Space Flight Mechanics

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Model No. 01
Lecture No. 37
Attitude Dynamics (Contd.)

We have been continuing with attitude dynamics so, in that context last time we derived the angular momentum equation. So, we continue further so, today will find out the equation for the kinetic energy of a rotating body and rotating rigid body and also the Euler's equation, Euler's dynamical equation.So, which describes if the torque is applied to a rigid body so how, the angular momentum of the rigid body will change. So, the equation derived the Euler equation it relates that.

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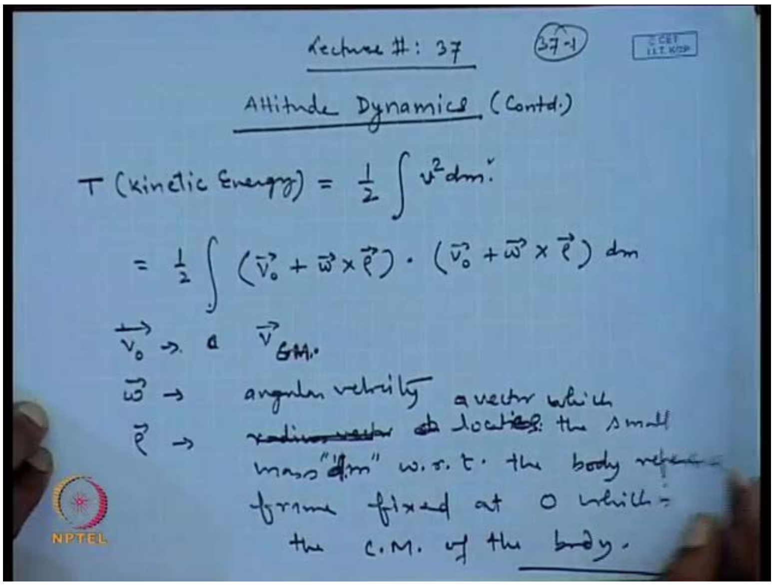

So, let us continue with the kinetic energy first. So, T let us say this is the kinetic energy. So, this can be written as 1 by 2 times V square d m where, V is the velocity of the mass d m.We can write this as 1 by 2 here, V 0 already we have seen this is nothing but V c m once we are considering the o to be located at the center of mass, this is V c m, omega this is the angular velocity and rho is the radius vector or say theit is vector, rho is a vector which locates the small mass d m with respect to the body reference frame fixed at o which is the center of mass of the body.

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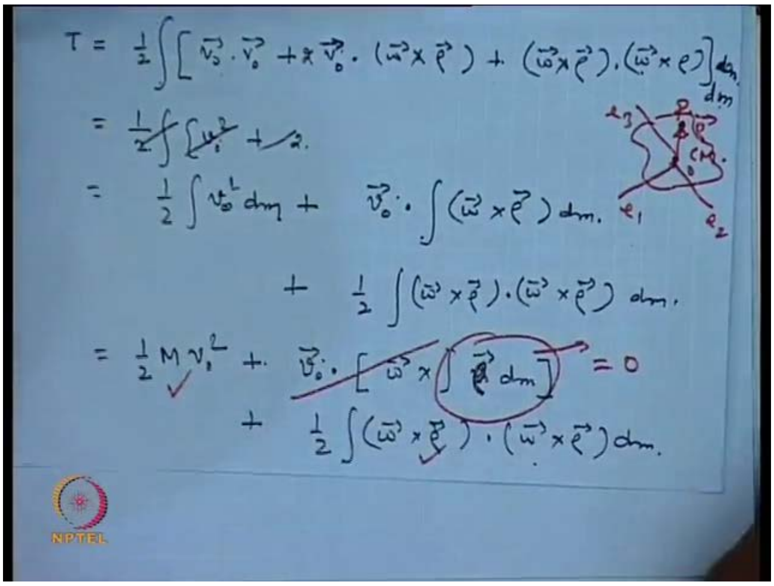

Now, we can expand the equation number.This is the equation that we have got, let us name this as equation number 1 so, we can expand it plus we can write here 2 times omega cross rho omega cross rho dot. So, from here we get V 0 square and this is 2 V 0 times 2 V 0 dot omega cross rho. So, V 0 is a V 0 is the velocity of the center of the mass and integration is being done over the mass of the body. So, we can take it outside the integration sign. So, let us write here also in this place, we will write 1 by 2 V 0 square d m plus V 0 dot.

Find this V 0 square this can be taken outside and directly we can write 1 by 2, the mass of the body times V 0 square plus V 0 this is the V 0 dot. So, V 0 dot we have already taken out now we can look into this moreover, we can put the whole thing in this fashion omega cross this is rho d m. So, this can be expanded and you can look from this place the quantity, rho d m this is nothing but because we are taking the point o to be the center of mass and rho is being measured from the center of mass.

We have the body here and this is the point o which is the center of mass and rho is the reduce vector of any point p that we choose earlier if you remember from our last lecture. So, this quantity will turn out to be equal to 0 because at the center of mass if you integrate or the moment of mass about the center of mass it will be 0. So, integration taken over the whole body. So, this term drops out. So, this term will drop out living us only these two terms.

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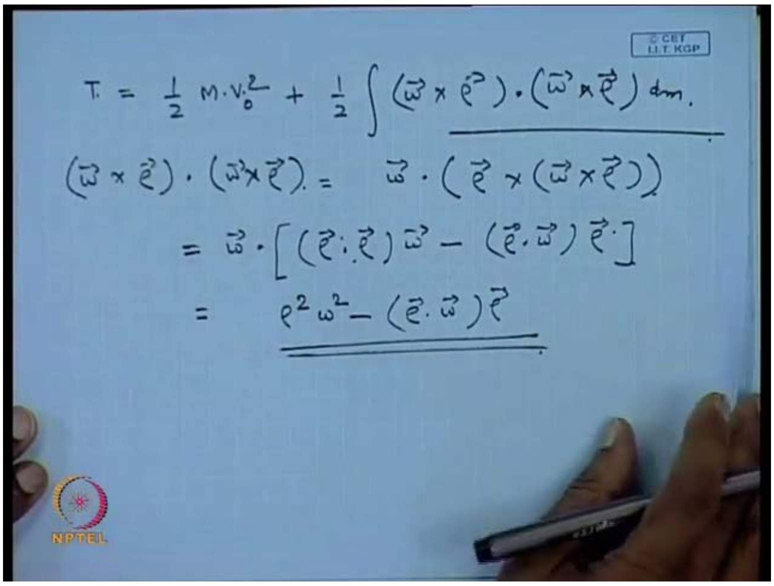

Therefore, the kinetic energy we can write as 1 by 2 M v 0 square plus now we need to find out what this quantity omega cross rho, what omega cross rho will be? So, we can write this as omega dot and then we can expand it further, this is the simple relationship here using for the dot and the cross product. So, this becomes rho dot rho omega minus and now we can take again this quantity becomes rho square this becomes omega square and minus rho dot omega rho.

