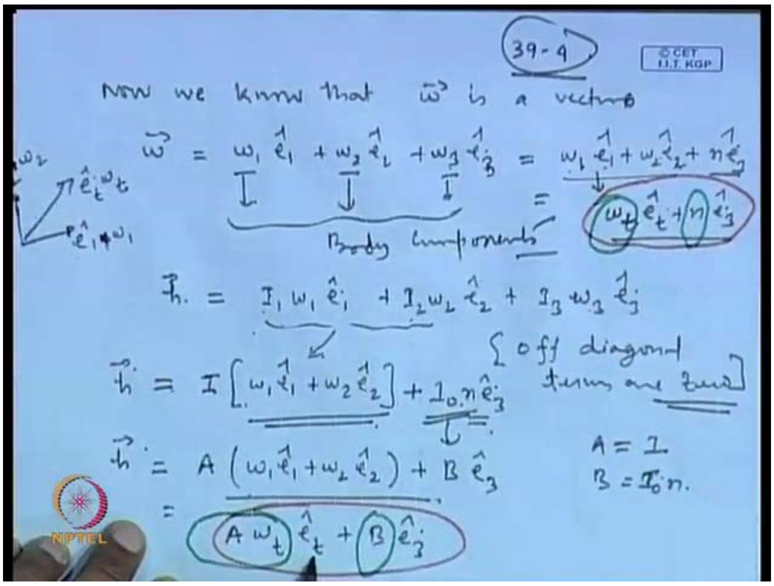
(Refer Slide Time: 11:28)



Similarly, the next equation by expanding this we can write the next equation so, total we are getting 3 equations from here. So, the next equation you can write as multiply the same this I 2, timesomega 2 dot, these elements are 0. So, they will be 0. So, I 2 times omega 2 dot and then the rest of the terms it will appear as omega 1, times omega 3, times I 1 minus I 3, this is equal to 0 and I 2 and I 1 we need to replace by I. So, this is omega 2 dot plus omega 1, omega 3. So, by rearrangement we can write this is our equation number 2.Next we take the third equation. So, third equation similarly, we can expand and write it here.The final equation I 3 times omega 3 dot.

Putting I 3 equal to I 0 omega 3 dot minus omega 1, omega 2, I 1 equal to I.I 2 is also equal to I. So, this implies I 0, times omega 3 dot plus. So, here you see this is I minus I equal to 0. So, this term becomes equal to 0. So, what we get from this place I 0 times omega 3 dot equal to 0 and this implies omega 3, this is equal to a constant and let us say this constant we denote by n. So, this is my equation number 3.

(Refer Slide Time: 14:18)

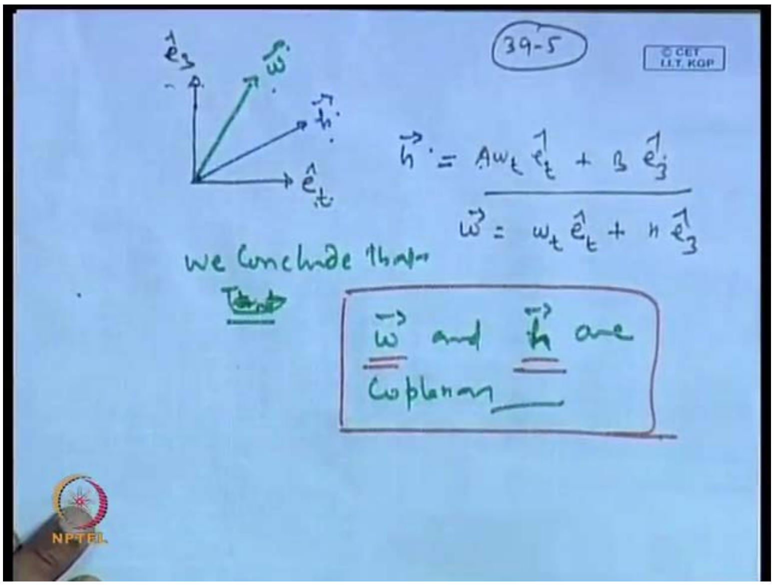


Now, we know that omega is a vector, that is omega equal to omega 1 e 1 cap plus omega 2 e 2 cap, omega 3 times e 3 cap. So, these are the body components the components of omega along the body axis. And what we are getting that omega 3 equal to omega 3 is a constant. So, we will have omega 1 e 1 cap, omega 2 e 2 cap plus n e 3 cap.Now, the angular momentum we can write as I 1 omega 1 e 1 plus I 2 omega 2 e 2 plus I 3 omega 3 e 3.In the options of this we have written for the off diagonal terms are zero. So therefore, what we get I 1 and I 2 are equal. So, we can take it out side and put here as I this 2 terms can be brought here. So, omega 1 e 1 plus omega 2 e 2 cap plus I 3 is I 0 times n e 3 cap.

So, now look into this equation carefully what we have done here.In this h can we write this as say A times omega 1 e 1 cap plus omega 2 e 2 cap and plus this quantity we write as B, times omega 3 cap where A equal to I and B equal to I 0 n.And obviously, here the something we have done in this place alsoand let us indicate this as a vector omega t times e t cap because these are the 2 vectors along the omega 1 e 1 is a vector, omega this e 2 is a vector. So, we have suppose e 1 is a vector and this is e 2 is a vector. So, this is multiplied by omega 1 and if you multiplied this by omega 2, these are not the cross product here. So, the resulting vector it will be in the same plane as in the plane e 1 and e 2 are there.

