Space Flight Mechanics

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Lecture No. 36

Attitude Dynamics (Contd.)

We have been discussing about the attitude dynamics. So, in that context we are working out the movement of inertia.So, we are looking into the principal movement of inertia.

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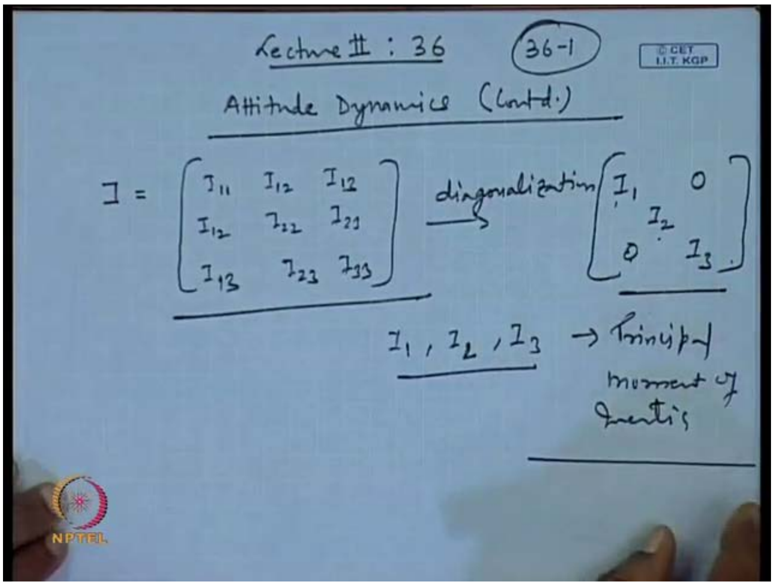

So, we were given that the movement of inertia matrix this is I 1 1, I 1 2 this was movement of inertia matrix and if the diagonalization is done. So, off diagonal terms then become 0, off diagonal terms will be becoming 0 and therefore, we get the we get only diagonal elements. So, this diagonal elements are nothing, but the principle movement of inertia I 1, I 2 and I 3 these are the principle movements of inertia. So, in this context we took an example of cuboid.

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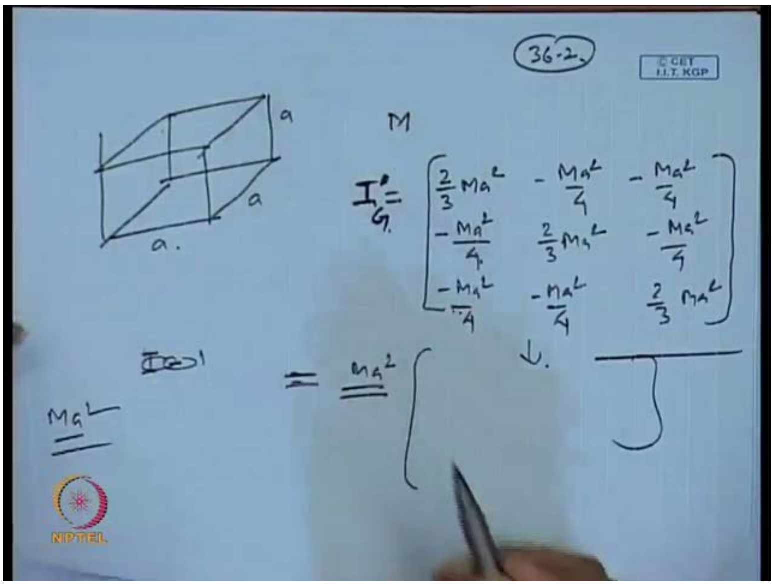

So, we had a cube of each of the side of the cube is a, and we assumed this mass to be m. So, ultimately what we got last time?That our inertia matrixwe put in the format of 2 by 3 M a square minus M a square by 4 minus M a square 4 similarly, other terms here because it is a real symmetric matrix.

And therefore, we wanted to (( )) find out the principle movement of inertia from this place. So, we wrote the whole thing as I minus if we can put here some other subscript to indicate, how does it differ fromthe other representation. So, in that context we have used notation first of all we took out the M a square outside and there after we kept the numerical terms inside only.And we said that the if we try to find out the movement of inertia so, the finally, whatever the numerical terms comes in the diagonal elements. So, those will get multiplied by M a square.

So, while working in that terms so, we had the movement of inertia matrix which we suppose we write it has I prime, not has I, because I most of in while writing the matrix in working in the terms of matrix notation. So, we indicate I is the identity matrix or may be we can keep here I G.

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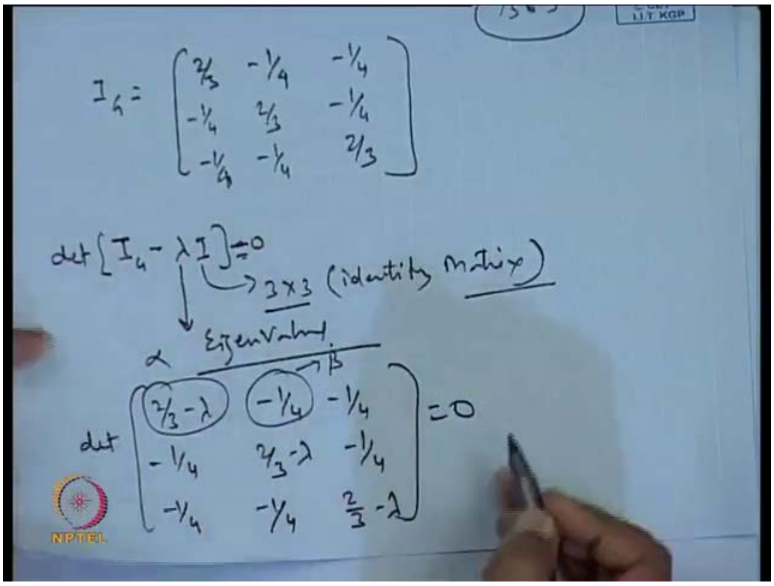

So, therefore,I G was written as 2 by 3 minus 1 by 4 minus 1 by 4 2 by 3 minus 1 by 4 2 by 3 and for finding out the principle movement of inertia terms. So, we have to write it in this way and determinant of this to be equated with 0, where this is the identity matrix of 3 by 3 size, identity matrix and this is what the we call as the Eigen value.Now, in solving this the whole thing got reduced into the format 2 by 3 minus lambda the determinant of 2 by 3 minus lambda minus 1 by 4 minus 1 by 4 minus 1 by 4 minus lambda this should be equal to 0 and this term we wrote as alpha and this term we wrote as beta.