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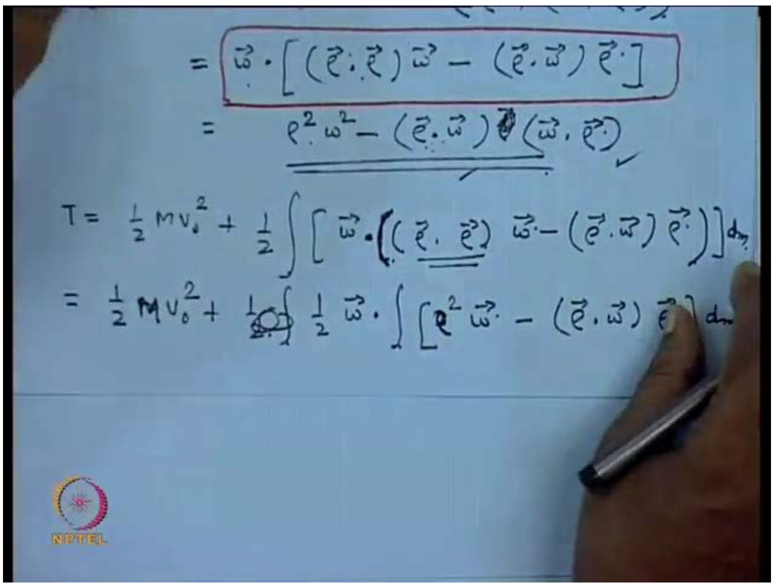

So, this relationship we are going to utilize later on where we are measuring the term omega dot rho rhorhorho square and this is rho dot omega times omega dot rho or rho dot omega it is a same.Now, the kinetic energy finally, we can write as 1 by 2 and V 0 squareplus 1 by 2 inside the integration sign we have omega dot we can choose from here times rho dot rho.

This omega dot we can take it outside. So, let us take it outside omega dot and then inside the integration sign this quantity will represent as rho square minus omega and this becomes rho dot omega.

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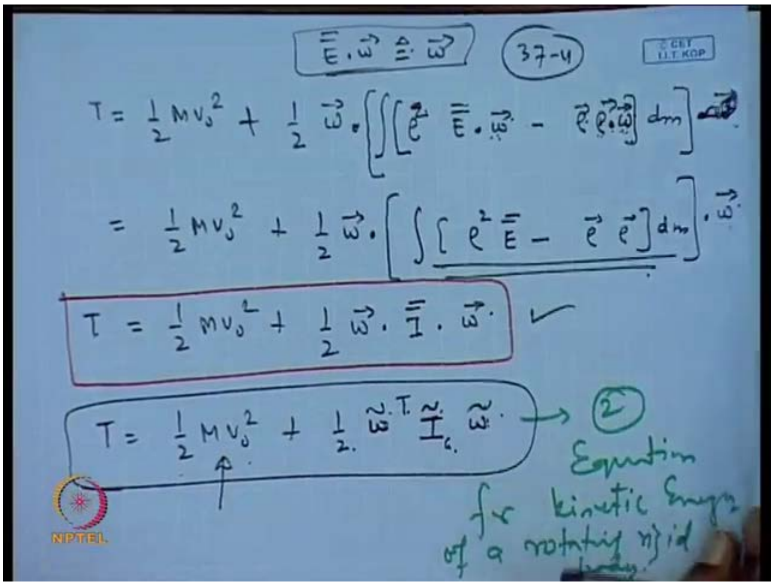

So, further working on this equation so, we can write T equal to 1 by 2 M v 0 square plus 1 by 2 omega dot and inside the integration now if you remember that any vector can be written in terms of a unit dyadic. So, our unit dyadic we wrote as E double bar. So, E double bar dot omega this is nothing but equal to omega this is identically equal to omega.So, therefore, this equation can be written in this fashion and here we can write this as rho times rho, this rho we can take from this place from the this side to this side and we can write in front of this bracket.And then we can write here d m and this whole thing we can put in a bracket and dot omega so, this omega we have taken it.

Now, this omega the omega, which was present here let us keep it here itself, right now we have rho dot omega. So, this dot product is between rho and omega, in the next step omega dot rho square E double bar minus now if from here we can write here rho rho d m dot omega.We are taking the dot product this omega outside this whole bracket from here and here both the places we are taken it outside.Now, if you remember this quantity is nothing but our inertia dyadic that we have defined earlier so, we can write this as I double bar and so omega. So, this gives the equation for the kinetic energy and the same thing if you try to write in the form ofin the matrix notation.

So, this will change to omega tilde transpose I tilde and I tilde I is indicating the inertia matrix or either we can put g to indicate that this is inertia matrix and so, omega tilde. So, this gives you the equation for kinetic energy.Kinetic energy of a rotating rigid body and it is easy to identify.This is the kinetic energy of the body rigid body, considering this as a particle or a of mass m and this term represent the energy due to the rotational motion. So, total energy becomes the translational energy plus translational kinetic energy plus rotational kinetic energy.

