Space Flight Mechanics

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Module No # 01

Lecture No # 35

Attitude Dynamics (Contd.)

Then in the last class we had been discussing about the attitude dynamics. So, continue with that and in that particular part we were discussing about the moment of inertia matrix out of inertia tensor. So, in that context we have been working out the properties of a real symmetric matrix.

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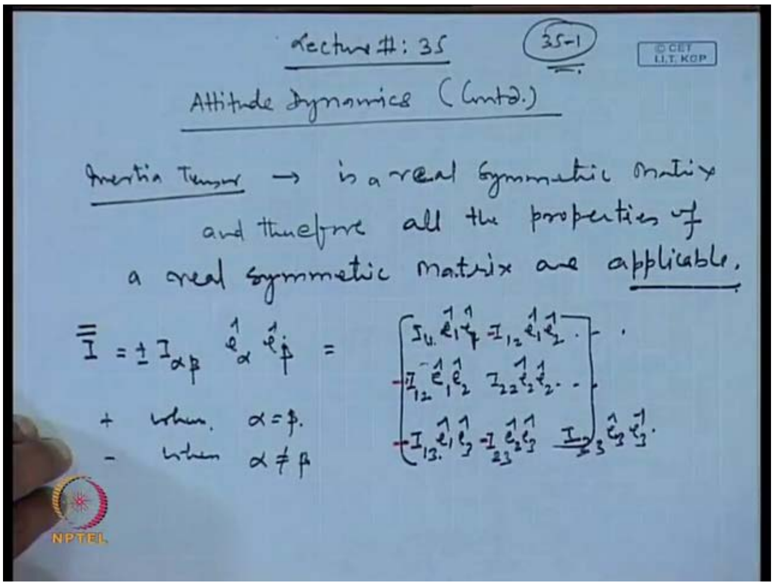

So, we saw that our inertia matrix or the inertia tensor is a real symmetric matrix. And therefore, all the properties of a real symmetric matrix is applicable here.All the properties of a real symmetric matrix are applicable.So, we had the; we wrote the inertia dyadic as I alpha beta times e cap alpha and e cap beta. And in that what we wrote as this I 1 1, e 1 cap, e 2 cap, I 1 2 e e 1 cap here and I 1 2 e 1 cap e 2 cap and so on. So, here in the next we had the I 1 2 e 1 cap e 2 cap and then I 2 2 e 2 cap e 2 cap so on. And the last entry here is I 3 3 e 3 cap, e 3 cap. And moreover as I told you that here minus sign you can observe in this place.

If you observe the minus sign so, what we need to do we can put here also the plus minus sign so, plus sign plus when alpha equal to beta minus when alpha not equal to beta.So, that way we will be able to observe the negative sign in all this places. So, only with the diagonal elements we have the positive sign and with the arc diagonal terms we have negative sign. So, here in this place we have I 1 3 with a minus sign here e 1 cap e 3 cap I am here putting a minus sign here, in this place and then we will have I 2 3, e 2 cap, e 3 cap with a minus sign.

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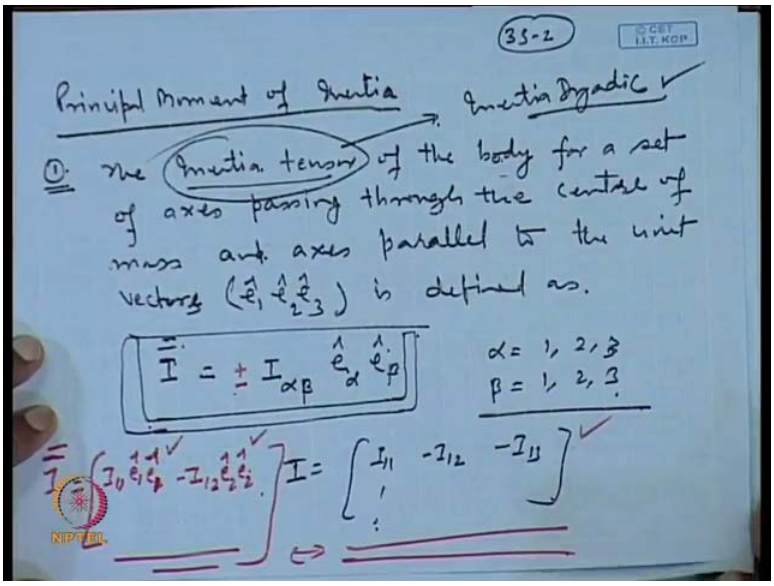

So, now are going to discuss about the principle movement of inertia

The inertia tensor of the body,

Through the center of mass, So, we have inertia tensor of the body for a set of axis passing through the center of mass and so, inertia tensor or either inertia dyadic and axis parallel to the unit vectors, e 1 cap, e 2 cap, e 3 capis defined as. And as I told you that for the out diagonal terms, if you want to remove use the sign in the for the order of diagonal time terms as negative. So, you can put here plus minus sign though this notation is not very usual, but this next is convenient in this case and a this notation becomes uniform with the other notations which are available in the literature.

So, here alpha equal to 1, 2, 3 and beta also equal to 1, 2,3. So, we if we are writing just inertia tensor means, inertia tensor is you can write as I 1 1, minus I 1 2, minus I 1 3 and so on.While we are talking about the inertia dyadic so, in that case you need to put here. So, if for inertia dyadic we put a double bar over this and then you need to put here I 1 1, e 1 cap, e 1 cap then I 1 2 and you can put a minus sign here as we have discussed here so, e 2 cap, e 2 cap and so on. So, this is the difference between these 2 here, the vector are also included while in this the vectors are not included.

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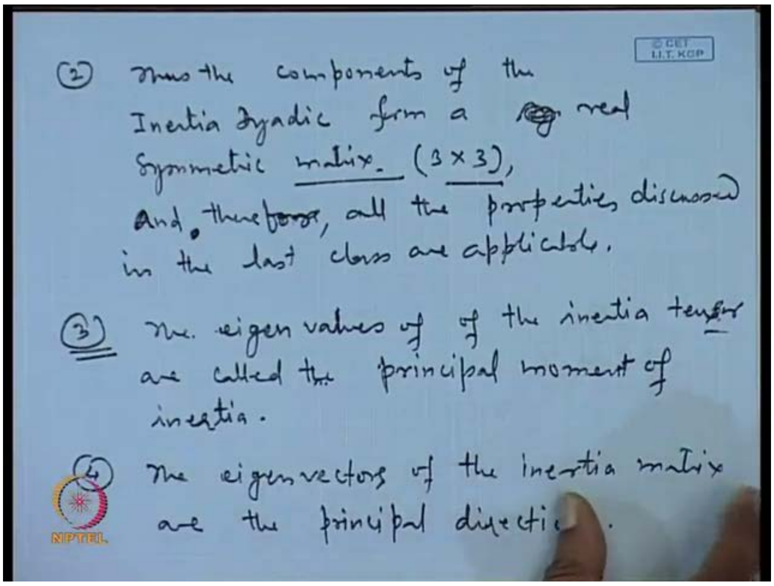

Thus the component of the inertia dyadic, form a real symmetric matrix. So, here we are talking about the magnitude part and this is a 3 into 3 matrixand therefore, all the properties discussed in the last class. Now, the Eigen values of the inertia tensor are called the principle moment of inertia and the Eigen vectors of the inertia matrix are the principle directions.