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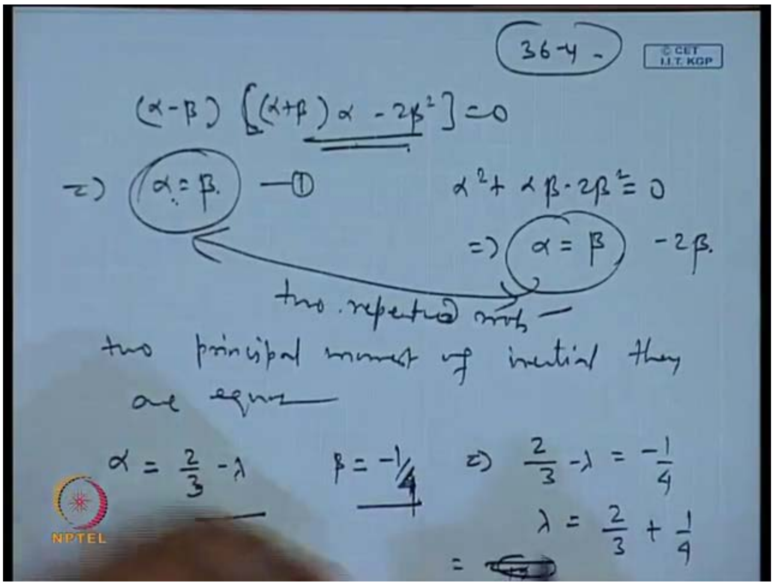

So, ultimately the polynomial that we that we got. So, it was written as so, the whole thing we wrote this as this diagonal terms as the alpha and this of diagonal terms as beta.So, the our equation got reduced into the format alpha minus beta alpha plus beta times alpha minus 2 beta square equal to 0. So, this implied that alpha equal to beta, this was 1 of the solution. So, other two we get from this part so, this alpha square plus alpha beta minus 2 beta square equal to 0 and this implies alpha equal to beta one solution another solution is minus 2 beta. So, what we are getting that?This alpha and beta there are two repeated roots. So, two principal movement of inertia they are basically equal.

So, alpha already we know that alpha equal to we have written as 2 by 3 minus lambda and beta we have written as minus 1 by 4. So, alpha equal to and so, this two imply that2 by 3 minus lambda equal to minus 1 by 4 and lambda equal to 2 by 3 plus 1 by 4.

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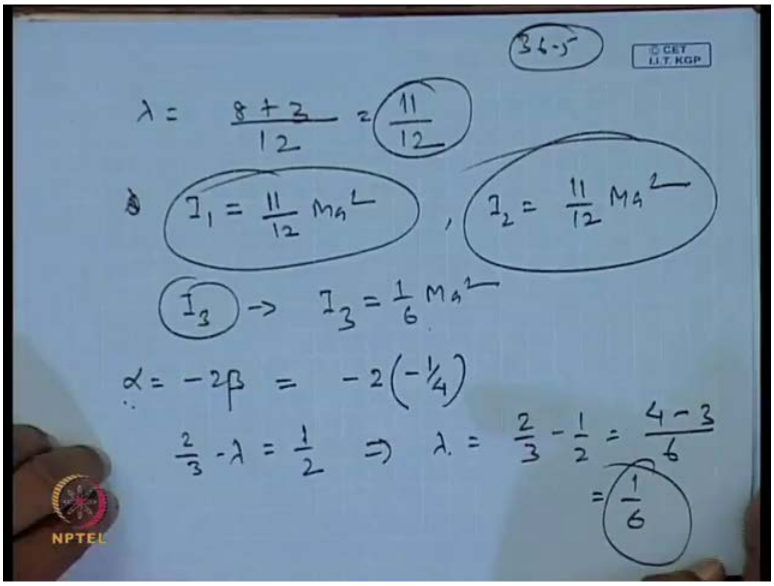

So, this we get as. 11 by 12. So, finally, this lambda we need to multiply by M a square to get I 1 so, I 1 becomes 11 by 12 M a square. Similarly, I 2 will be equal to 11 by 12, M a square and I 3 now, we have to calculate. So, for calculating I 3 we have write alpha is equal to minus 2 beta which is this part we are using now. So, minus 2 beta minus 2 times minus 1 by 4 and obviously, alpha on this side we have 2 by 3 minus lambda so, this becomes 1 by 2 and this implies lambda is equal to so, 1 by 6. So, I 3 we get from there so, I 3 will be equal to 1 by 6 M a square. So, these are the 3 principal movement of inertia that we have just now worked out.

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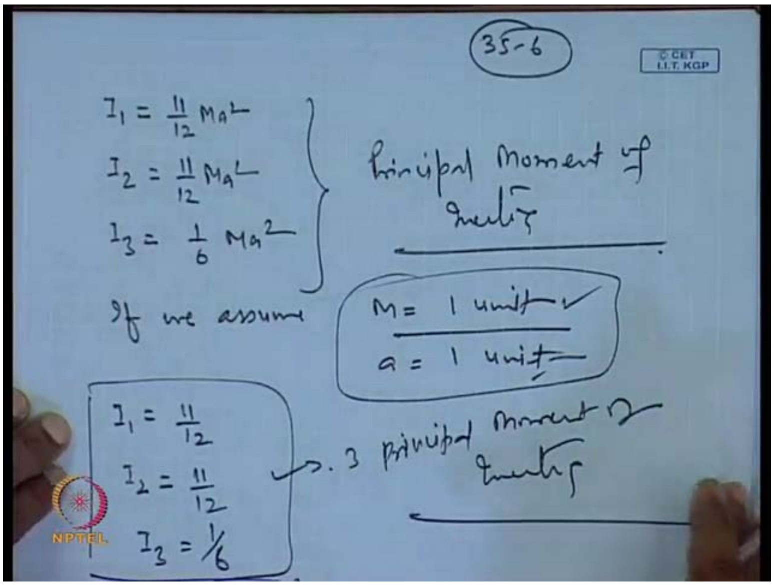

So, I 1 we are getting as 11 by 12 M a square I 2 as. Now, if we assume m to be 1 unit and a to be 1 unit. So, it may be either you can consider in c g s unit or a si unit so, I am not naming it either the m as kg and a as meter or may be gram 1 centimeter. So, this a generalized notation I am using here so, m is equal to 1 unit a is equal to 1 unit here. So, therefore, I 1 will be 11 by 12 I 2 will be 11 by 12 and I 3 will be equal to 1 by 6. So, this is yourall these are the three principal movement of inertia.