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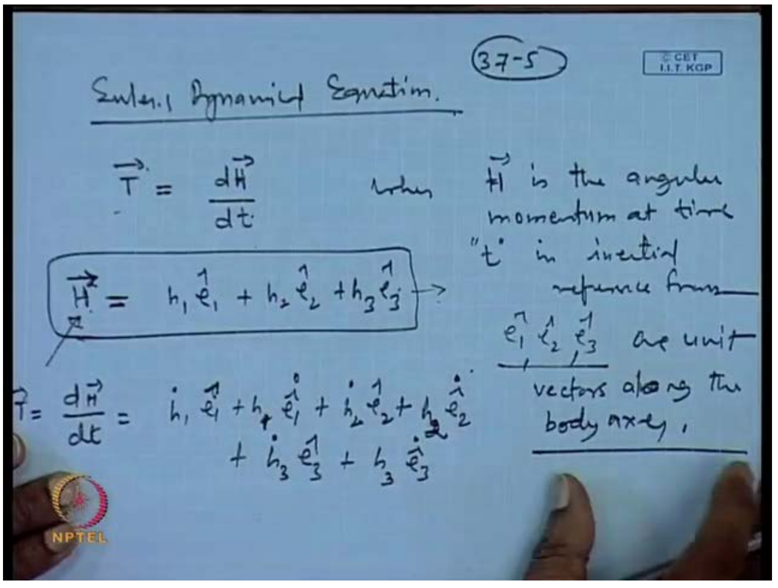

So, we have completed this. Now, we go to the next step of deriving the Euler's dynamical equation. So, we know from our elementary physics that the if torque T is applied on a rigid body,then this can be written as torque is equal to dH by dt where, h is theangular momentum at time t in inertial reference frame.Now, this h this can be either capital H or a small h, last time we have used the small h notation. So, any one of them you can use. So, this can be written as h 1 e 1 plus h 2 e 2 plus h 3 e 3.

What we have doing here that the angular momentum vector of the rigid body, this is in the inertial reference frame, but we can take the components of this angular momentum along the body axis. So, this is broken along the body (( )) e 1 e 2 e 1 e 2 and e 3 cap, these are the are the unit vectors along the body (( )).So, if we differentiate this so, we can write this as h 1 dot e 1 cap plus h 2 h 1 times e 1 cap dot plus h 2 dot times e 2 cap plus h 2 times e 2 cap dot plus h 3 dot e 3 cap plus h 3 times e 3 cap dot.

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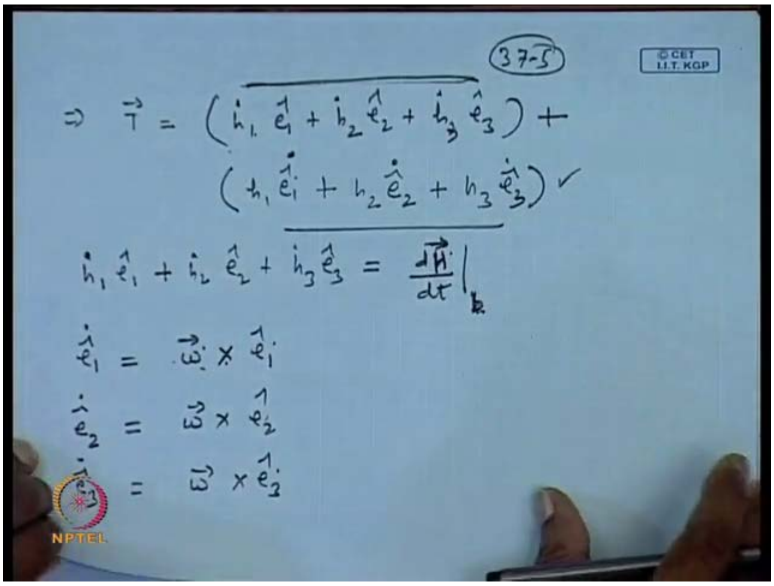

So, this implies torque is equal to h 1 dot e 1 plus h 2 dot e 2 cap plus h 3 dot e 3 cap plus h 1 times e 1 cap e 3 cap dot. Now, in this two this is one part and this is the second part.This we can identify as this can be written as d H by dt in the body (( )) or we are using capital notation for H. So, we can write in this way dH by dt with respect to the body (( )).And this part we need to explore. So, now, we have to first evaluate e 1 dot. So, e 1 dot will be nothing but the angular velocity of the rigid body times e 1 cap and it is a very easy to prove we have earlier proved also this things e 2 cap dot similarly, this can be written as e 2 cap and e 3 cap dot omega times e 3 cap.

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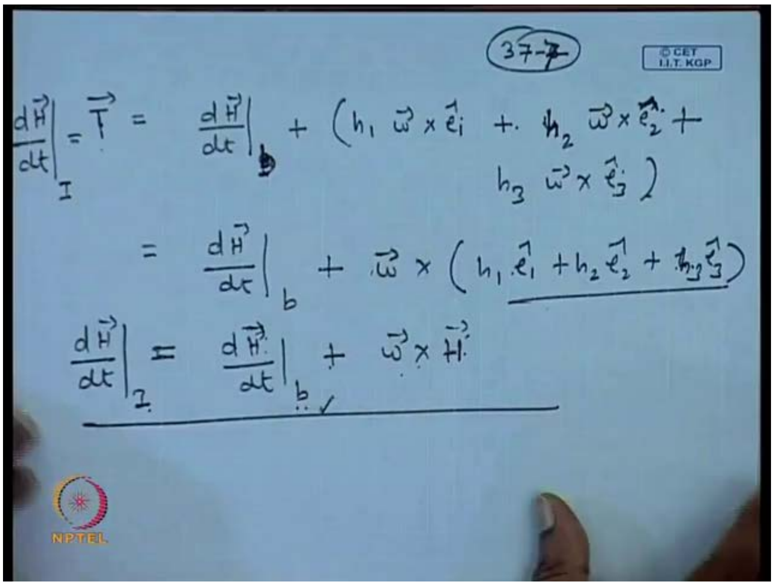

So, therefore,let us say this is equation number 3. So, equation number 3 gets reduced to dH by dt is in the inertial reference frame, we can put here I to indicate this is with respect to the inertial reference frame.This is equal to torque applied and this is with respect to the body (( )) and the rest of the things you can write as h 1 times omega cross e 1 then h 2 times omega cross e 2 cap plus h 3 times omega cross e 3 cap, this is b. So, now, you can recognize that this thing can be written as h 1 e 1 cap plus h 2 e 2 cap plus h 3 e 3 cap and this quantity is nothing but h.