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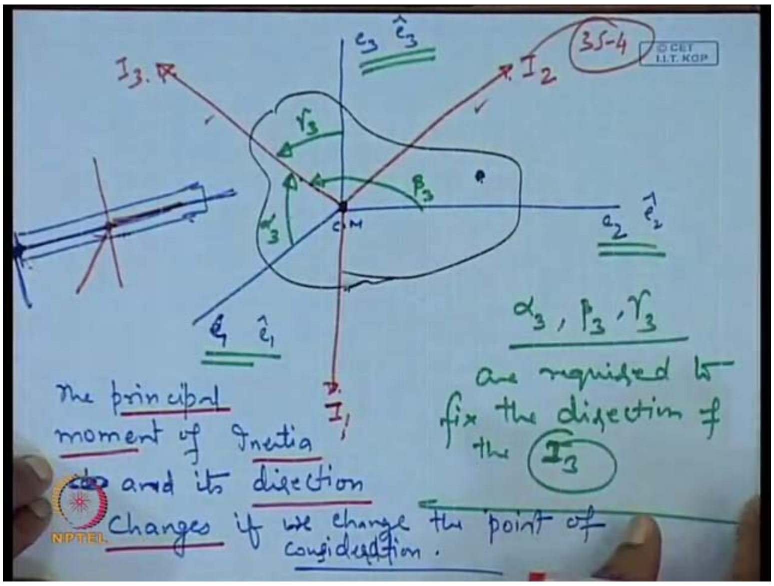

Like you can consider it in a way say, this is your bodyin which you have set body x at the center of mass where you are showing the e 1, e 2 and e 3 x and the unit vector in this direction is e 1 cap, e 2 cap and e 3 cap these are the unit vectors. Now, say your principle moment of inertia so, if for a real symmetric matrix you will get 3 values for the 3 Eigen values, you will get 3 Eigen values. So, they are indicating the 3 principle moment of inertia and the corresponding Eigen vector will tell you the direction of the principle moment of inertia.

So, say that you show one direction like this, another direction like this, another direction like this. So, this is your principle moment of inertia. Now, instead of writing I 1 1 we writing as 1, to indicate this is a principle moment of inertia I 2 and I 3. These are the 3 principle moment of inertia and they are directed along this 3 x now you can see from this figure that the orientation or say I 3. So, the orientation of I 3 can be defined using the angles, this angle from here to here, angle from here to here and angle from here to here.

So, defining this 3 angles are required and let us say that this angle is alpha 3 and this is beta 3 and this is gamma 3. So, angle alpha 3, beta 3 and gamma 3 are required to fix the direction of the I 3. So, I 3 is the moment of inertia and its direction is directed here and you need at least 3 angles to fixed this direction.So, similarly, you will have to define the angles for so, if these are your fixed states. So, these are the body state body axis which you are fixing in the body and this can be change an orbital.

And in this case the center point we have chosen as a center of mass, we chooses another point as the center of mass. So, about that point the principle of moment of inertia will become different it is direction will also will change, sorry the principle of moment of inertia direction will differ if you choose any other point we are considering here and if we take another point and about this point if we try to find out what will be the directions some of the principle moment of inertia they will be totally different.And similarly, the directions are different and also the moment of inertia, principle moment of inertia value will also change.

So, if here I am not drawing the lines and not too complicate this figure. So, to start with this is our figure. So, one thing you should remember here that what I stated the principle moment ofinertia and its direction changes if we change the point of consideration. That is very obvious that, whatever the inertia you are having about this point you cannot have this point. That is similar to say that if you are taking the moment of inertia about this axis which is the we are choosing first the point here. And then we are measuring the moment of inertia about 3 perpendicular axis, 1 axis is this and this and if you instead of doing this if you choose the point here and you have the directions like this.

So; obviously, had these 2 point the moment of inertia will not be same. So, moment of inertia along this 3 axis will be different and here it will be totally different from here. And correspondingly the moment of inertia principle moment of inertia also changes principle moment of inertia and this direction changes. So, here the principle moment of inertia will be something else and at this point the principle moment of inertia will be something else. Direction in this at this point for the principle moment of inertia as we have shown here it will be different at this point also this will be different.

So, the Eigen vectors are the principle directions as we have written and the fifth point we have discussed here. And this was the fifth point that has the point of consideration changes the principle moment of inertia and it is corresponding directions also change.

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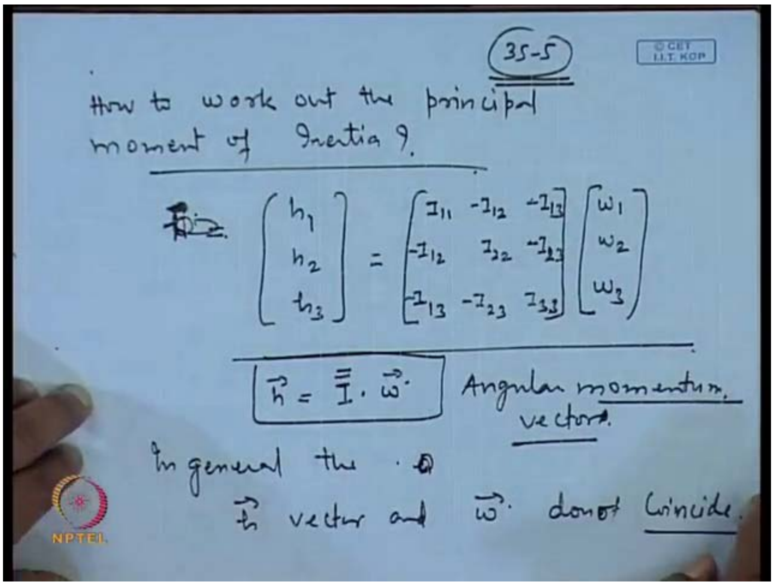

Now, how to work out the principle moment of inertia?

So, the angular momentum vector if you remember we have written as for the matrix notation the same thing h 1, h 2, h 3 this can be written as omega 1, omega 2, omega 3 and I 1 1.

So, in other word what we have done that, h can be written as this is what the notation we have written in terms of inertia dyadic. So, this is your angular momentum vector. So, in general the h vector and omega vector do not coincide.