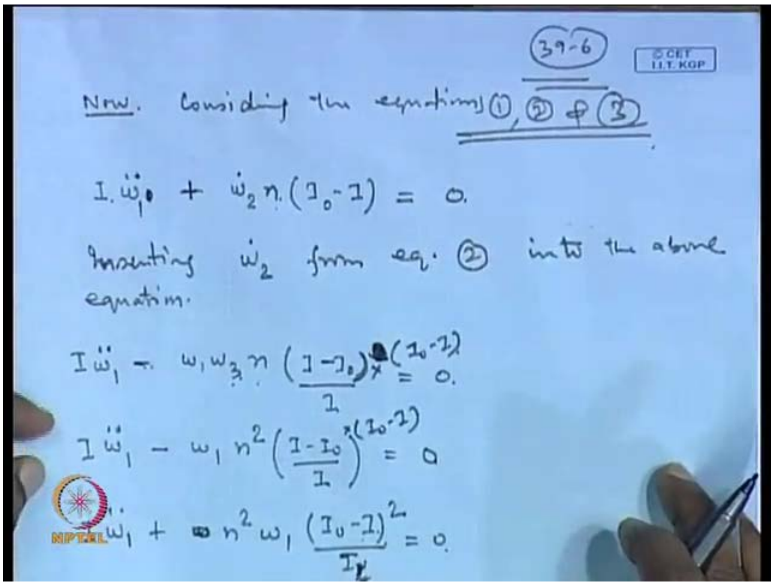
So, this vector we can write as e t cap times say the omega t is the magnitude of this vector and e t indicates its direction so, this becomes n times e 3 cap.So, the same way here we can write as this A times omega t times e t cap plus b times e 3 cap.Now, look into this equation and this equation, in this two equations you have this part is a simply a scalar, here this is a scalarand both are multiplied by e t cap and e 3 cap e t cap e 3 cap.

(Refer Slide Time: 18:56)



So, what does say it simply implies that you have two vectors, e t cap and e 3 cap, they are orthogonal vectors. So, if you multiply some quantity like the A omega t and times e t cap plus Btimes e 3 cap. So, it will constitute a vector which will be appearing as let us say this vector we write finally, this is basically we are writing as h, this is your h vector so, this is your h vector here.And the omega vector you have indicated as omega as omega t times e t cap plus n times e 3 cap so, the omega vector it is it may appear like this.What is the important thing here?This vector and this vector they are coplanar.This constitutes one plane and a linear combination of these two vectors, it will be lying in the plane constituted by e t and e 3 therefore, omega and h both are coplanar. So, what we conclude?The conclusion is we conclude that omega and h are coplanar this is an important conclusion which we are going to use later on omega and h they are coplanar.

(Refer Slide Time: 21:04)

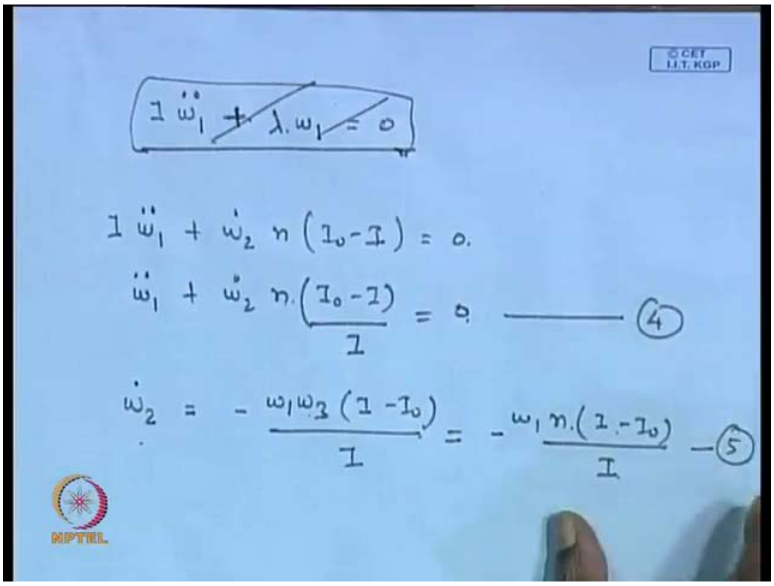


Next we take the equation that we developed equation number 1, equation number 2 and equation number 3 so, now considering the equations 1, 2 and 3. So, what we do? We will differentiate this equation first. So, if we differentiate this equation number 1. So, we will have I times omega 1 double dot plus omega 2 dot, omega 3 remember this is constant and we are writing this as n. So, simply we will write here as n, omega 3 we are replacing by n. So, omega 2 dot times n and this equal to 0 we have missed earlier. So, we put a 0 in this place I 0 minus I this is equal to 0.

Next we take this equation number 2, in this we have the omega 2 dot available omega 2 dot is appearing so, omega 2 dot we can insert from equation number 2. So, inserting omega 2 dot from equation 2 into the above equation, I times omega 1 dot 1 double dot and omega 2 dot will be we need to replace this so, it will appear with a minus sign here. So, omega 2 dot will be omega 1 times omega 3 divided by I 3 times I minus I 0. So, this is omega 1 times omega 3 and naturally omega 3 will replace later on by n in 1 more a step. So, this is n times I minus I 0 divided by I and this quantity is equal to 0. So, here we carry out 1 more a step and write this as omega 1 times n square I minus I 0 divided by I equal to 0.

If we do little more manipulation in this by re writing in this way n square times omega 1, I 0 minus I, we have taking the negative sign out from this place by taking negative out of the bracket by I this equal to 0.One more thing here we have I minus I 0 so, I 0 minus I was already present so, we need to multiply here one more term.So, omega 1, omega 3 times n that we replaced from this place and I minus I 0 we need to bring in this place. So, we will put here as I minus I 0 times I 0 minus I, this is multiplied by I 0 minus I. So, while we are putting it in this format this term will become square.