So, dH by dt in the inertial reference frame, this can be written as d H by d t in the body reference frame. So, rate of change of the angular momentum in the body reference frame plus omega cross the angular momentum vector.

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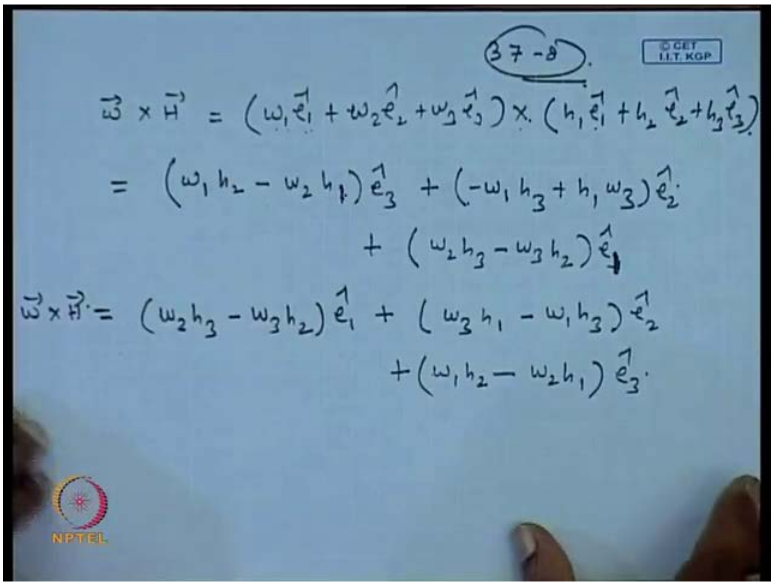

So, we know that omega cross H can be written as omega 1 e 1 cap plus omega 2 e 2 cap plus omega 3 e 3 cap.And we have to take the cross product of this, with h and h we can write as h 1 times e 1 cap h 2 times e 2 cap and h 3 times e 3 cap. Expanding this we can re write them as omega 1 h 2. So, what we have to do once we take the cross product the self e 1 cross e 1 this will vanish only thing it will adjust with the other 2. So, this can be written as omega 2 h 1.

Times e 1 cap.Re writing the whole thing we bring in the third term make it the first term, omega 2 h 3 minus omega 3 h 2 e 1 cap from here omega 3 h 1 minus omega 1 h 3 e 2 cap plus now taking this term e 3. So, this is omega cross H.

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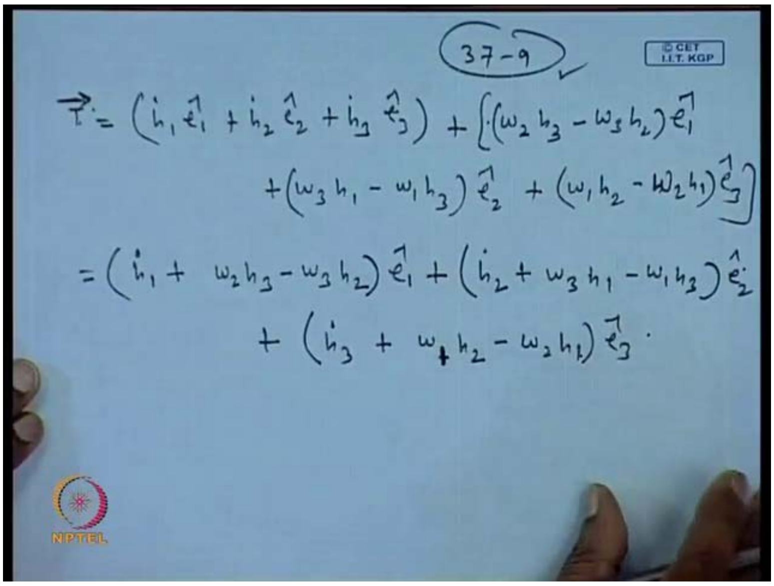

Therefore, this T what we have written as h 1 dot e 1 plus h 2 dot e 2 cap plus h 3 dot e 3 cap and the omega cross h now we put here in this place omega 2 h 3.

We can combine the terms with e 1 e 2 and e 3 separately. So, this will become h 1 dot plus omega 2 h 3 minus omega 3 h 2 times e 1 cap h 2 dotplus omega 3 h 1 minus omega 1 h 3 times e 2 cap plus h 3 dot plus omega 1 h 2 minus omega 2 h 1 times e 3 cap.

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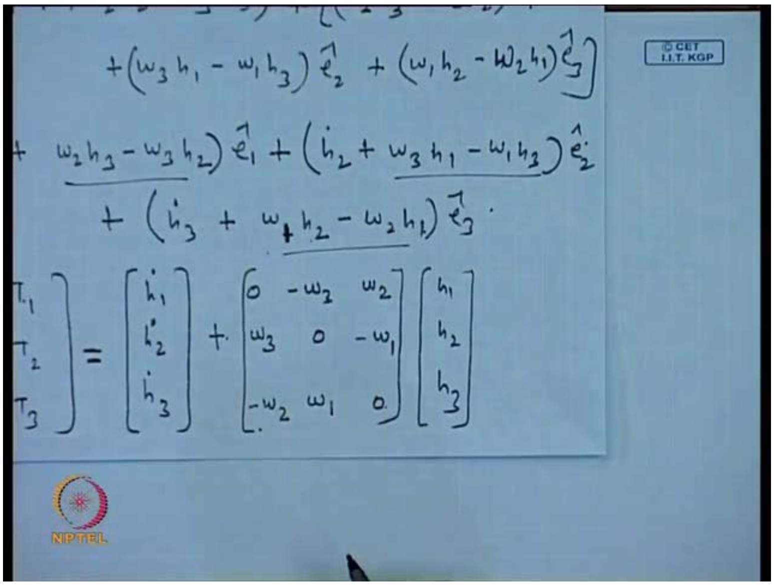

If now, if you look into this equation so, this T if you try to write in the matrix notation. So, you can write this as T 1 T 2 T 3and here you will have h 1 dot h 2 dot h 3 dot and plus the other terms which are present. So, similarly, you can represent them. So, for representing them in a very simple format, there are compact notations for representing them. So, let us say that there are two ways of doing this omega 2 omega 3 like this terms are present here. So, first we take the omega outside this and write here h 1, like h 1 h 2 h 3 and omega we write here in this place and later on the h 1 can be expanded in terms of the moment of inertia matrix and the angular velocity vector components along the body (( )).