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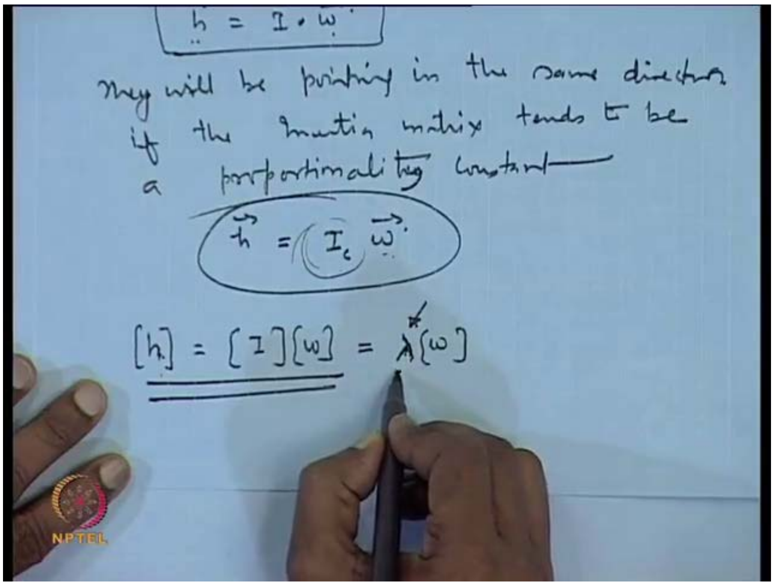

So, h is equal to I double bar, omega in this h vector and omega vector they are not the same.They will be pointing in the same direction if the inertia matrix tends to the proportionality constant.And in that case you can write h is equal to say some I c, this is the to indicate this is a constant times omega. So, in this case you can see that this is just appearing as proportionality constant and therefore h vector and the omega vector, they are pointing in the same direction.

So, h is a vector here to indicate we can write h is a as a vector here in this place and I is the inertia matrix and omega is a vector. So, to find out the principle directions, we can write this a as lamda omega where now,lamda is indicating a quantity by which if the vector is multiplied this will be equal to this, hence this will happen.

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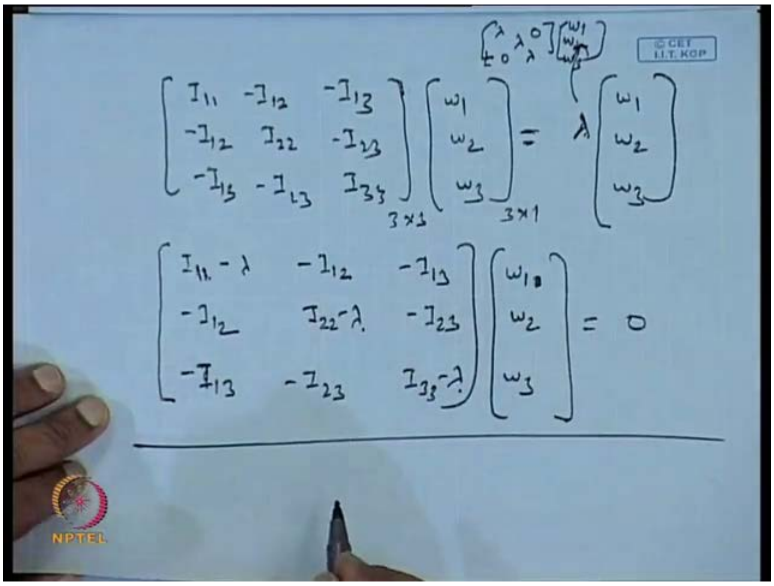

So, you can solve this problem. So, if you solve. So, you have the inertia matrix here I 1 1, minus I 1 2, minus I 1 3, omega 1, omega 2, omega 3 and this times lamda and this is omega 1 omega 2 omega 3.So, bring it on the left hand side. So, this whole problem can be indicated as I 1 1, minus lamda, minus I 1 2 it is a very easy doing this, omega 1, omega 2, omega 3.Because you remember this product this is a 3 by 3 matrix and this is 3 into 1 so that, this is a vector and here this also is a vector and therefore, you can subtract.

So, what you need to do? Just put this in the form of a matrix like lamda,lamda,lamda of diagonal terms being 0 and this multiplied by omega 1, omega 2, omega 3 and the subtract on this side.So, basically you are multiplying with identity matrix. So, if you multiply with your identity matrix you change nothing in that. So, this matrix is equivalent to this vector multiplied by lamda. So, no difference now you subtract it. So, this will quantity will turn out to be 0 on the right hand side. So, I 2 2 minus lamda, minus I 2 3 and minus I 1 3, minus I 2 3 and I 3 3 minus lamda.

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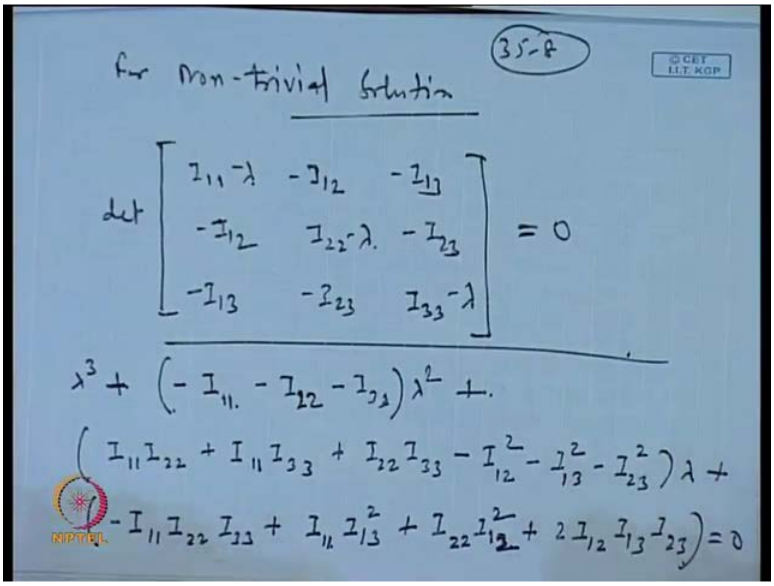

So, for now this is 35357 now, for non trivial solution the determinant of I 1 1 minus lamda, minus I 1 2, minus I 1 3 this must be equal to 0. So, now we can determinant of this matrix must be equal to 0. So, expand it and by expanding you can solve. So, you can write here, lamda cube This is what you get solve this?This will give you 3 real roots.