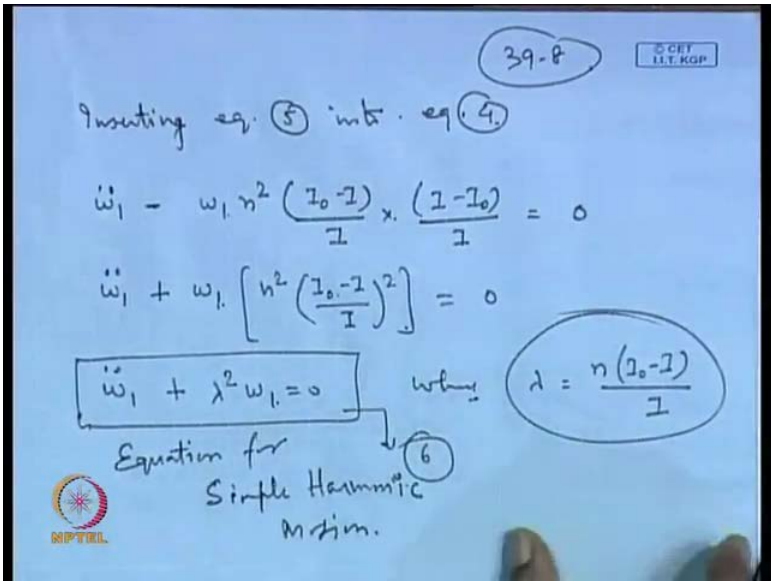
(Refer Slide Time: 25:56)



So, what we have done? That we have got this equation I times omega 1double dot plus we can write here the term n square times this quantity which is present here as lamda say we are putting lamda for this. So, lamda times omega 1 or let us make this as lamda square. I will carry out this step again once more this is I times omega 1 double dot plus omega 2 dot times n times I 0 minus I, what we wrote in the equation number 1?This is the equation number we have chosen. So, omega 1 double dot this becomes omega 2 dot n times I 0 minus I divided by I equal to 0, let us go step by a step because some of the terms we miss.

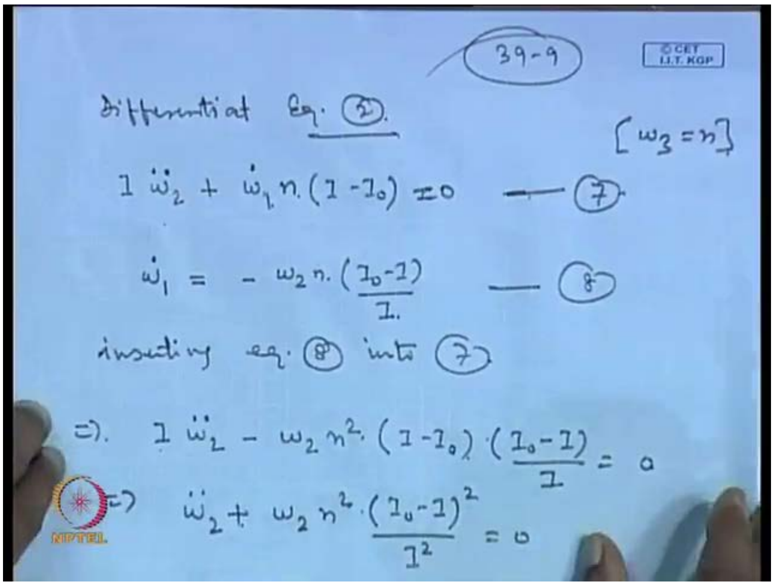
The next equation is we are using the omega 2 dot from this equation. So, omega 2 dot this is minus omega 1, times omega 3, times I minus I 0 divided by I. So, here we can put omega 3 is equal to n in this step I minus I 0 divided by I.Now, we will use this equation and we will name this equation as equation number 4 and this as equation number 5. So, you can see that what we are working out so, we missed out the square term here. So, this is also coming with divided by I here so, ultimately you are getting the square term here in this place this is a square term. So, I am canceling out this page and what we have done is fresh derivation we are putting in and this is 39.

(Refer Slide Time: 28:54)



Now, inserting equation 5 into equation 4, equation 4 this gives us omega 1 double dot minus, omega 1 times n square, I 0 minus I, divided by I, times I minus I 0 divided by I equal to 0 which can be re arranged to give omega 1 double dot plus omega 1, times n square times, I 0minus I divided by I, square equal to 0.And compact notation we can write as lamda square times omega 1, this is equal to 0 where lamda is equal to n times, I 0 minus I divided by I.Now, this equation is correct and this we name as this equation as this is our 6th equation. So, you can recognize this is equation for simple harmonic motion.

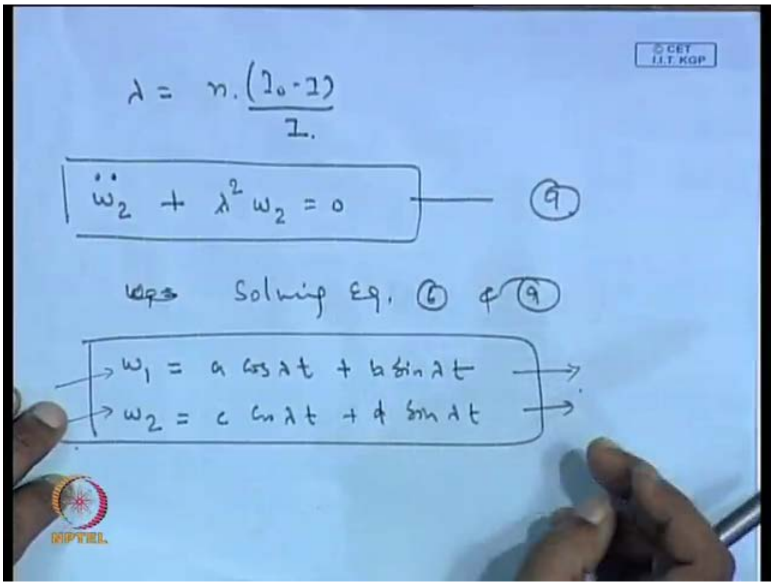
(Refer Slide Time: 30:40)