So, h 1 here if you take the first term here. So, you can see from here this the omega 2 is related to h 3 and positive sign while h 2 is related with omega 3 and this is having a negative sign.And omega 1 is not present in this so, we can write here 0.Similarly, we can take the next term here so; next term is h 1 times omega 3 and minus omega 1 times h 3. So, omega 1 will come with a minus sign and here we will have 0, next we take this term, but this is related to h 2. So, with h 2 we have the omega 1 term present and while with h 1 we have the minus omega 2 term present. So, we represent it in this format.

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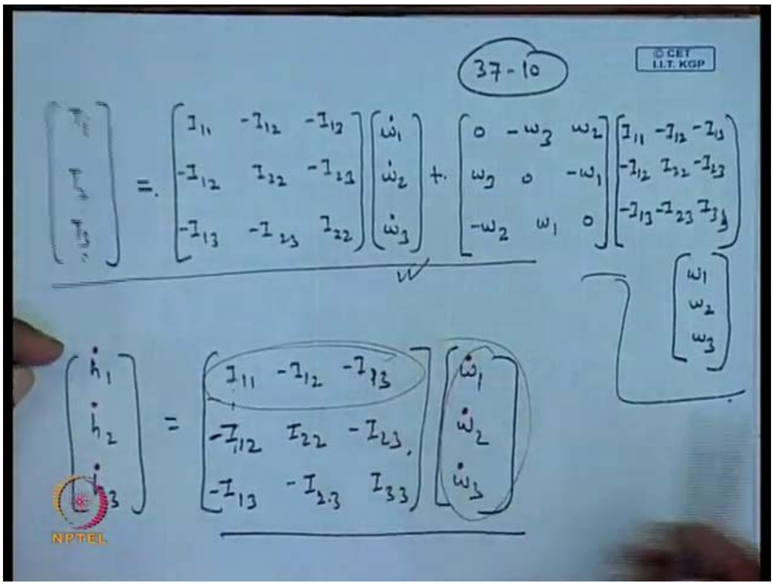

So, thisequation so, this you can write further as T 1 T 2 T 3.

Now, look into this equation, once we wrote the h 1 h 2 h 3 dot so, what we assume that our body axis fixed in the body, rigid body itself.And therefore, the moment of inertia will not change.If you do not fix the body (( )) in the rigid body so, moment of inertia will keep changing and that is what happens with respect to the inertial (( )).In the inertial (( )) the moment of inertia of the body l because the moment of inertia we are defining with respect to the 3 (( )) and if the body orientation is changing so, with respect to this (( )) moment of inertia will change.

So, whichever point you choose the moment of inertia of the body changes. So, we need to fix the body (( )) in the rigid body. So, once we fix it then h 1 dot can be simply written as h 1 dot we have h we have written as let us write below first h 1 h 2 h 3 this we have written as the I matrix I 1 1 minus I 1 2 so, here minus I 1 3 minus I 2 3 I 3 3 and here we have I 1 2 with minus sign I 2 2 minus I 2 3 omega 1 omega 2 omega 3. So, h 1 dot is the product of this vector and this one.So, this is basically we are taking you multiply each of the term here.So, if you differentiate you take write h 1 dot if you are writing h 1 dot so, these terms are not changing because the body axis is fixed in the body itself and therefore, only thing will happen that this omega 1 dot omega 2 dot omega 3 dot will keep changing with time. So, here we can put the dot for that.

So, therefore, T 1 T 2 T 3 this will become equal tonow this h 1 dot h 2 dot h 3 dot in this equation need to be replaced by this equation. So, we insert here in this phase this is I 1 1omega 1 dot omega 2 dot omega 3 dot and plus the matrix just now we have written here. So, 0 minus omega 3 omega 2 and then this is omega 3 0 minus omega 1 you can see this is anti-symmetric matrix or asymmetric, this is basically asymmetric matrix.And then h 1 h 2 h 3 we can similarly, write here h 1 will be I 1 1 minus I 1 2 minus I 1 3 minus I 1 2 I 2 2 minus I 2 3 with minus sign and I 3 3 and this 2 be multiplied by omega 1 omega 2 and omega 3.

So, whatever we have got it represents that if we apply a torque to a rigid body. So, the torque is directly taken in the inertial reference frame. Now, T 1 T 2 it’s a taken the inertial reference frame, but you can consider that we have the components T 1 T 2 T 3 along the body axis. So, these are we are writing we have broken along the body (( )) because all these omega 2 omega 3 they are all a about the along the body (( )). So, if you apply the torque whose components in the body (( )) are T 1 T 2 and T 3, then you get this dynamical equation which relates the rotational motion of the body. So, how the angular velocity will change it will be obtained from this equation.