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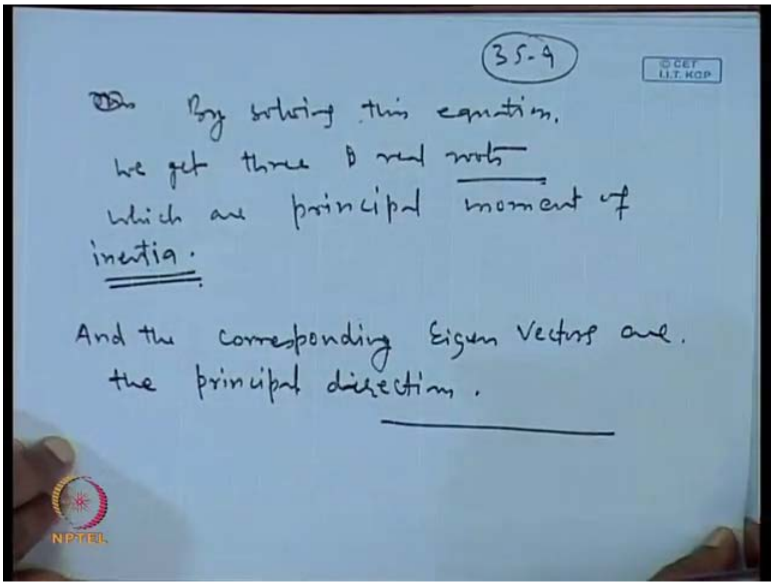

So, by solving this equation we get 3 real roots which are the principle moment of inertia. So, this 3 roots are our principle moment of inertia and the corresponding Eigen vectors and the corresponding Eigen vectors are the principle directions.

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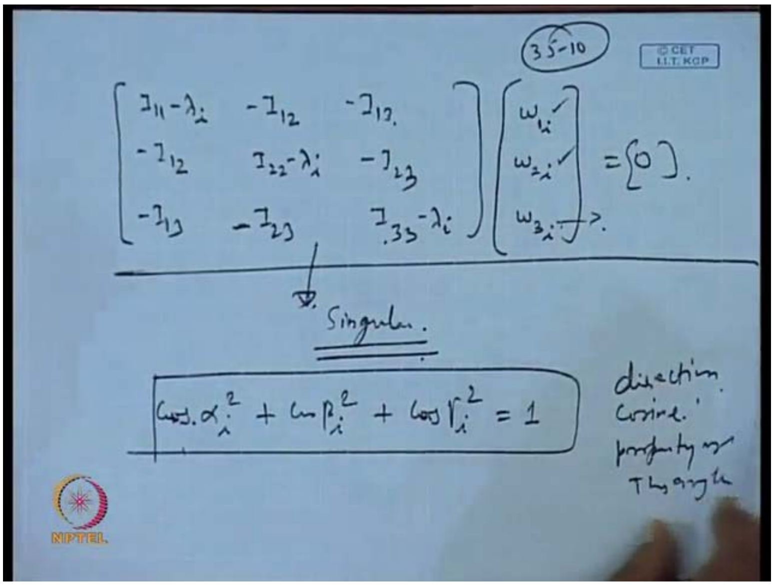

So, if this sec equation that we wrote earlier I 1 1 minus lamda and let us put here instead of lamda now, the lamda I this is minus 1 2, minus 1 3, minus lamda I, minus 2 3 and thenomega 1 we put the subscribe I that is the Eigen vector corresponding to the I t h Eigen value. Now, this matrix, this is similar what is mean by similar?Similar implies that the determinant of this matrix will be 0 and for the non trivial solution that is what we did here. We found the determinant of this equated or determinant of this 2 0 and then worked out the principal Eigen values.

So, if this is a similar matrix so; obviously, you cannot solve for omega 1 omega 2 and omega 3 simultaneously. Only two of them you can find and third is to be fixed because the third equation will turn out to be a linear function of, linear combination of the other tool. You can say in other way, that this two can be combined to give you the third equation in third line here. So, now, this is the situation so, in that situation how you will find the principle movement of inertia direction?

So, as I explained you earlier that the principle movement of inertia direction these are the 3 directions which are available to us. But 1 favoring thing is available that Cos alpha e square plus Cos beta I square plus Cos gamma I square this will be equal to 1 this is follows from this the cosine property. So, these are the direction cosines. So, square of them will turn out to be equal to 1. So, now, we if imposed this condition so, you will be able to fixed the third direction.So, it is telling that here that only 2 can be worked out only 2 elements omega 1 I and omega 2 I can be worked out and 3 I cannot be worked out because basically this is singular. So, to fix it up if we assume that this is to be followed then the whole thing gets solved.

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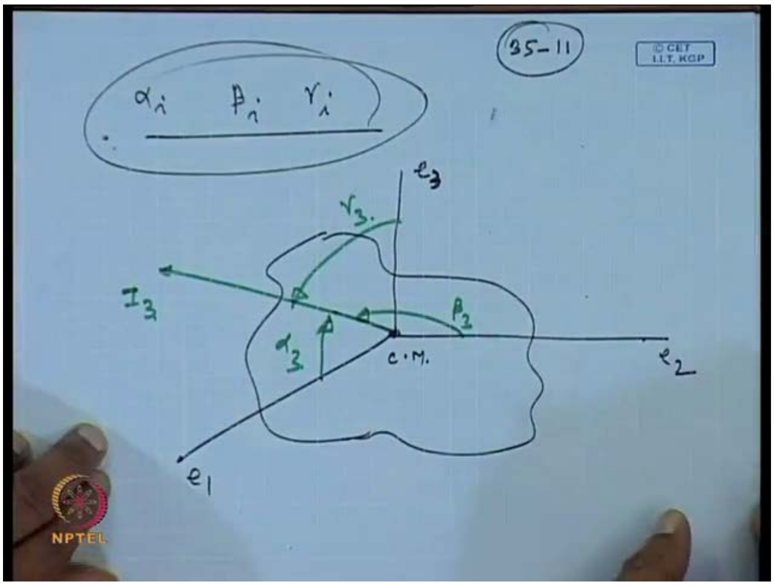

So, now, let us say see that how to write in the form of Cos alpha e, Cos beta e? So, if we have to find out the angles here, alpha I, beta I and gamma I. So, to find out the angles we go back into our original picture e 1, e 2 and e 3 these are the body h direction, this is the center of mass let us say that this is the principle inertia I 3 and this is the corresponding direction.We need to work out this angles we will write this as a alpha 3, this angle as beta 3 and this angle as gamma 3. Now, these are the 3 angles we have to decide and this 3 angles give the direction.

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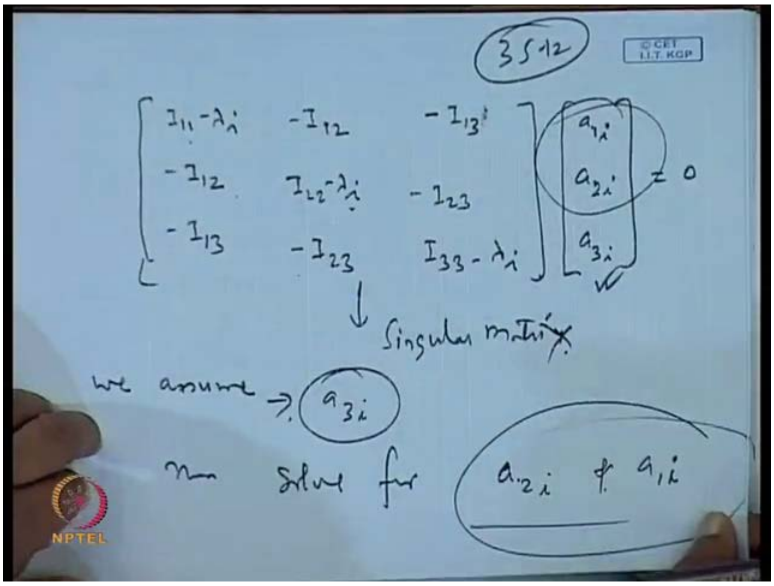

So, the matrix that we have used I 1 1 minus lamdaI, minus I 1 2, minus I 1 3 and instead of using here the Eigen vectors as omega 1 omega 2 omega 3 not to confuse with because this notation may confuse. So, let us choose here as omega 1 I which is corresponding to this particular I t h Eigen values. So, this is the corresponding Eigen vector I 2 i a 2 i and a 3 i. So, we can this associated with the omega for the time being So, this quantity equal to 0 and we have to we have seen that this is a singular matrix.And therefore, only 2 of them 2 equations will be independent if you form the equations 3 equations you can from here, but only 2 will be independent, 1 will turn out to be a linear combination of the other tool.

