**Space Flight Mechanics**

**Prof. M Sinha**

**Department of Aerospace Engineering**

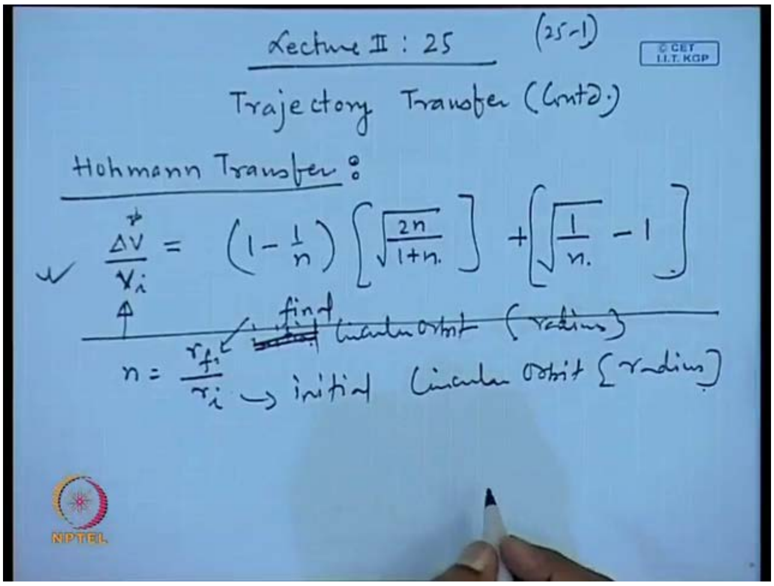
**Indian Institute of Technology, Kharagpur**

**Lecture No. 25**

**Trajectory Transfer (Contd.)**

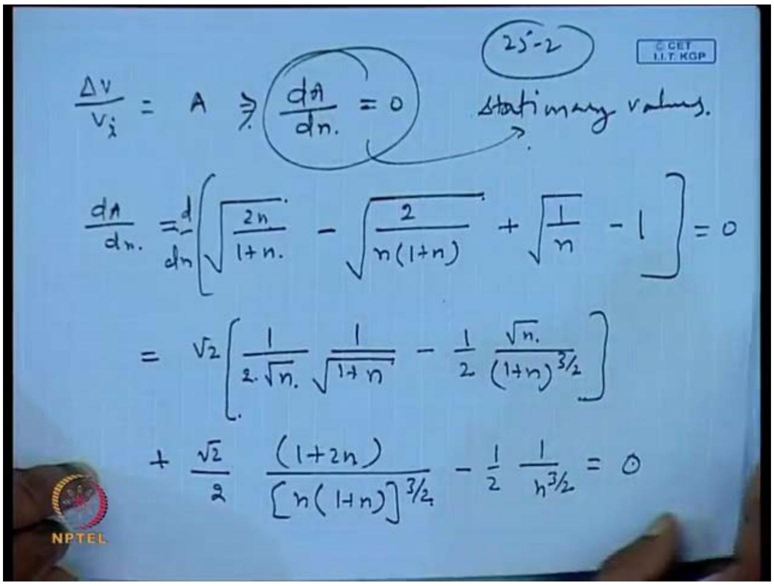
So, we have been discussing about trajectory transfer, so, we continue with that. So, last time, we had started working with the efficiency of Hohmann transfer and Bielliptic transfer. So, Bielliptic transfer we are still did not start.

(Refer Slide Time: 00:33)