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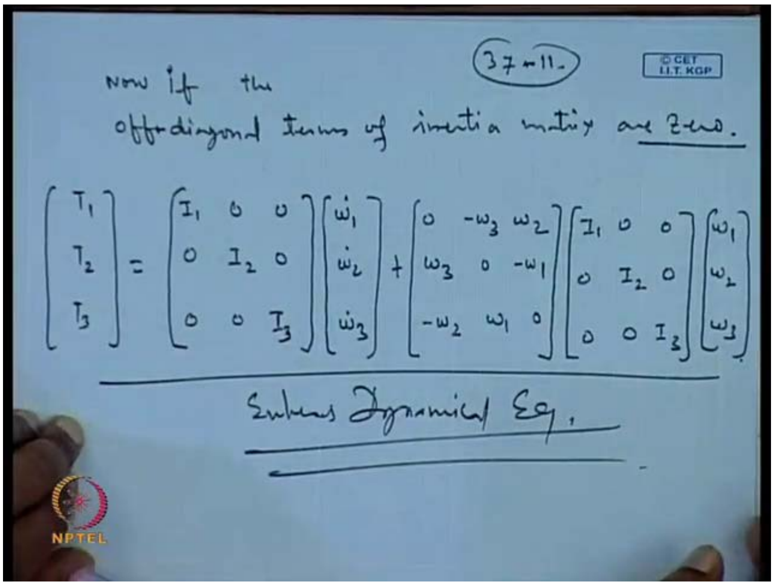

Now, if the off diagonal terms of inertia matrix are 0then what we get T 1 T 2 T 3 now we write as I 1 I 2 I 3 to indicate that these are serving as the principal moment of inertia along the 3 (( )) omega 2 dot omega 3 dot.And similarly, here we have 0 minus omega 3 I 1 I 2 I 3 and this is nothing but your Euler's dynamical equation.This is what we intended to derive you can break it down and write the each of the component separately.

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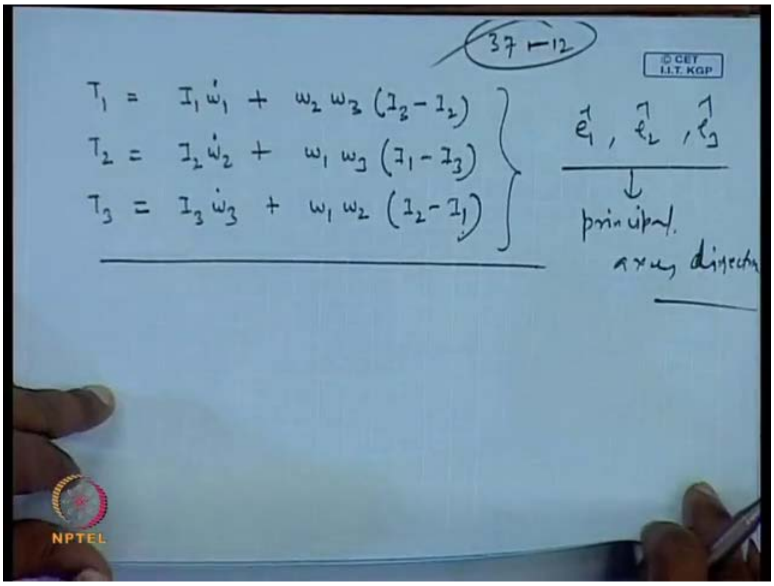

So, the first equation if you break it up first equation can be written as I 1 times omega 1 dot multiplying this row by the whole column. So, we get I 1 times omega 1 dot and then we have to multiply this similarly, and write here. So, what you get? I 2 times omega 2 dot plus I 1 minus I 3.

So, here you your directions e 1 e 2 and e 3 they are basically serving as the principal (( )) directions because the off diagonal terms are 0 here.

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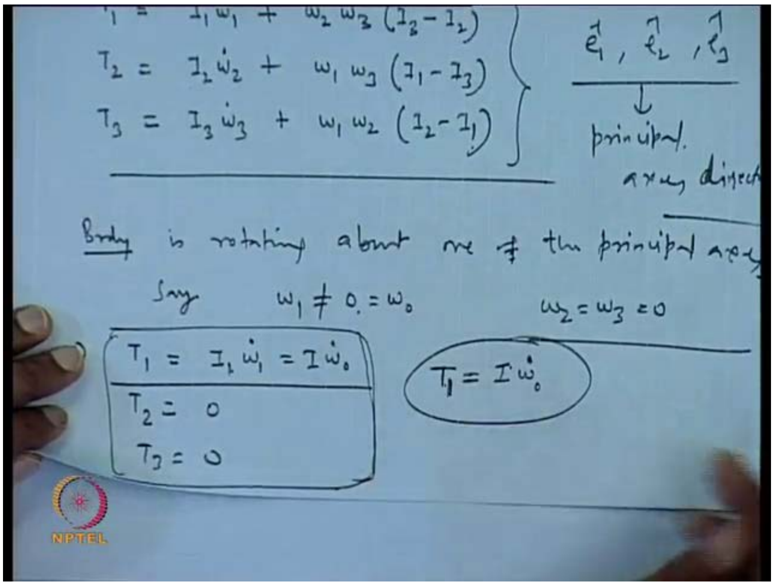

Now,we in this case, if you considered that body isrotating about one of the principal (( )), then let us say this is omega 1 which is not non 0 and this we are putting as omega 0 and omega 2 and omega 3 equal to 0. So, the above equation then will get reduce to T 1 equal to I 1 times omega 1 dot and omega 2 omega 3 they are 0. So, therefore, those quantities vanish.Similarly, T 2 equal to now omega 2 itself is 0 is not present. So, this term also vanishes omega 3 is 0 so, this will vanish.And all similarly, other terms will become 0.This is the very simple equation that we have got T 1 equal to I times omega 0 dot.And earlier I have told you that the angular velocity vector and the angular momentum vector, they will coincide if and only if rotation is about 1 of the principal (( )).

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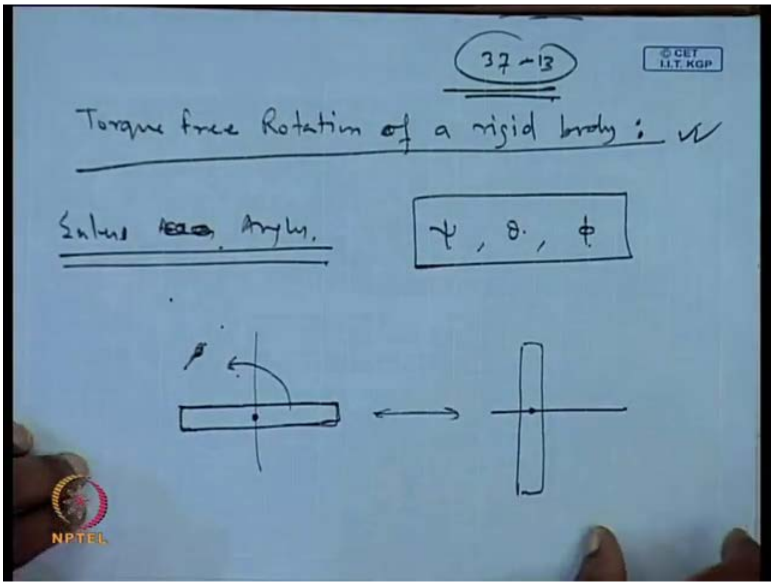