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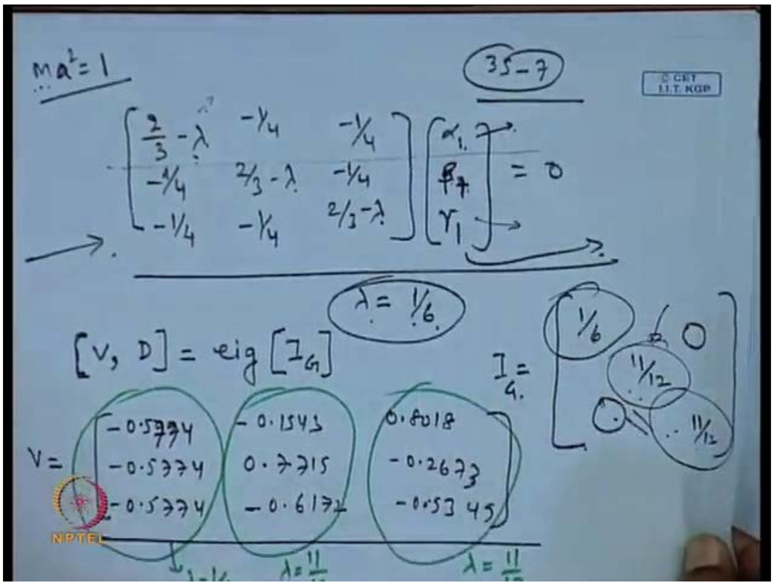

Once we have done this so, next our objective will be to find out their corresponding direction. So, we know already that we have to solve for M a square, M a square now it as got reduced equal to 1 because of 1 unit choosing a as 1 unit and M as 1 unit . So, therefore, we have we can write the equation as 2 by 3 minus lambda and 2 by 3 minus lambda 2 by 3 minus lambda here minus 1 by 4 minus 1 by 4 minus 1 by 4.And let us say alpha 1 alpha 2 or alpha 1 beta 1 and gamma 1 these are the 3 Eigen vector corresponding to lambda equal to the first 1 we can take as first 1 we can take as 1 by 6.

So, if we kept this value to be 1 by 6 and then we need to solve. So, basically your inertia matrix is now, looking as this I G this is looking as 1 by 6 11 by 12 11 by 12 off diagonal terms are 0. So, we are choosing the original equation where we try to find out the that the find out this Eigen values? So, once Eigen values are determined now we have to we are set for finding out the Eigen vectors which are given by alpha 1 beta 1 gamma 1 this is 1 Eigen vector whose components alpha 1 beta 1 gamma 1 are to be determined.So, for doing this first you need to insert lambda is equal to 1 by 6 her in this place and then you need to solve and last time I explain how to do that you can choose one here you have two independent equation.

So, if this value will turn of this matrix will turn out to be similar once you put 1 by 6 into this you can check it. So, therefore, it indicates that only two independent equation you will get. So, for doing this what you need to do?You assume one and find other two and then solve for this Eigen vector. So, similarly, next time you take this one and in that case because this is the repeated Eigen value so, you will get only one independent equation for this case you will get two independent equation.While for these are the two repeated roots so, you will get only one independent equation and then you will you can find still you can find two independent vectors Eigen vectors.

So, if even if you have the repeated Eigen values so, you can have two independent Eigen vectors, but if you have the independent, but the converse is basically not true. So, here what we are going to do?That instead of working out the whole thing because it is going to take time our time for this particular part is limited. So, I will suggest you look into the wave lecture and for the time being you can use the command in the mat lab say V comma d inside the large bracket and this you can write as Eigen and the Eigen value matrix you can write here as I g. So, this will list you the V matrix as minus 0.5774 and this is minus 0 1 5 minus 0.6172 0.5345.

So, this vector this is one vector this is another Eigen vector this is another Eigen vector so, this Eigen vector it corresponds to your Eigen value of lambda is equal to 1 by 6.While this two they corresponds to lambda is equal to 11 by 2 11 by 12.

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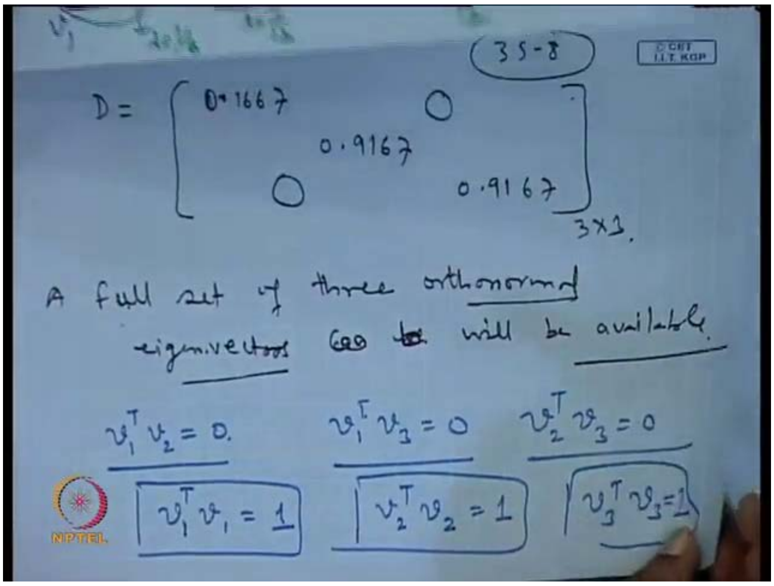

And the mat lab the output it will give the another output that will be d and you will find d to be 1 by 6 so, instead of 1 by 6 it will list it as 0.1667 the off diagonal terms being 0 this is 3 into 3 matrix. So, you get the Eigen values simultaneously by using this command?You will get the Eigen values and also the Eigen vectors. Now, as we listed in the properties of the real symmetric matrix we stated that, a full set of three orthonormal Eigen vectors will be available. So, of real symmetric matrix will have a property which states that it will have a full set of three orthonormal Eigen vectors.

So, this implies that that the Eigen vectors that we are getting here, this was let us say this is this is new 1 this is new 2 new 3. So, here new one new two new three, these are the Eigen vectors now, full set of orthonormal Eigen vectors implies that new 1 transpose new 2 this will be equal to 0 similarly, new 1 transpose new 3 will be equal to 0 and new 2 transpose new 3 will also be equal to 0. So, this is the orthonormality and orthonormal why this is orthonoraml because you can check, this is the V 1 transpose V 1 this you will find equal to 1 similarly, you will find V 2 transpose V 2 this is equal to 1 and V 3 transpose V 3 this is also equal to 1 so, this is the normalized Eigenvector.

So, as we stated that herewe can get a number of solution for alpha beta 1 gamma 1, but in the normalized form it will represent a unit direction and that is the direction of the principal Eigen value and which is the principal movement of inertia in our case.