So, the way we proceeded, we got one equation we which is in the form of simple harmonic motion now we can take the equation number 2.The way we have done the differentiated equation number 1 sosimilarly, we differentiate equation number 2. So, this will give you I times omega 2 double dot plus omega 1 times, omega 1 dot times, omega 3, omega 3 is a constant this is a constant and therefore, we just put for this as n here.We can note it down omega 3 is equal to n, this is I minus I 0 this is equal to 0 now, we need to replace omega 1 dot in this equation. So, omega 1 dot we can replace from equation number 1.

So, in the equation number 1 this is the equation number 1 we have written earlier. So, omega 1 dot this can be written as minus omega 2 times n, I 0 minus I divided by I.And let us name this equation as this is equation number 7 and this is equation number 8.So, inserting equation 8 into 7, this will imply I times omega 2 dot minus omega 2 times n square I minus I 0, times I 0 minus I ,divided by I equal to 0.We bring this I on the right hand side. So, omega 2 double dot and the minus sign we can take out from this place this minus sign we can take it out and put it in the format where it will appear as I 0 minus I. So, this becomes a square divided by I square equal to 0.

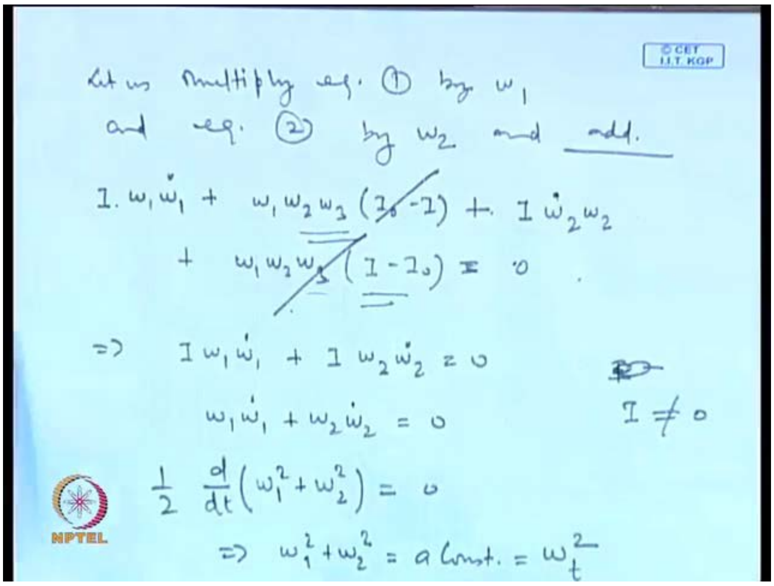
(Refer Slide Time: 33:33)



And the compactly in the same way because we have already defined lambda equal to n times I 0 minus I,divided by I. So, using that notation we can write this equation as omega 2 double dot plus lambda square times omega 2 equal to 0 and this is our equation number 9.So, we have got 2 harmonic equation now, this can be solved easily so, omega 1 can be written as solving equation 6 and 9.This gives us omega 1 is equal to a Cos lamda t plus b sign lamda t similarly, omega 2 we can write as c Cos lamda t plus d sign lamda t. So, it simply indicates that the body component the angular rates of the body component, they are nothingbut the they can be represented by this equations.

So, because this rates itself they constitute the simple harmonic motion. So, once we are representing in this format.Now, to get little more information's we need to do some more processing. So, we get back to our equation number 1 and 2. So, these are our equations number 1 and 2.

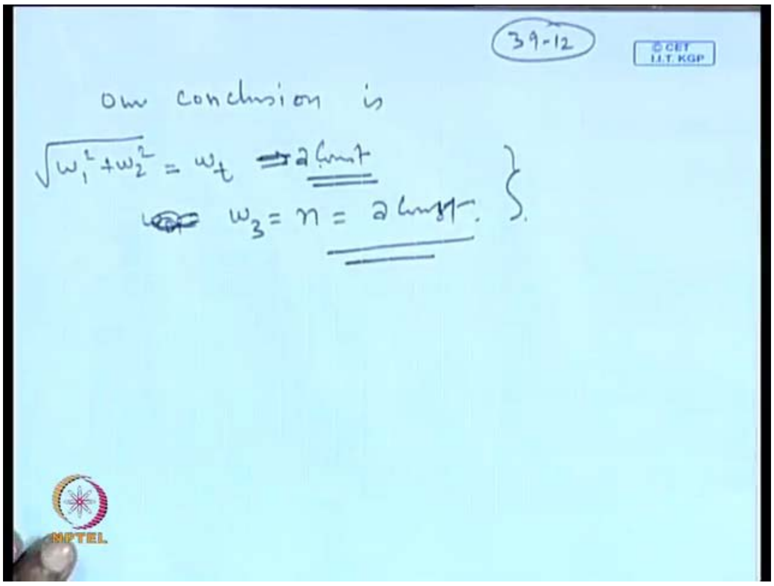
(Refer Slide Time: 35:39)



So, let us multiply equation 1 by omega 1 and equation 2 by omega 2 and add. So, that will give us this equation is to be multiplied by omega 2 and this equation is to be multiplied by omega 2.And the equation number 1 which is present here this we are multiplying by omega 1. So, this indicates I times, omega 1 times, omega 1 dot plus omega 1, omega 2, omega 3 times, I 0 minus I, plus now check the equation number 2. So, equation number 2 is here, we multiplied by multiplied by omega 2. So, I times omega 2 dot times, omega 2 plus omega 1, omega 2, omega 3, times I minus I 0, this is equal to 0.