So, we go to the next step. Now, we consider the we will consider the torque free rotation of a rigid body. So, why this is important for our study that say if we have in this space some satellite is left alone and no torque is applied.The torque’s whatever the torque’s magnetic torque’s or the rockets which are on board are the reaction will nothing is working neither the suppose the solar pressure is also not there. So, torque due to solar pressure is also not present. So, if it is the satellite is free from the all sort of torques, then how the satellite is going to behave? So, if we orient the satellite say this is my satellite and this is moving in the orbit. So, I can have the orbit like this in which the satellite has to move in this way, always pointing in one direction, always pointing towards the earth.

So, is it possible that once we are putting in this configuration that it is going to remain in this configuration. So, in the case of because it is a long body and we have seen that because of the gravity gradient it tends to stabilize in this configuration only. But we will do a general derivation right now and we will see that if the bodies now, if the satellite is left in the orbit and depending on it is a moment of inertia along the different axis how much it is.So, whether the satellite will remain in the same configuration or means the same orientation or the orientation will change. So, this is related to stability of the rotational motion.If it is rotating in this direction like this, if it is rotating like this so, whether it will maintain that direction that becomes important to know or satellite orientation will or either it will change over a period of time.

So, this we can derive if we work on the torque free rotation of a for a rigid body.So, first of all we need to know about the few the details like the Eulerangles. So, Euler angles are represented by psi theta and phi. So, in general if you see anybody we are having so, it can be oriented in the space, say if I have any object which is lying like this. So, you can bring it to this configuration by rotating it about the axis which is passing through this page.If you rotate anti clock wise by 90 degree so, it will come in this configuration.So, the next rotation you give so, the configuration will change. So, for any orientation of the body this three rotations are three consecutive rotations can be given and it will get oriented in certain direction.

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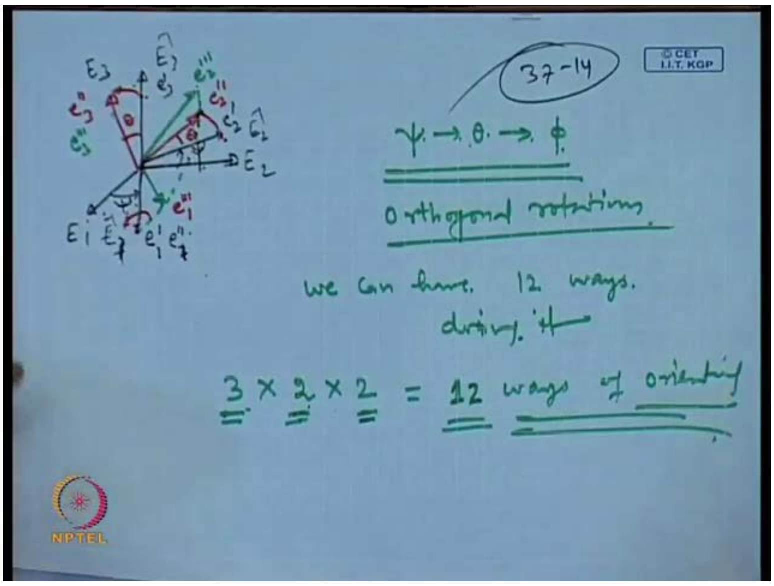

So, let us say that E 1 E 2 and E 3, these are the 3 axes with respect to which I have to orient the body. So, let us say that these are the vectors E 3 cap E 2 cap and E 3 cap E 1 cap along this direction.And if I give a first rotation by angle psi about this z axis so, this is my new position and new position let us write this as e 1 prime and from here we write this as e 2 prime and e 3 prime remains along the same direction and this angle is basically your psi this angle is psi.The next rotation so, we have the new configuration in which this axis came to this position this axis moved to this position. So, we can give a rotation here about this by angle theta.

So, if we rotate it, this will move from this place to this place and this vector will move from this place to this place. So, our new vector will be e 3 double prime and here we can write as e 2 double prime and this is our new vector.And by how much angle we are rotating here?This by angle theta and the you see that they are not in the same plane.First we rotated about the z axis then we are rotating about this is e 3 axis is serving as the z axis here and even is serving as the x axis. So, next we are rotating about this axis. The third rotation we can give in the plane about e 3 double prime axis.

So, e 1 will move in this plane from here to here, the e 1 prime and e 2 prime now is the same.e 1 prime and e 2 prime it would e 1 double prime they are the same. So, e 1 double prime will move from this place and it will come to e 1 triple prime here in this place and this will move in this plane from here and it will go into this place and here,we write as e 2 triple prime.While our e 3 triple prime will remain in same direction because we have rotated about this axisso, we are basically giving three consecutive rotation which are indicated by psi and then we are getting rotating bytheta and then we are rotating it by phi.And these are called orthogonal rotations. So, for orienting a body in the inertial reference frame, we can have 12 ways of doing it. So, the first rotation I can choose among any of the 3 axis.

So, the first rotation can be given in three ways, once we have given the first rotation. So, the next rotation can be given only about the remaining 2 axis.We cannot give about the first at (( )) like here we choose the third axis to rotate it so, there after we are left with e 1 and e 2 (( )) to rotate it.We cannot give the next rotation about the e 3 (( )) again if you do that so, it is the same rotation nothing it is a rotating in the same plane. So, only thing you are adding to this psi delta psi more.Some more angle you are adding so, the first rotation can be given in 3 ways, the next rotation can be chosen in two ways once you are chosen the next rotation about any remaining of the 2 (( )). So, suppose that we choose the e 1 x. So, next rotation can be chosen among the rest of the two xso, the total of 3 into 2 into 2 so, total twelve ways of orienting. So, giving the different magnitude of the psi theta and phi and reshuffling their order, you can rise till the same orientation in 12 different ways.