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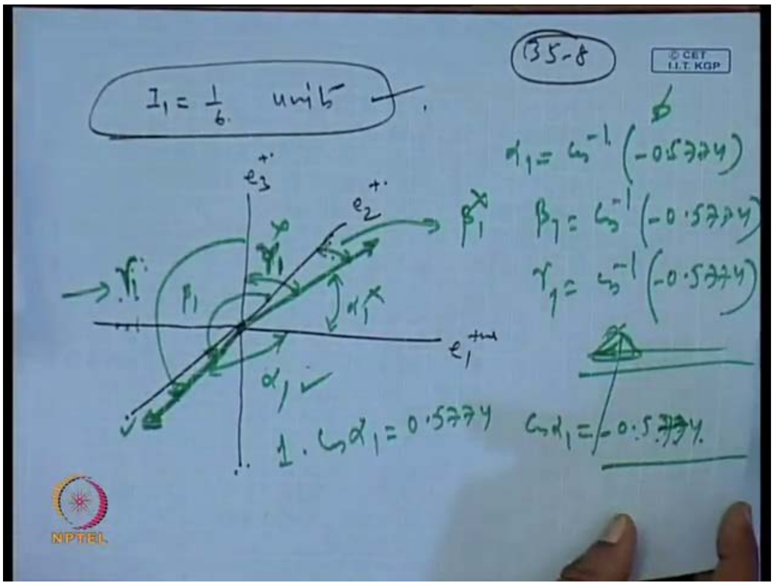

So, whatever, we are getting here you look into this Eigenvector. So, this is corresponding to lambda is equal to 1 by 6 so, this is the principal movement of inertia. So, I 1 is equal to 1 by 6 units so, this our first principal movement of inertia and the corresponding Eigenvector is indicated here and if you look from say this is e 1 e 2 and e 3 vectors. So, principal movement of inertia direction this is all with a negative sign here it is all shown with a negative sign. So, we will make it here, the this part these are the positive directions all of even e 2 e 3 positive it will go in this you can assume that there is a vector which is unit in magnitude.

So, unit magnitude vector will say this vector is unit in magnitude and the angle from here to here, this is say alpha, alpha 1 this angle sorry this angle is alpha 1 and this is beta 1 this metimeti 3. So, we have written this as the gamma 1 in the last lecture and this angle from with e 2 this is beta 1. So, alpha 1 beta 1 gamma 1 and this all three angles will be equal to they are all equal.So, now we do not have to take in this direction, but rather it is a all the signs are negative this basically, these are all cosigns alpha 1 beta 1 gamma 1 whatever, we are getting they are the all cosigns value. So, working out all of them you have to just tick in the negative direction. So, what is the part minus 0.577? So, you take along the negative X axis minus 0.577 along the negative Y axis this is the negative Y axis. So, it tick minus 0.577 and along the negative Z axis. So, you will get just a vector opposite to this and this is the principal direction which is being shown and now, you can see the angle that is now we can cut this. So, your angles are now from here to here this is your alpha 1 so, this is your direction e 1 so, with e 1 we are measuring alpha 1 and with e 2 directionthis is e 2 direction. So, with respect to e 2 we are measuring the angle then this angle is your beta 1 and from here we are measuring so, this angle is your gamma 1. So, that is how you locate?

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So, these are all here one thing I would like to say that these are all cosign values means, alpha 1 we have written. So, instead of writing alpha 1 we could have written here as cosign value as Cos alpha 1 as Cos beta 4 and Cos gamma 1 beta 1 and Cos gamma 1. So, these are whatever you are getting these are all the cosign values. So, if you write Cos inverse minus 0.5774. So, this gives you the value here alpha 1 and similarly, beta 1 you can get as Cos inverse minus 0.5774 and so on gamma 1 also you can write as Cos inverse minus 0.5774. So, so here you remember these are the cosign these are along with the cosign actually, I wanted to show you this is just a vector. So, this vector corresponds to this cosign value last time I have stated so, here I missed it so,now this part is finished.

You can similarly, in the same way you can take the next vector which is now here 0.1543 minus 0.1543 0.7715 minus 0.6172 and this corresponds to lambda is equal to 11 by 2.In the same way you can find out this angles so, take care of that these are all negative. So, they are lying in the if it is negative means lying in the second quadrant and in properit proper sign value actually, you should always take while resolving it so, you take all the proper values and work it out.Another we have doing the same thing because you can see that these are the cosign magnitudes that is Cos alpha 1 equal to0.5774 if we are writing like this means, if I have a unit vector here if this is a unit vector.

So, one times Cos alpha 1 this will give you 0.5774 and this is basically, the intercept it is intercept on the X axis. So, this case it will come out to be negative from here to here and it is a value will be minus 0.5774. So, basically all this quantities whatever your getting here? These are the normalized values and therefore, they are indicating the intercepts on the X Y and Z axis. So, similarly, these are the intercepts on the X Y and Z axis so, if you use the proper placement of all this points. So, you will get the location of the principal inertia direction and principal inertia magnitude we have already get in this units in generalized units lambda is equal to 1 by 6 11 by 12 and 11 by 12 . So, this completes our discussion about the movement of inertia.

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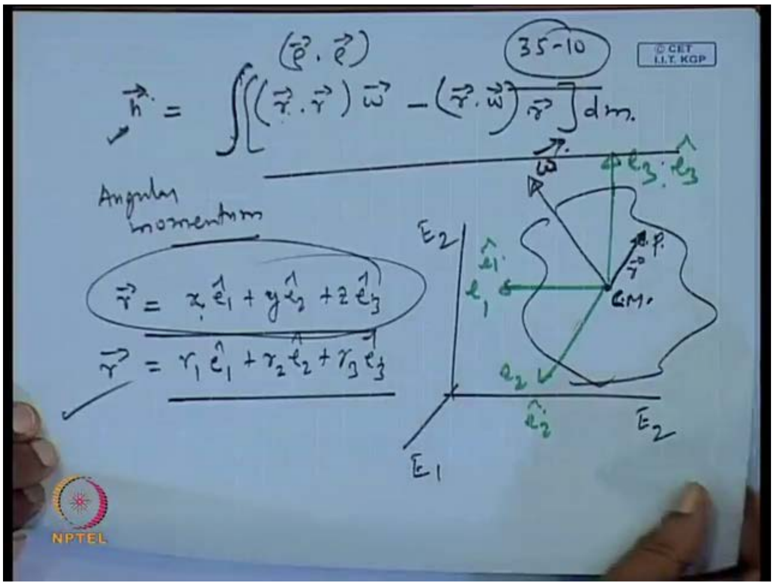

Now, if you look for the equation that we have started with h equal to r r dot and this we integrated over the whole body. So, this gives us the angular momentum vector, but we did not expand this equation completely and got it in a real form in which we have written the we try to work out the movement of inertia of this. So, this time I am going to finish this part say r equal to r r either we choose for as on the last lecture as row. So, either you can use the r or row we are we are considering this body any point p is there and this is the vector r from here to here and omega is the angular momentum with angular vector angular speed.