So, that simplify implies that only 2 of them you can find out independently. So, for the third one you cannot solve. So, in this case we assume the third value. So, we assume a 3 i or say you can assume a 1 i. So, in this case let us say that I am assuming a 3 i and then solve for a 2 i and a 1 i. So, once we solve for this. So, now, after fixing this value p you have a the particular equation coming from here lamda is known to you. So, you can fix this value and the solve the equation after solving you can solve the other two also.

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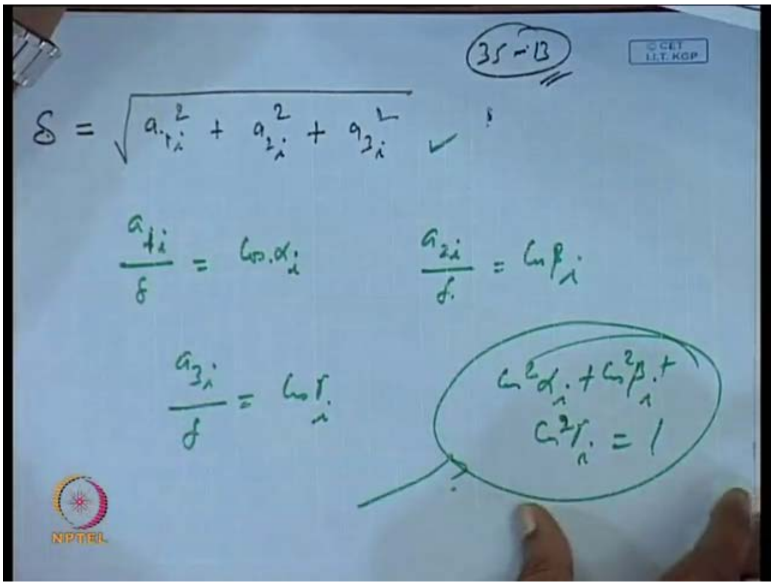

And thereafter what you need to do that, u s square a 1 i a 2 i s square and i 3 s square take s square root of this and write this as let us say a quantity which is delta . So, if in this equation if I divided by a quantity delta, it will not affect the whole thing this is various very obvious from here delta is a non 0 quantity. So, this delta we are using and if we divided by aa 1 i divided by delta. So, this we can write as Cos alpha i we have used.

Similarly, a 2 i divided by delta we can write as Cos beta i and a 3 i divided by delta this we can write as Cos gamma i.Now, you can check that by a squaring this quantities and adding this will give you 1. So, ultimately you are going to get Cos square alpha i plus Cos square alpha a plus Cos s square beta e plus Cos s square gamma e this is equal to 1. So, you have done basically the normalization of this quantities. So, in the normalized form this is what will appear. So, a i gets a i 1 the value which you choose the value a 3 i.

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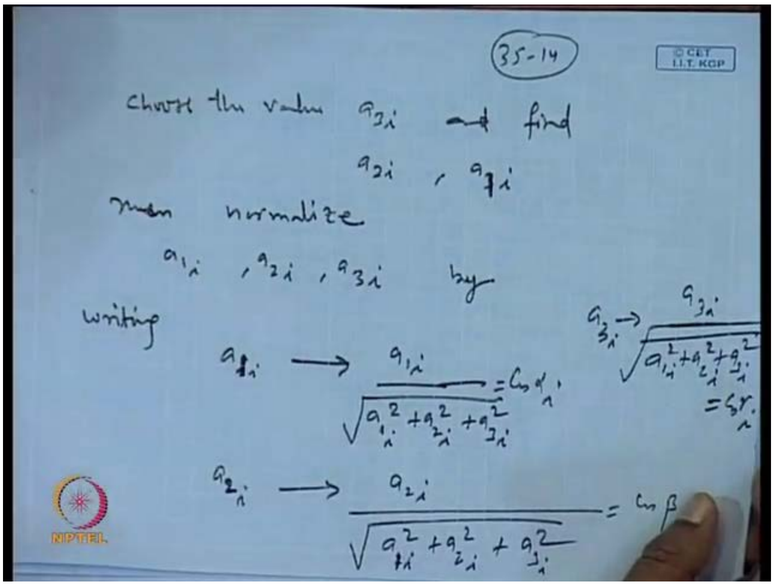

So, here let me write choose the value a 3 i and find a 2 i a 3 i a 1 i then normalized a 1 i a 2 i a 3 i by writing a 1 i you will replacing by i 1 i divided by a 1 i square plus a 2 i square plus a 3 i square under root.Similarly, a 2 i you will replacing as a 2 i divided by a 1 i square plus a 2 i square plus a 3 i square and similarly, a 3 i will be replacing with a 3 i divided by a 1 i square plus a 2 i square plus a 3 i square. So, this 3 quantities they are nothing but you are the cosine quantities Cos alpha a, Cos beta a and this quantity is Cos gamma a.

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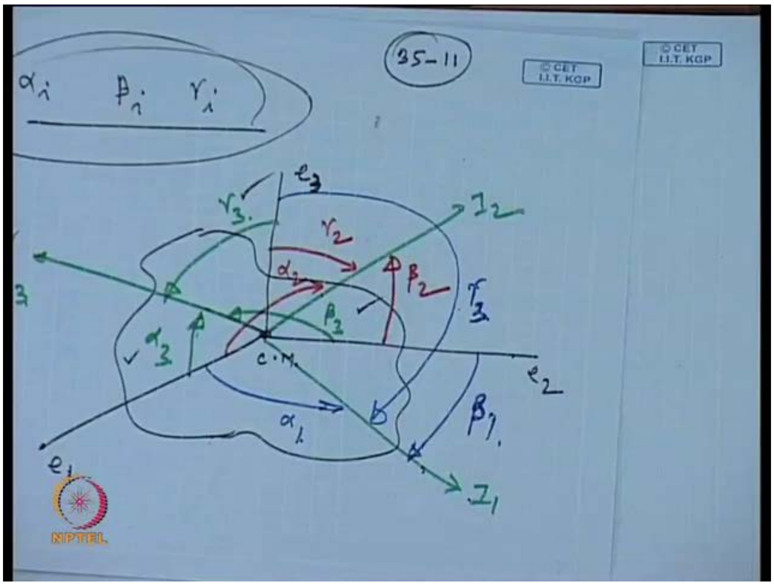

So, once this values are known means, your this 3 angles get located. So, this way you have located the 1 principle direction the i 3. So, in the same way you can choose the next direction. So, we can choose the next direction here let us say this is your i 2 and similarly, then you have work out for all this angles. So, here now this angle will be your beta 2 and this angle will be gamma 2 and the angle from here to here this will write as alpha 2.And then finally, you may have i 1 in this direction so, this I 1 also you need to fixed up.