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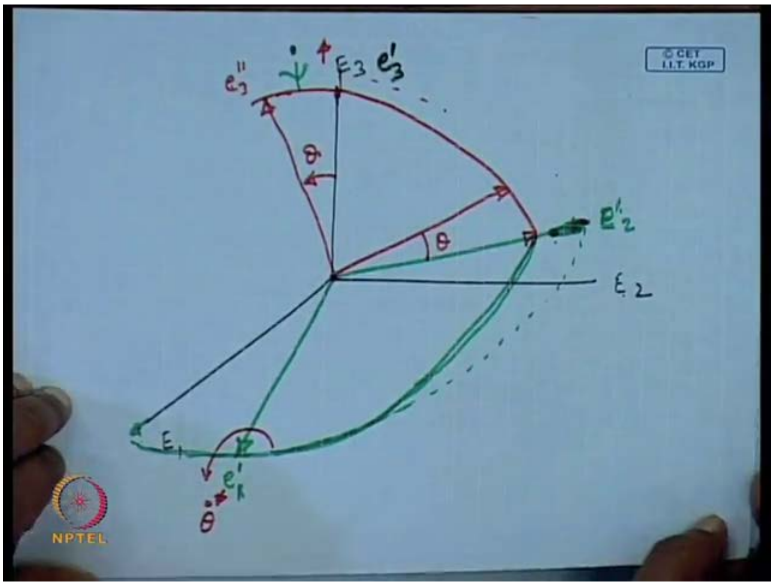

So, what we have done just now, we can explode it little more.This is our E 1 E 2 and E 3 and first we are rotating about E 3 xso, and rotating about by angle psi. So, the rate of change of this angle psi dot we indicate here so, the next vector is from here to here. So, this we write as e 1 prime.And similarly, in the same plane e 2 will move to this place and this we have indicated as e 2 prime. Next, the rotation we gave about this x by theta. So, theta dot can be shown along this direction this is the direction for psi dot. So, if we are rotating about this. So, this x will rotate, it will move from this place to this place. So, we take it up and this will go from here to here.

So, this vector will move from here to here, this vector is moving from this place to this place. So, here you have e 3 e 3 prime and in this place you have e 3 double prime.And this vector will also move from this place to this place so, this is your let us put this vector here. So, first of rotation we show by this line so, this has rotated from here to here by theta.You see these are the different planes, they are not in the same planerather I will make another figure to make with more clear, this figure is not very good.

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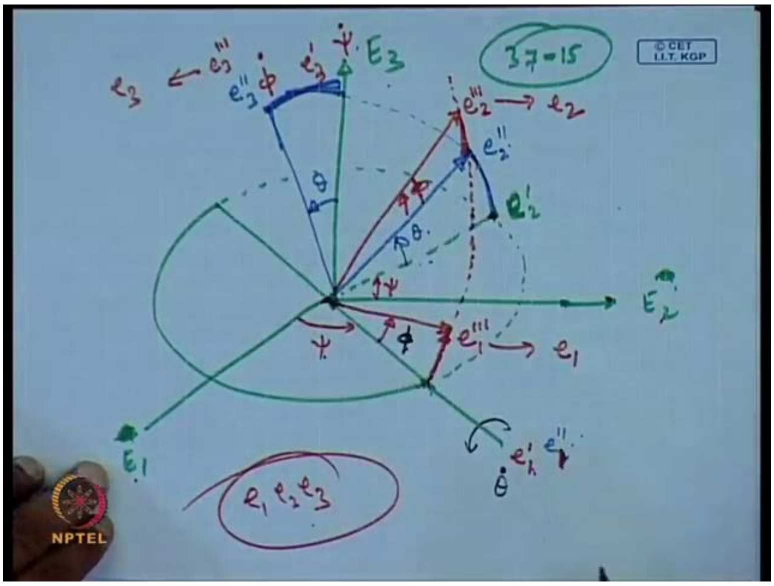

And this is E 1 and this is the direction E 2, this is the direction E 3. So, the first rotation we give by angle psi. So, psi dot is along this direction and your new orientation is e 1 prime and E 2 prime similarly, it will rotate in the E 1 E 2 plane, E 2 will move in the E 1 e 2 plane and let us say this gets oriented from this place to this place.Obviously, goes into this place, this is E 2 prime and here we have the e 1 prime.The next rotation we are giving is about this by angle theta so, theta dot will point in this direction.So, if you rotate it so, this vector and this vector now it is a forming a triode e 1 prime e 2 prime and here we have the e 3 prime.So, it is a forming a triode. So, this e 2 prime and e 3 prime will rotate about the e 1 prime. So, we can make it rotate. This angle is turned to this psi and let us say, that we rotate it so, it rotates like this.

And a move from this has been shown by dotted line to show that this is in the e 1 e 2 plane.It moves from here to here, this is here e 2 double prime, e 2 double prime is along this direction itself.And this will move from this place to this place. This is the angle theta and here we will write as e 3 double prime.Next rotation we can givewhich we will be contained in e 2 double prime, this is e 1 double prime here and e 2 double prime plane. And we will rotate about this xso, e 3 triple prime will remain here and we are rotating about and this angle will rotate by phi. And we can show it that this will move from this place to this place. So, this is angle phi.

And this will move from this place to this place, this is angle phi. So, they are not in the same plane you can see. So, these are the three orthogonal rotations. So, any set of vector is once need to be converted from the inertial reference frame to the body reference frame so, say the final orientation which we have got as e triple prime we write as e 3 and here this is moving from this place to this place. So, we write this as e 1 triple prime which we will show as e 1 and e 2 moves from this place to this place. So, we write as e 2 triple prime which will write as e 2. So, e 1 e 2 e 3 they constitutes the body reference frame and capital E 1 capital E 2 and capital E 3 it constitutes the inertial reference frame. So, changing from the inertial reference frame to the body reference frame it requires this three rotations and they can be represented in terms of the matrix. So, we will continue with this next time.Thank you very much.