This is basically, the angular velocity vector and this we are talking as the center of mass and this was our principal inertial reference frame and we had another frame it was fixed in this body at the center of mass this was e 1 e 2 and e 3. So, these are the body axis and unit vectors in this direction accordingly we choose e 1 cap e 2 cap and e 3 cap. So, this was e 2 cap e 1 cap unit vectors and e 3 cap in this direction.So, r we can write as either as in the notation of x e 1 cap y e 2 cap and z e 3 cap we are representing this vector r in the body reference frame.What is the components of this?It is a written here or either the same thing we can write as r 1 e 1 cap plus r 2 e 2 cap and r 3 e 3 cap both of them are same it is a nothing difference only thing notations we have changed.

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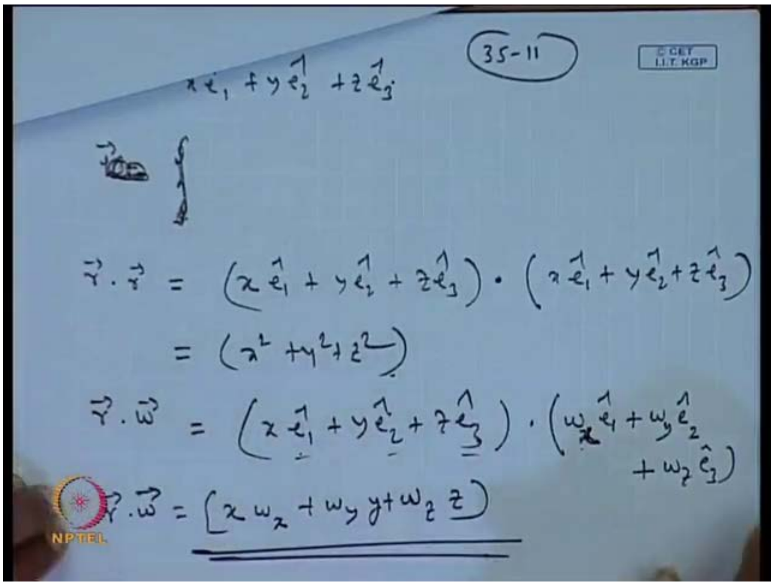

So, now taking of considering that r we are writing as x e 1 cap plus y e 2 cap plus z e 3 cap. So, the h vector can be written as, first of all we will work out the inside material the inside the integrand this integrand we first of all we will work out and then we will do the integration part. So, let us do that part r dot r this is x e 1 cap plus y e 2 cap z e 3 cap dot y e 2 cap plus z e 3 cap take the dot product. So, this indicates into this form similarly, you have r dot omega. So, r dot omega e 1 y e 2 z e 3 cap dot omega 1 e 1 cap or omega 1 instead of writing omega 1 I can write also as omega x to indicate. So, basically if you use this as small x small y and this is a small z.

So, we can use this notation omega x omega y and the unit vectors are a still we can keep the same notation quite of in the unit vectors are in the x direction indicated like x cap in the y direction as y cap in the z direction as z cap. So, what I was following with e 1 cap e 2 cap e 3 cap these are the generalized notation rather than writing in terms of x y and z. So, they represent any three orthogonal direction.So, any way if which ever we write it does not matter if the result will be same. So, this is omega z e 3 cap. So, this will give you x times omega x plus omega y times y and plus omega z times z.

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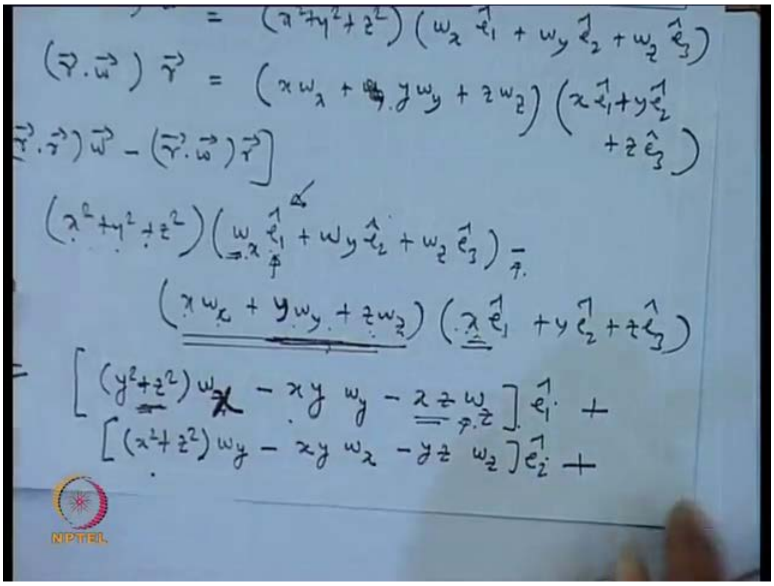

Finally, r dot r times omega this will be x square plus y square plus z square and from omega, omega x e 1 cap omega y e 2 cap and omega z e 3 cap.Similarly, we have r dot omega times r this is x omega x plus omega y times y of y times omega y and z times omega z times x e 1 cap plus y e 2 cap plus z e 3 cap and then we need to subtract them. So, r dot r times omega minus r r this quantity will be equal to x square plus y square plus z square e 2 cap plus omega z e 3 cap minusz e 3 cap and then this can be simplified.

So, if you simplify them this equation so, this will turn out to be y square plus z square equilip the terms similar, similar terms means you have to collect all the terms corresponding to e 1 cap and similarly, corresponding to e 2 cap from both this terms and write together.So, this will result in omega x y square plus z square omega x minus x y times omega ye 2 cap.

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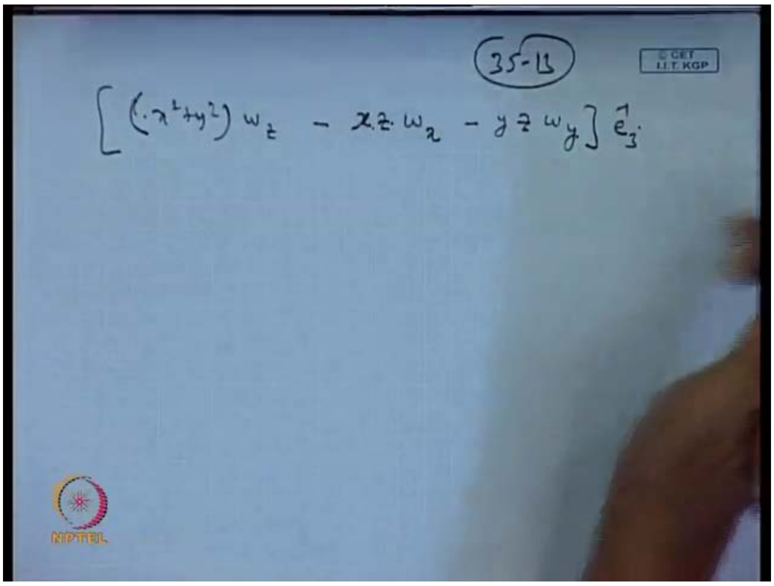

And the third term we write on the next page the third term will be so, y z x square plus z y square z square we have taken x square plus z square. So, this is basically a cyclic thingx times z.