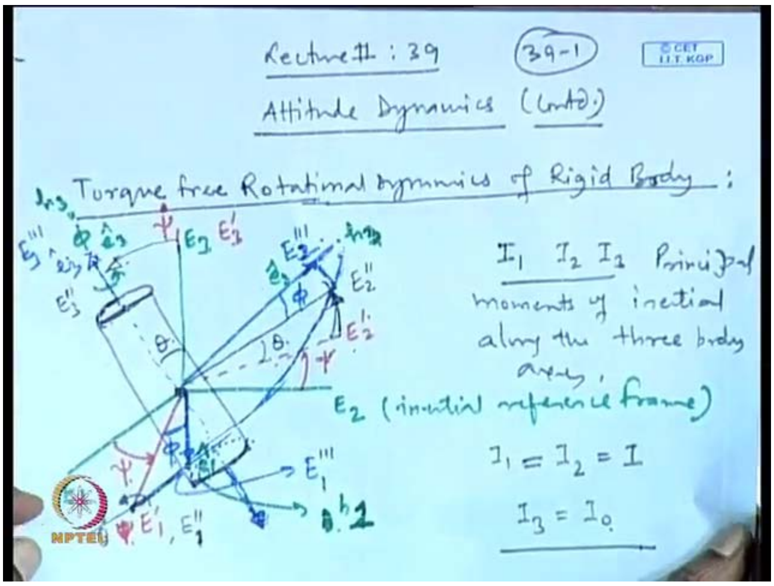
So, in this we can see that this term and this term it canceld out. So, this 2 terms will canceled out leaving us with I times omega 1, omega 1 dot plus I times omega 2 times, omega 2 dot, this equal to 0. We can take it outside because the quantity I is not equal to 0 therefore, we can write as omega 1 times, omega 1 dot.And this simply you can recognized that, we can write this as and this implies omega 1 square plus omega 2 square. This is a constant and this we can indicate as omega t square. So, what we have concluded till now.

(Refer Slide Time: 38:17)



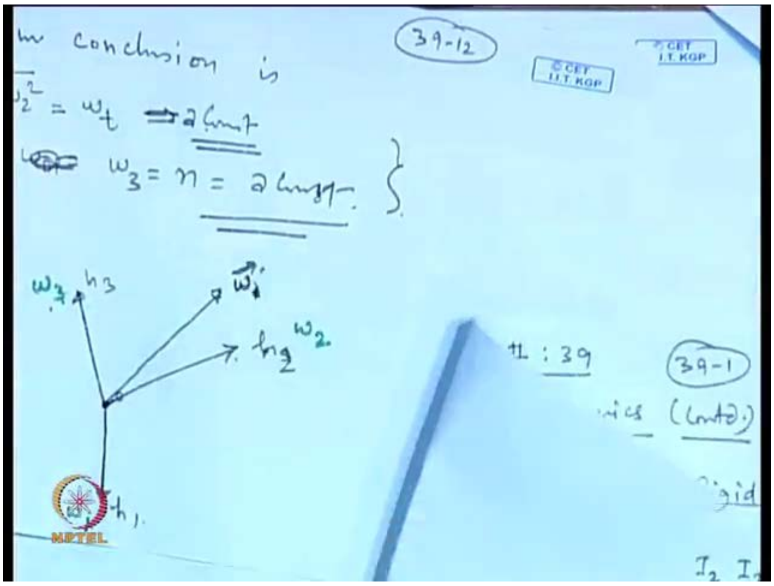
So, our conclusion is omega t, this is a constant means omega 1 square plus omega 2 square under root is equal to omega t, this is a constant.And also omega 3 which we are writing as n this is also a constant. So, these are two very important conclusions which will give us lot of insight into the equation of motion. So, we go back to our original figure. So, this body was rotating so, the body components we have already written here in this place.

(Refer Slide Time: 39:40)



So, the angular momentum vector it is a components can be taken in the body reference frame as h 3 and the another one is here in this place. So, here we indicate this is h 1, the body axisone and here in this direction we have h 3.

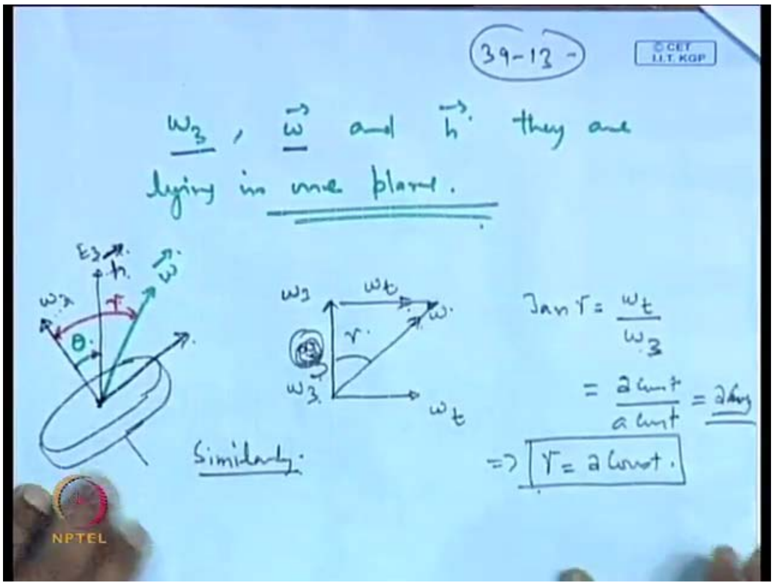
(Refer Slide Time: 40:18)