So, here you will have this as alpha 1 and which is the direction angle from e 1 and major 2 i 1 and you have this angle as alpha 1, beta 1 and with e 3, the angle may this angle from here to here this is your gamma 3. So, by following the procedure that we have just describe, it is a possible that all the angles you can write here

Once we have done this.So, a very simple example can be taken out.

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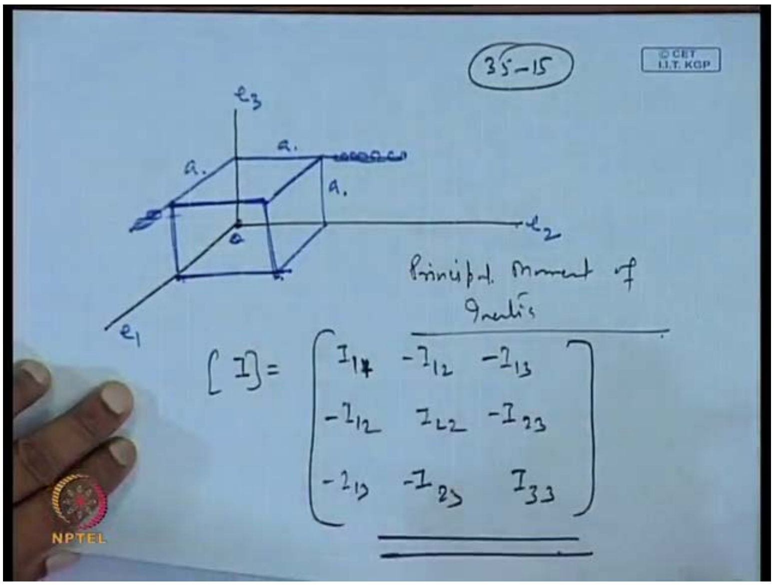

Let us say this is rectified and these are the body axis that we have chosen e 1, e 2 and e 3 this is the point out. Now, this point is; obviously, not the center of mass here in this case.And we need to work out the movement of inertia principle movement of inertia.So, finding out the principle movement of inertia first you need to write the inertia matrix. So, for writing the inertia matrix you have to get all this quantities I 1 1, I 1 2, I 1 3. So, first we need to work out these quantities.

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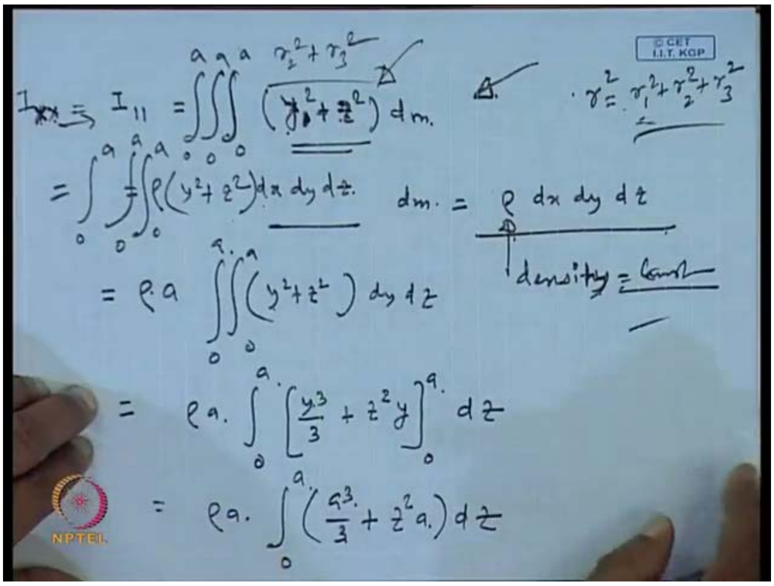

So, I 1 1 we can write as h 2 s square and x square plus sorry this is y s square plus z s square and d m and integrate it. So, we are going to develop this things later on we have not still developed how in the angular movement equation this term from, but the I 1 1 form can be written as this. So, once we know that little let us take it for granted that this term is being written in this fashion. So, once we have taken it up for d m can be written as along the 3 different axis you have for the; we have e 1 or here let us say this is we write as x y this as the z.

So, d m you can write as rho time dx,dy and dz, where this quantity is the density and this is the constant and say uniformly uniform density material you are taking. So, it is not varying along 3 axis, rho is constant throughout. But the same thing you could have written in terms of if you write in terms of say the r is r 1 square plus square r 2 square plus r 3 square, which are the component along the e 1 e 2 e 3 axis. So, you can also write the same thing instead of using this you could have written r 2 square plus r 3 square and it could have develop the whole thing.

So, let us follow this. So, once we do this you can also turn this as I X X. So, inserting this into this place y square plus z square rho will come here and dx,dy, d z. So, here we have all this quantities. So, at least, at first we can integrate how x out of this? So, rho we are taking out and integration we can put in the limits you can put the limits for all of them. So, integration limit is 0 to a, 0 to a, 0 to a. So, rho a and y square plus z square divided dz integrate between 0 to a 0 to a.Now, let us suppose that we integrate for y. So, rho a 0 to a this becomes y cube by y 3 plus z square y 0 to a 0 and while putting this value this a here in this place this becomes a cube by 3 plus z square a d z.