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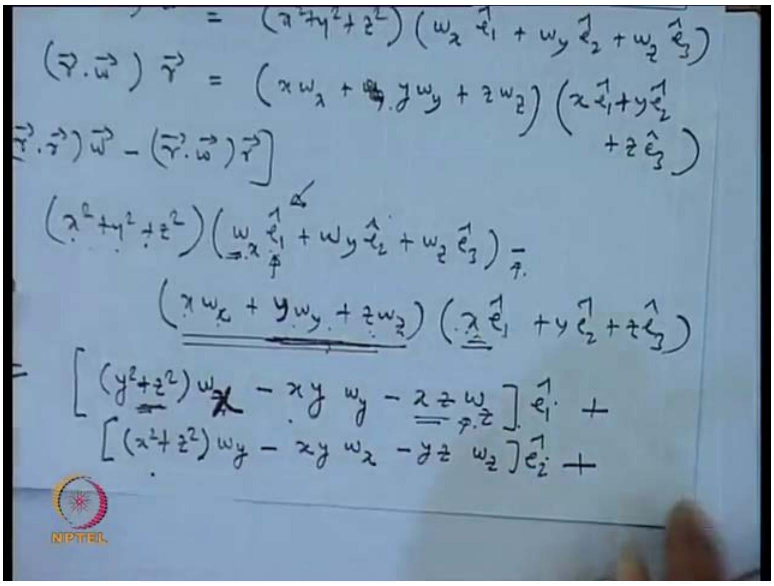

So, it is easy doing from here what we have done e 1 cap term we have to take out. So, you multiply this whole thing with omega x so, e 1 cap will come from this place and this term the whole thing multiplied by this x and it will come on this side. So, you will see that, here x square e 1 term is appearing times omega x and also, here you have the term omega x times x square. So, this two terms canceled out and you get y square plus z square which was appearing here as the first term, multiplied by omega x and rest other terms which come from this place.

So, y omega y?This multiplied by x so, x y times omega y so, x y times omega y with a minus sign which is appearing here and similarly, x z times omega z. So, x z times omega z which is appearing here. So, we are just collecting the terms together and writing here in this place.

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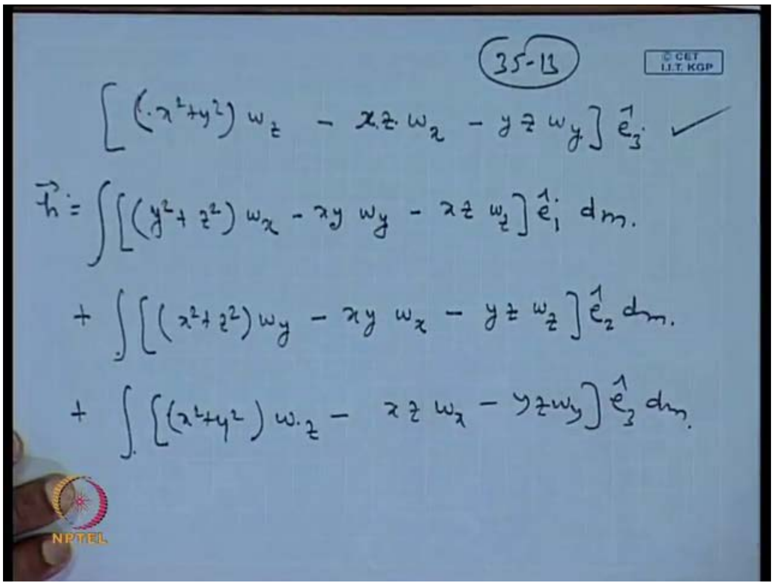

So, once we have written this soh can be written as inside the integration sign y square plus z squaree 1 d m e 1 cap d m plus,

to with last term we wrote as this is the last term. So, x square plus y square omega z minus x z omega x minus y z omega y e 3 cap d m.Now, the quantities which are present here so, you can easily identify them.

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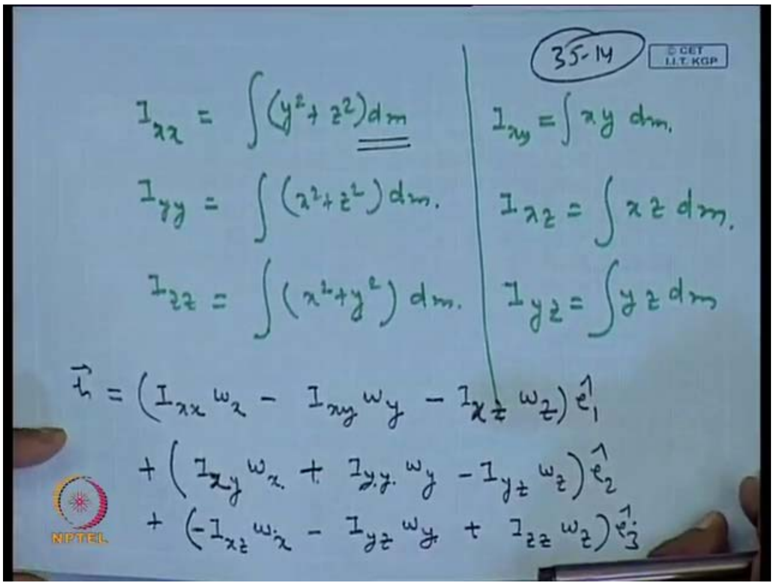

So, you can write I x x equal to y square plus z square d integration over the whole body similarly, you can write. So, this is the first term we have picked up this is the second term so, this x square plus z square. So, this is the term which indicates I y y. So, x square plus z square times d m similarly, I z z this is this term other you can identify as the cross product of inertia. So, the other terms, we will write as like the x y times d m this will write as I x y and similarly, I x z we write as x z d m and I y z we write as y z d m. So, this equation that we have written from here to here this will get reduced to h you can write as.