So, thus we see that moreover what now we have concluded that the vector will make a separate figure for this.Say we have the h 3 vector is along the third body axis which is here and we are getting in this direction, the h 1 and h 3 we have shown as this.Omega 3 is in this direction only and omega 1 is also in this direction and this is omega 2, and omega 2 is in this direction.Now, what we concluded in our earlier discussion?Look into this, what we concluded here?We concluded that omega and h are coplanar. So, we have omega and h so, omega vector and h vector they will be coplanar.

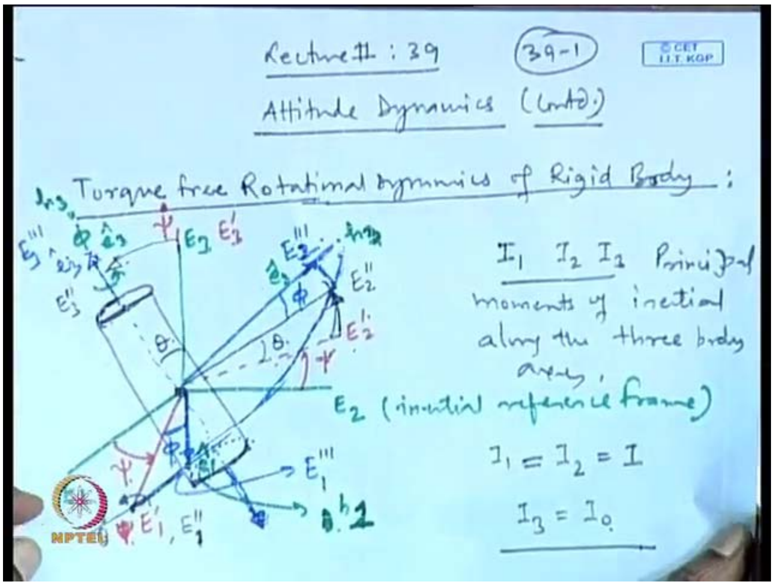
So, vector composed by omega this components omega 1 omega 2 and omega 3 it will give you some vector let us say which is being indicated as, this is your omega 1 not omega 1, this is omega, this is your vector. So, in this figure as become very complicated, but what we are trying to represent here that this is e 1 cap and e 2 cap, these are the 2 orthogonal axis making 90 degree with each other.And perpendicular to the plane of this is the third axis which is e 3 cap which is shown here.And omega t will lie in the plane constituted by e 1 cap e 2 cap, this we have seen here while discussing sorry this figure.

(Refer Slide Time: 44:04)



Now, omega t and e 3 it constitutes another plane and in that plane h and omega are they are lying.Now, what does it indicate? It indicates that the along the direction e 3 we have the omega 3 and here the omega vector is there and this is the h vector. So, this simply conclude that omega 3, omega and h they will lie in the same plane.So, what really happens? Then going back to this figure.

(Refer Slide Time: 45:35)



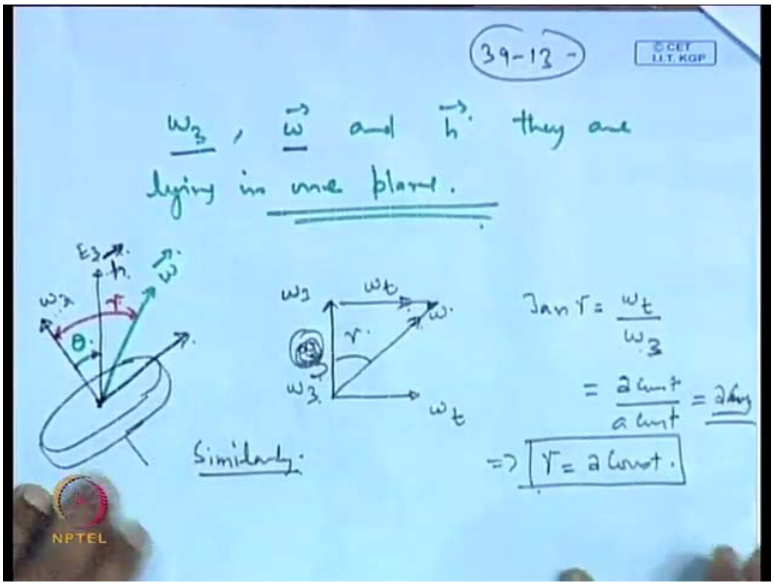
So, this body is rotating about this and simultaneously this is processing about this axis. The vertical axis that we have showing here in this place so, it will undergo procession like this. We have to conclude many more things, but still we are in the middle stage right now. So, say this is the main axis along which the psi dot, we are showing here and the body is rotating about this. This is our main axis and body is rotating about this axis, it is having this spin about this axis and then also its a processing around this. So, from here what we conclude that the omega 3 which is the vector omega 3 is the body component of the omega vector. So, omega 3 and the h vector.

So, h vector we can always oriented orient with for our convenience with along the e 3 direction.We can always orient our inertial reference frame such that it concedes with e 3. So, right now suppose that h is along this direction, omega 3 is along this direction and what we are just now concluded that this omega which is the vector. So, these three are lying in the same plane. So, if the body is rotating this in this way and also it is spinning on a shown axis like this. So, you can imagine what is happening, this is your h vector and this is the omega 3 vector is along this direction.Omega 3 is along this direction which is lying along the third body axis and omega is another vector whose component actually the omega 3 is.