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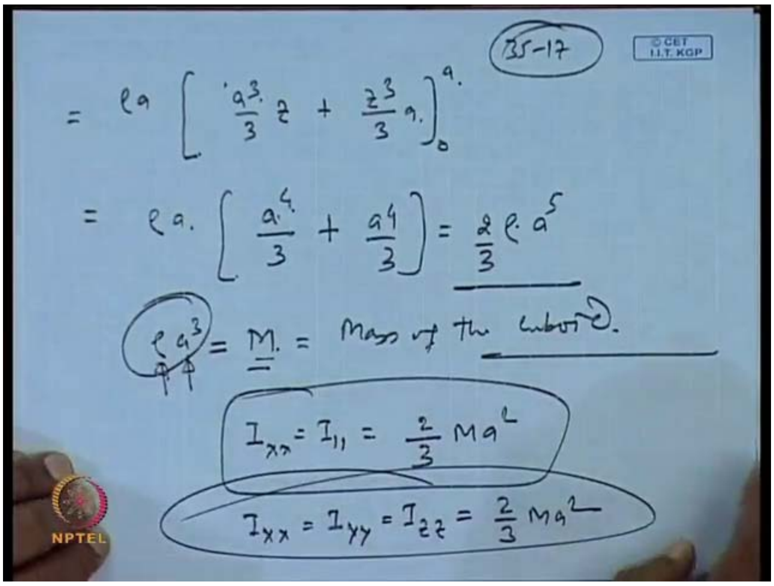

So, after integrating this will yield a cube by 3 z plus z cube by 3 a in putting z is equal to a and putting this limits. So, what we get here?a to the power 4 by 3 plus a to the power 4 by 3, it is basically 2 by 3 a to the power 5. Now, rho times a cube, this is nothing but your mass of the cuboid.a q is your volume multiplied by density this is at mass.So, we can write here that I X X equal to I 1 1 this is nothing but 2 by 3 M a square and by symmetry you can check that I X X equal to I Y Y equal to I Z Z equal to 2 by 3 M a square. This is obvious from the symmetry because its required if heard it when you parallel type or any other figure. So, this result you would not have got.

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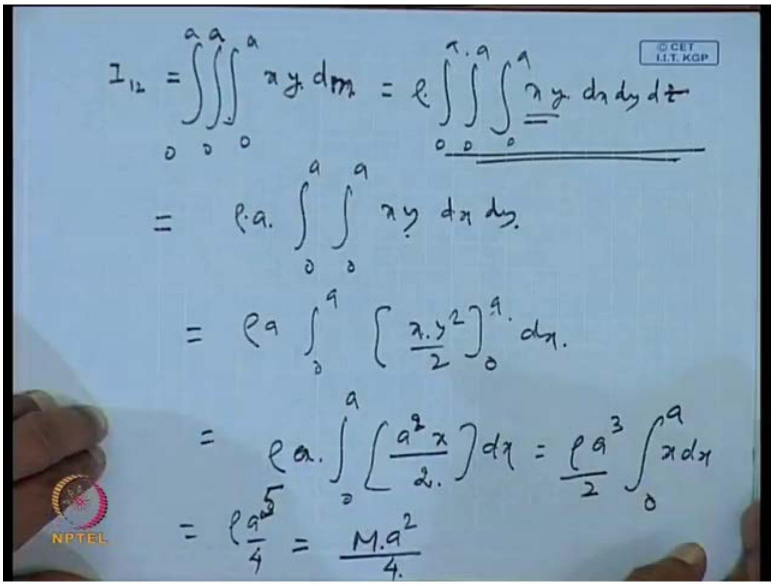

Next you need to work out I 1 2. So, I 1 2 here we can write as x y d m and d m; obviously, we have written as an integration limit remain same.x y rho we can take outside and this dx dy d z.

So, integration can follow the convinces so, here because z is not present in the integrands so, we can take out the z first. So, we can write this as a rho a 0 to a 0 to a x y dx t next we can integrate for y. So, if we integrate for y this is 0 to a x times y square divided by 2 0 to a d x. So, putting this values so, what we get here a s square x y 2 e s square we can take it outside, this is rho a q divided by 2 x dx and this is 0 to a. So, ultimately this will also yield a square by 2. So, this becomes rho a q by 4. So, a to the power 5.So, this becomes n a s square divided by 4 this is what you are I 1 2 and from the symmetry.

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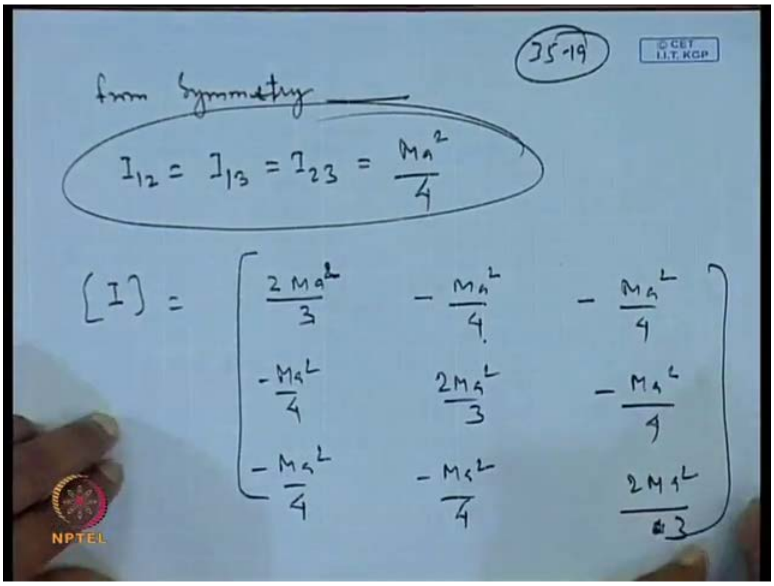

Simply we can write immediately from symmetry we can write I 1 2 equal to I 1 3 is equal to y 2 3 equal to M a square by 4. So, this is the first product of inertia. So, your inertia matrix is now available. So, your inertia matrix is basically 2 M a square divided by 3.

Now, this is your inertia matrix now; obviously, here you can see that M a square is common everywhere.So, for mathematical simplicity we can take out this as common.

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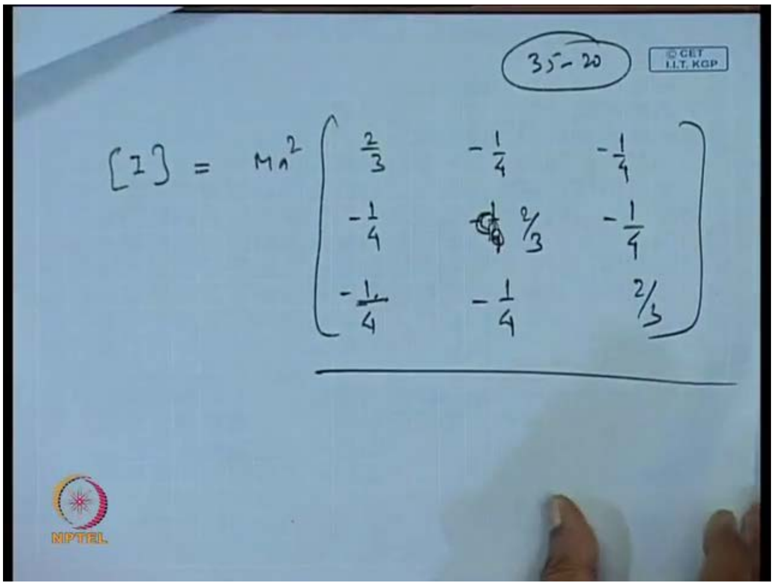

So, I can be written as M a square 2 by 3 and this is minus 1 by 4 minus 1 by 4this is 2 by 3 2 by 3.Now, this is your inertia matrix and you have to solve for the Eigen values.