So, omega your doing the integration over the body not over the omega you remember. So, this can be taken out so what we can write here? I x x times omega x minus I x y times omega y minus I x z times omega z times e 1 cap plus again you have to take this term will be taking first I x y times omega x plus.Then this term we choose and here we write as I y times omega y and then the last term here it is I y z. So, this within negative sign times omega z e 2 cap and similarly, the third term we can write asI x z with a negative sign here omega x minus I y z times omega y plus I z z times omega z e 3. So, we have just re arrange the terms which are present here and written it in this format. So this what appears in the vector notation?

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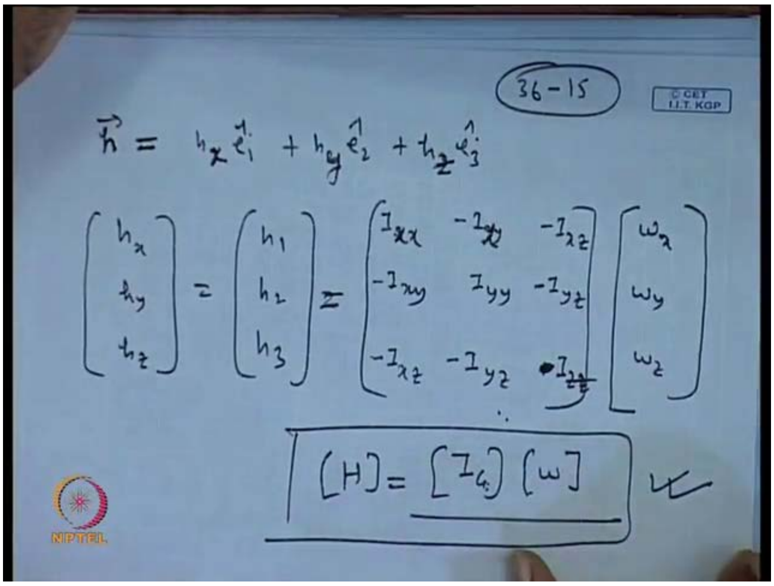

So, h can be written as h 1 times e 1 plus h 2 times e 2 plus h 3 times e 3 cap or either you can write this as hx, hy and hz. So, these are the 3 components of the angular momentum vector and in matrix notation the same thing you can write as hx, hy, hz or either in terms of h 1 h 2 h 3 this as omega x or omega 1 which ever you want to use and put all the terms from here to here in this place. So, this will give you I x x minus I x y minus I x z here you get minus x I x y I y y and minus I y z similarly, here you get minus I x z minus I y z and plus I z zso, this is your inertia matrix. So, this is your inertia matrix I G times this is the omega vector and this you can write as the angular momentum vector. So, matrix notation this is a very compact representation and this is what we have written earlier?

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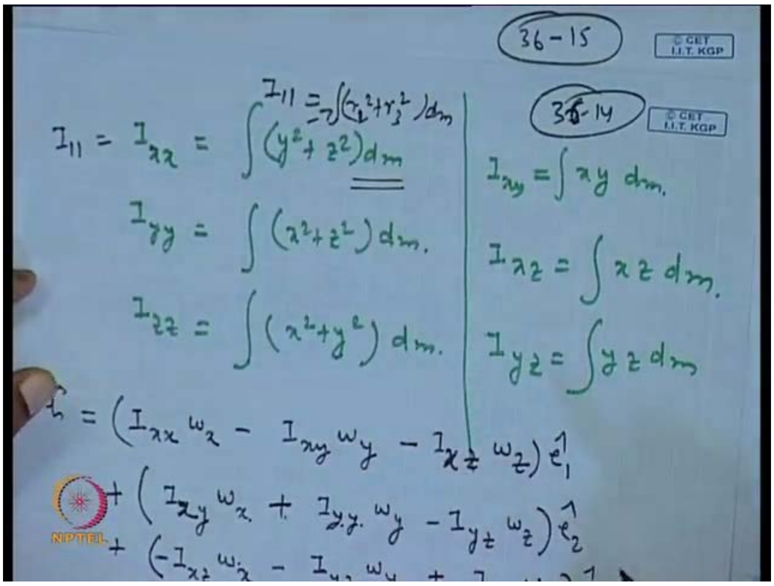

So, earlier without deriving this equation we have used it to find out just we wrote this as I 1 1 and then we worked out this as writing this as r 1 r 2 square plus r 3 square dm and this we integrated and wrote as I 1 1.So, same notation you can do for all other things. So, what we see that whatever, we used earlier this can be derived in this way.

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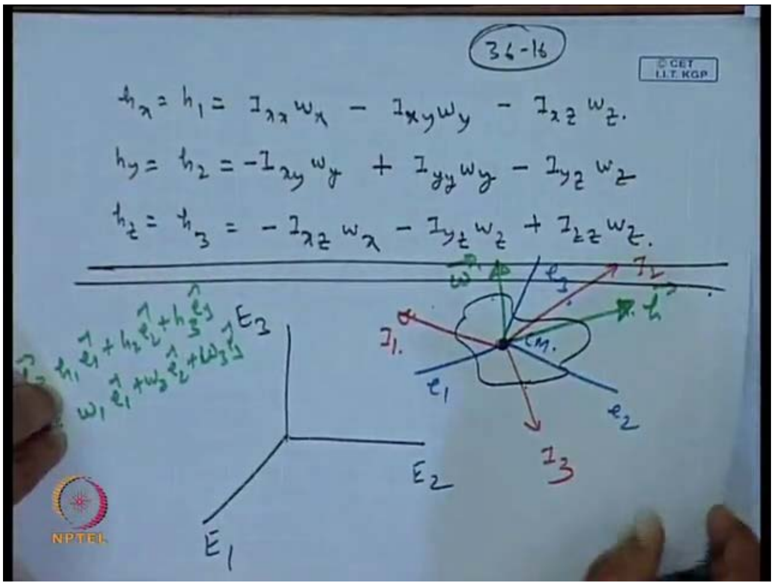

So, you see that from this place yours h x or either h 1 which your writing this is nothing, but this particular quantity from here to here. So, this is your I x x omega x minus I x y omega y minus I x z omega z and this you can also get from here if you do the matrix multiplication this is 3 into 3 matrix and 3 into 1. So, working out in this way we have goty times omega y minus I y z times omega z and omega x minus I y z times omega z and this is plus y z z times omega z.Now, once we have done this so, our objective main objective is objective is to find to compute the Euler equation to find out how the Euler equation you might have hard earlier.

So, if a torque is acting on the rigid body a rigid body is given an if you applied torque to this. So, what will be the change in the rotational rate of the body? So, we say we have the inertial acts here E 1 E 2 and E 3 and I have a body here in which a set of x e 1 e 2 and e 3 are fixed this is the center if mass. So, if I apply a body first of all you remember that we have got the moment of inertia this angular momentum about this center of mass.This is our angular momentum vector and this is the angular velocity vector.So, in general your angular velocity vector and the angular momentum vector they are not in the same direction which is evident from.