So, now let us indicate it like this. So, this is your omega vector this is your h vector and this is the omega 3 which is representing the spinning the body component of the angular velocity along the third axis. So, it is a rotating like this and finally, this body is also processing. So, the omega vector it will process in this manner, this is your h vector and this is omega vector, this is the omega 3. So, it is processing like this you see here it as it is suppose it is starting in this wayso, it is a rotating like this. So, always they are lying in the same plane this is what we conclude from our mathematical derivations still now we have done.So, this is our very important conclusion once we have done this.

So, let us consider a case of a disk.

(Refer Slide Time: 48:35)

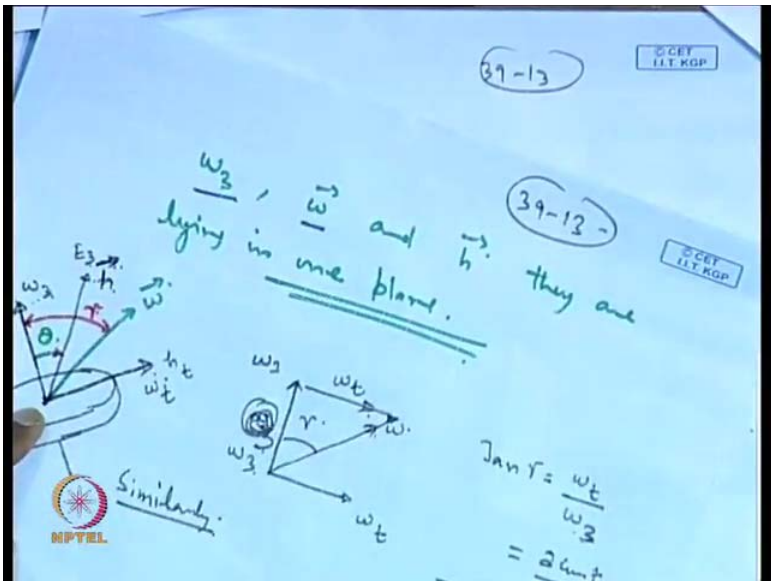


This is your omega 3 shown this way so, it is rotating about this axis, h vector as usual we have indicated along the e 3 direction. So, this is the h vector and this is the e 3 direction, this is obituary h vector, this is the angular momentum of the body.And if you remember from our earlier lecture, that h vector and omega vector in general they do not concede until unless the rotation of the body is along one of the principal axis. So, only along one of the principal axis so, here in this case what is happening we have the h vector here omega 3 is here in this place and omega vector is somewhere like this. So, they are constituting one plane so, in one plane this angle we have indicated by theta and let us say the angle from here to here, this is gamma.

So, you have omega 3 here and omega t is lying in also in the same plane because we have just now looked earlier that, omega is composed of omega t and n is nothing but omega 3. So, omega 3 and this is your omega t they are lying in the same plane.In the same plane your omega vector also lying and h vector is also lying.So, you can see from this place and this angle we are showing between the angle between omega 3 and omega h gamma.And this is the component we can show as omega t and this is the component you can show as omega 3. So, tan gamma we can write as omega t by omega 3.

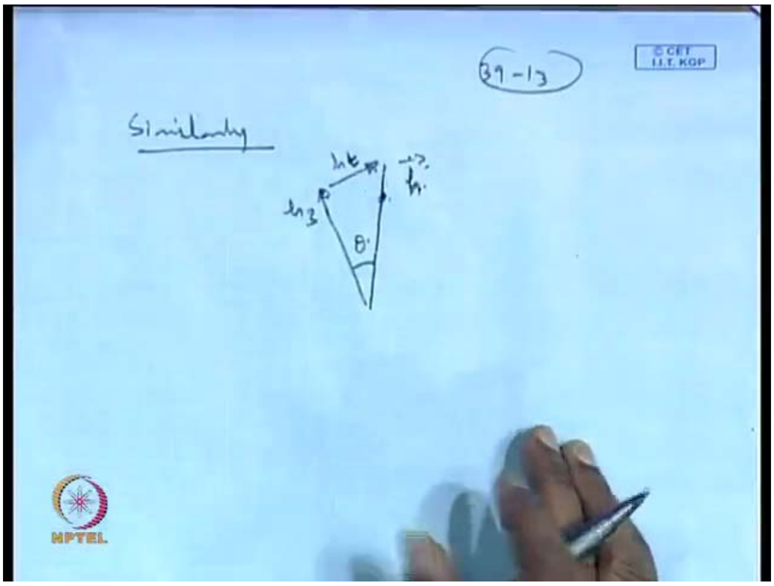
Now omega t this implies the omega t is a constant and omega 3 is also a constant therefore, this is a constant. So, this implies gamma this is a constant. So, this angle the angle between omega and omega 3 this remains constant.Similarly, The h we have written as combination of e t in terms of e t and e 3.