So, for solving for the Eigen values what we can do? We can work only with this inertia this matrix here, which are having the numerical values and whatever the Eigen values results if we multiplied by M a square. So, result will be will we will obtain the required result or either we can keep it inside itself then subtract with a the corresponding lamda in the diagonal terms solve it and solve for the determinant.And there you get the Eigen values which are the lamda, where the M a square term will be directly appearing. So, both ways you can do, but this way; obviously, working with this is much simpler than looking in other way.

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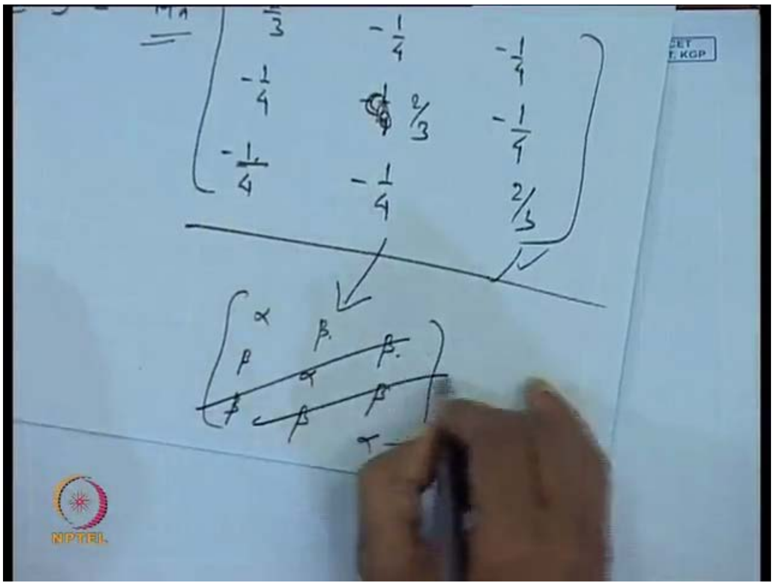

So, let us say that we write the quantity which is present here, this quantity as alpha, beta, beta sine we are observing inside this is beta, alpha, beta and beta, beta, alpha. So, this is the matrix now. So, we have to work out a for we will go one step more ahead.

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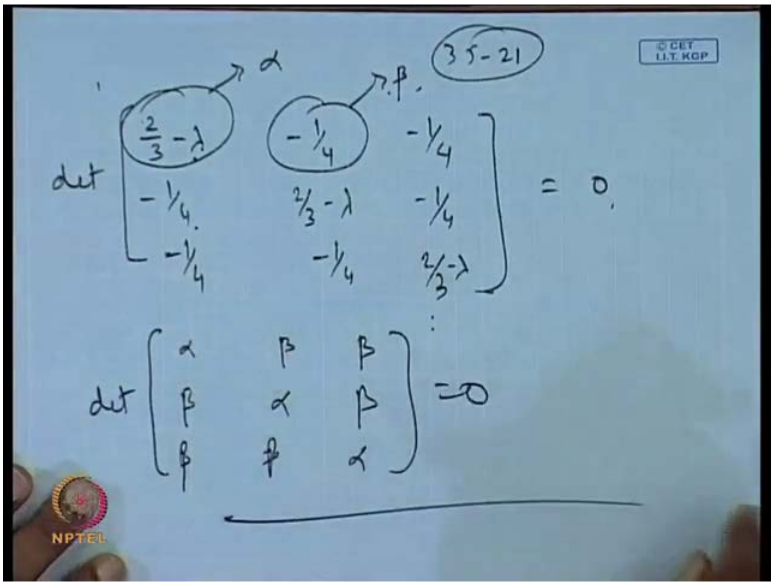

Will write here as 2 by 3 minus lamda, minus 1 by 4, minus 1 by 4 then minus 1 by 4, 2 by 3 minus lamda, minus 1 by 4, minus 1 by 4, 2 by 3 minus lamda.So, in this format so, we have take the determinant of this matrix and solve it 0 so, this quantity we can put as alpha and the quantity which are present here this we can write as beta. So, lamda already got embedded in this. So, we have the matrix now this is alpha, beta, beta and then beta, alpha, beta, beta ,beta ,alpha and fed to 0 solve it.

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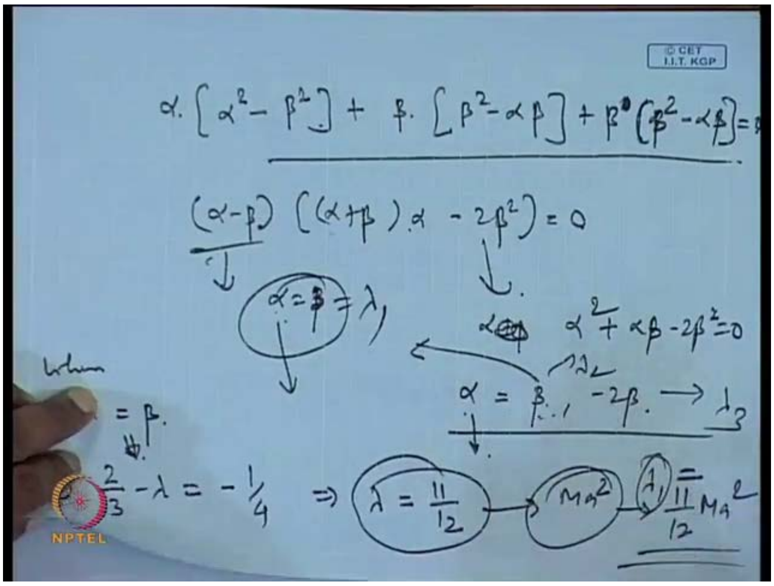

So, if you solve this determinant you can write this as alpha times alpha square minus beta square the taking the first term and then alpha square minus beta square plus beta times then we have this term beta square this is, the mid between we are taking. So, beta times beta square. So, beta square minus alpha beta and the third term with respect to this we write as beta square minus alpha beta plus beta s square minus b times beta s square minus alpha beta and this is equal to 0 solve it. And once you solve it what you will find that? You will get this equation and look this gives you alpha equal to beta and this will yield you alpha plus from this will yield you alpha square plus, alpha beta minus 2, beta square equal to 0.

So, alpha will be equal to either beta or minus 2 betaso, you have got this 3 quantities alpha equal to beta or beta minus 2 beta. So, alpha is nothing but when alpha equal to beta so, you can see that alpha is nothing but 2 by 3 minus lamda and beta you have taken as minus 1 by 4. So, alpha equal to beta implies, this quantity and this implies lamda equal to 11 by 12. And then if we multiply it by M a square so, the principle movement of inertia you get as 11 by 12 M a square. So, this is your say the lamda 1,lamda 1 equal to this quantity similarly, we can solve the other one. So, here we have the other two remaining.

So, this 1 is alpha equal to beta is lamda 1, this we can write as; obviously, here also this alpha equal to beta is coming and this is the minus beta. So, this is lamda 3, this is your lamda 2 so, rest we do in the next class thank you very much.