So, from this equation itself you can see here, your h vector is given by this here the omega is getting multiplied and there after your writing, while your angular velocity vector is written differently. So, h vector your writing as h 1 e 1 cap plus h 2 e 2 cap plus h 3 e 3 cap while omega vector you can write as omega 1 times e 1 cap plus omega 2 times e 2 cap and omega 3 times e 3 cap and where this h 1 and omega 1 they are not equal in general. So, therefore, this h and omega vector they are not in the same direction when they will consider?They will consider only if the rotational axis x is also the principal x. So, principal x fundamentals we have already got that if in this matrix this is the inertia matrix if this off diagonal terms they all become zero.

Living only the diagonal terms then we call them as the principal directions. So, you given a this is a real symmetric matrix so, given any matrix you can always reduced into a diagonal matrix and you can live the off diagonal that is you can live the off diagonal terms 0. So, in that condition now, you have this body and then the principal x direction you can show as say let us say this is the I 1 direction this is the I 2 direction and this is the I 3 direction.So, until unless your rotation is along any of the principal x. So, if this body is rotating about this principal x here this I 1 we have written so, if it is rotating about this x so, omega vector is conceding with this only then the angular momentum vector and the angular vector both will be same.

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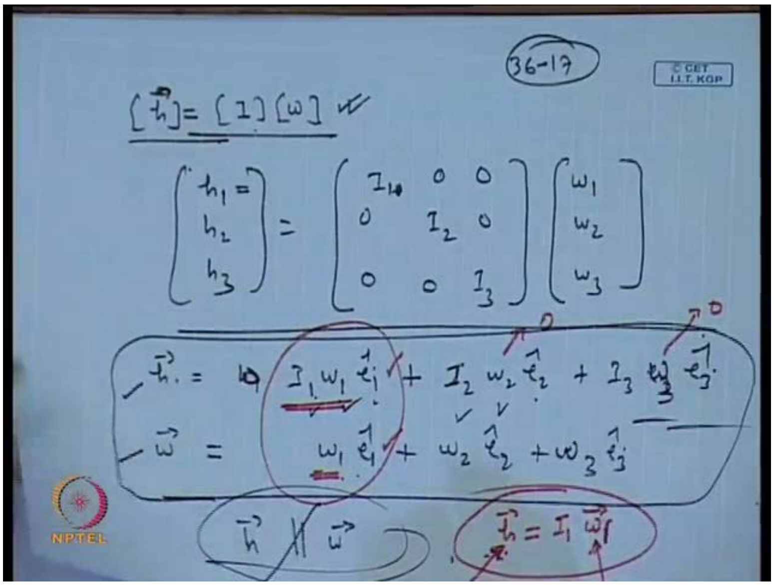

So, here in general we write this as h we are writing as I matrix and times in this format; obviously, this is not conducive for here this is the format what we have used here. So, in this format you can see that the off diagonal terms are present so; obviously, you do not have the principal x consideration here. So, we need to reduce it first into that form. So, let us write this as h equal to h 1 h 2 and h 3 and we write this as I 1 I 1 I 2 I 3 rest other terms 0 and here we write as omega 1 omega 2 omega 3. So, still you can see that h vector you can write as I 1 times omega 1 e 1 cap I 2 times omega 2 e 2 cap plus I 3 times e 3 omega 3 times e 3 cap.And from where we are taking this?

We are taking from this place this is what we are using herethis is finally, what we used. This is your h 1 term this term is your h 2 term and this term is your h 3 term. So, if your off diagonal terms are 0 means this goes this goes so, only I x x times omega x. So, I x x means instead of writing this we have written as the principal inertia as I 1 and this omega x we have written as omega x we have written as omega 1. So, this 2 terms drop out living at e 1.

The next term we taken this term will be 0 of diagonal term this will represent. So, this we have written as I 2 times I and omega y 2 we have written as omega 2 times e 2. So, if you see this while this is the angular momentum vector and omega vector you can see we write as this we have written always in this format omega 3 e 3 cap. So, from here itis a very clear that the h vector and the omega vector they are not parallel. So, h vector and omega vector they are not parallel.For them to be parallel means, the condition in which you can write h equal to omega and let us say I defined this as I 1 times omega 1 h is equal to I 1 times omega 1.

So, what does it imply?If I try to write like this, if I can make omega 2 0 this become zero this become zero. So, thistwo terms will vanish what you get?Here I 1 times omega 1 e 1.Similarly, here omega 2 is zeroomega 3 is zeroso, you can see that omega equal to omega times e 1. So, in that condition these are components here and here this is the unit vector multiplied by so, both are in the same direction. So, the angular momentum vector and the angular velocity vector they will get align or they are in the same direction if and only if the rotation is along the principal x.

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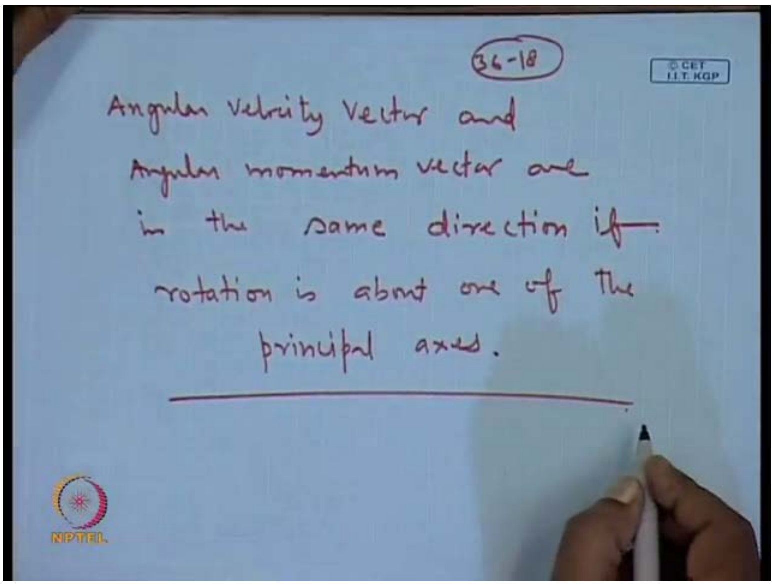

So, here we can state angular velocity vector and angular momentum vector are in thesame direction, if rotation is about one of the principal axes otherwise not. So, we will conclude our discussion here so next time we continue with this. So, next time we will develop the Euler's equation and also the kinetic energy of a rotating rigid body. So, these are the targets for the next and then we look into some small a stability problem for a torque free rotation rotating rigid body and the that will conclude our discuss in about this attitude dynamics. So, thank you very much.