(Refer Slide Time: 44:04)



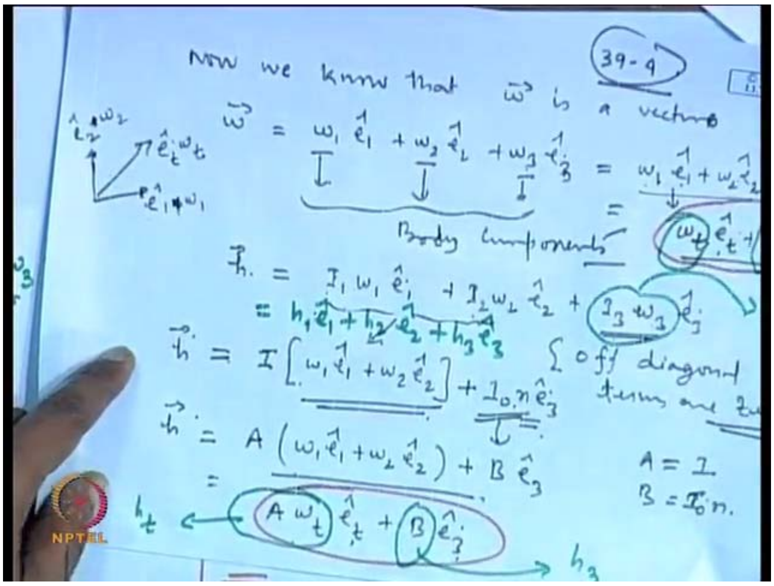
Similarly, you can write this angle theta. So, along this along the third direction we have h 3 and h t is obviously lying in this plane because h is lying in this plane. So, the component of h t we can write in this direction also. And omega t is also along this direction itself.

(Refer Slide Time: 52:45)



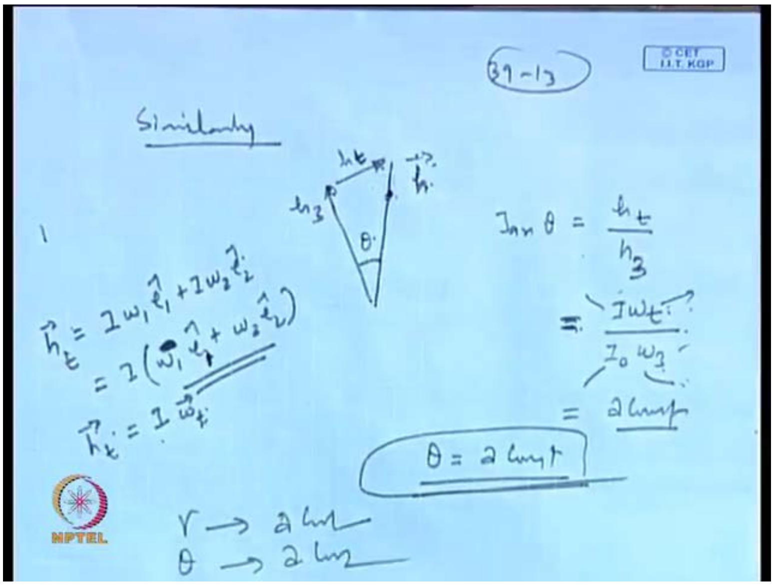
So, the same kind of figure we can draw for h t and omega t also h 3 and h and h t. So, we have this angle this is your theta angle, this is your h 3 and this is h.Now look from this place, along this direction you have the h t. So, we can join from here to here to complete this vector, this constitutes your h t and this equation what we have done this is the quantity which is present here this is nothing but your h T.And b which we have written here this is nothingbut your h 3.We can look back into our old equations, see what is the quantity here this quantity is nothing but h 3.

(Refer Slide Time: 53:47)



This can be written as basically h is nothingbut h 1 times e 1 cap, plus h 2 times e 2 cap, plus h 3 times e 3 cap. So, this is your h 3 and this together it constitutes the h t vector.So, this component is h 3 and this is h t.This is obvious and this we reduced into this form so, this is your h t and b is your h 3. So, that is why I told that this result is very important and based on this the whole structure is based.

(Refer Slide Time: 54:35)



So, once it is a written in this format you can see from there, the tan theta this is nothing but h t by h 3

And h t is how much?

h t this. We are once we are writing in terms of e t. So, we can indicate this see the h t part this was basically your I times omega 1 e 1 cap plus, I times omega 2 e 2 cap if you remember what we have done here in this place. So, taking I outside this is omega 1 cap omega 1 times e 1 cap plus omega 2 times e 2 cap. So, this is I times and this is nothing, but, your omega t cap omega t vector. So, h T is your I times omega t. So, then the magnitude wise you can write as I times omega t and what was your h 3, h 3 is already you have written here is I 0 times n. So, this is I 0 times n or omega 3 omega 3 or n.

So, now see here this is a constant this is a constant, omega t is already a constant we have proved it omega 3 this is also a constant. So, all this 4 quantities are constant therefore, this also implies this is a constant. So, what does it imply ? That theta is also a constant. So, by our conclusion till now what it is, gamma this is a constant and theta this is a constant. So, ultimately what conclusion we have drawing that this angle theta and this angle gamma, they remain constant. In since if this is your omega 3 this is the omega vector and this is the h vector. So, this whole plane is rotating like this, but the angle this angle from here to here which is the gamma angle.

And this angle from here to here which is the theta angle, both of them they remain constant means there is no change. Only the plane is rotating but the position wise, position wise there is no change.They will maintain the same position throughout.So, from our conclusion we can tell that the torque free rotation of a body once we are taking it and the body is obviously in this case we are taking such that, the body axis as aligned with the principal axis. So, in this case the theta angle it remains constant the gamma angle remains constant later on we will see that the theta angle we can call this basically we call as the notation angle. So, notation angle remain constant and the body which is rotating like this call the processing motion and this suppose a top is rotating.

So, it is rotating like this. So, this is the spinning motion if the top you disturb it so, it will get tilted.And then it will a start doing like this. So, in that case settle is the gravitational motion is acting, but this motion is call the notation motion and the rotation which is being done like this is call the procession motion. So, we stop here in this place and we continue in the next lecture thank you very much.