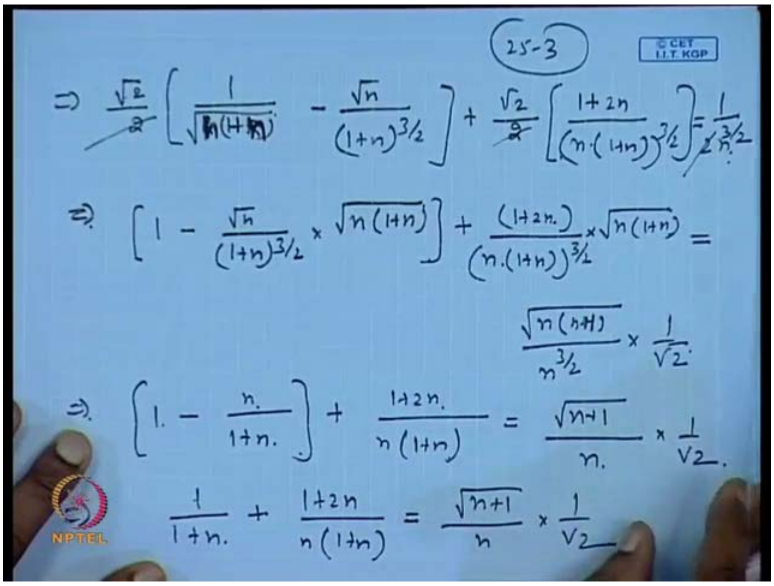

So, about the Hohmann transfer, we have worked out the equation that delta V, the amount of impulse required, that is given by delta V By V I, in fact, that was given by 1 minus 1 by n times 2 n divided by 1 plus n…So, this was the equation that we derived last time. Now, in this equation, delta V is the total impulse required, and V i is the velocity in the initial orbit, and where n, we defined as r f by r i, where r f is the radius of the initial circular orbit, initial circular orbit and (( )) the r f is the final circular orbit radius and r i is the initial circular orbit radius. Now, this equation gives you the amount of impulse required. So, when this impulse is going to be maximum, that we can deduct from this equation and that will give a, us idea that, the Hohmann transfer we are using, how useful this is.

(Refer Slide Time: 02:12)



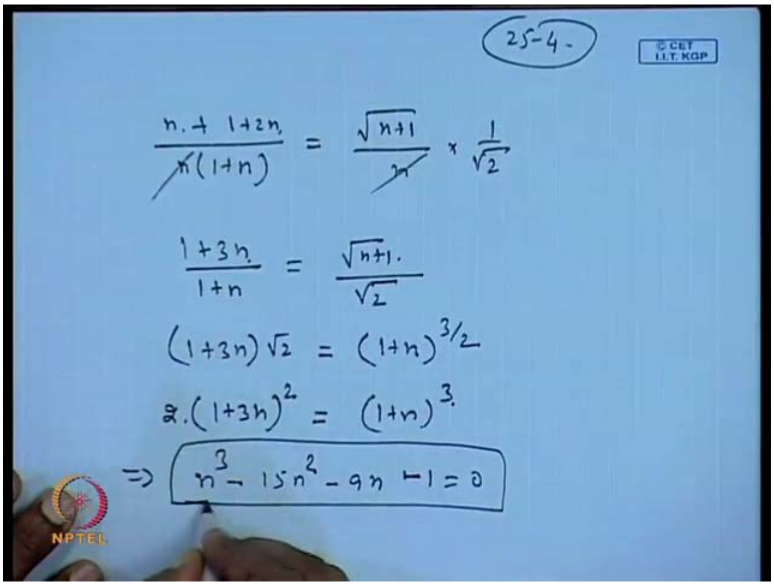

So, if we suppose delta V By V i, if we write this as a quantity A and then, differentiate this quantity with respect to n, and set it to 0, so, this will give you the stationary values. So, this gives equation, then, gives you the stationary value. So, we had started doing this. So, we will have d A by d n. So, by differentiating this, equation is very complex. So, let us work it out patiently and then, look into the result. So, first, we have the term, which is written like this. So, we will break it, to simplify the whole thing. So, by breaking, we can write, 2 n by 1 plus n under root, minus…So, the, the second term we are taking inside the square root sign. So, we have to differentiate with respect to n, this quantity. So, from here, what we get...So, we differentiate this first, taking this as the first term and 1 by 1 plus n as the second term, and this is under the square root sign. So, accordingly, you get this quantity here. So, next, we take this term and differentiate this, so, the, the whole term here, this, we can take as a single quantity and differentiate. So, this will give us under root 2 by 2…And next, taking this term. So, this, we are setting equal to 0. So, this quantity becomes equal to 0.

(Refer Slide Time: 05:37)



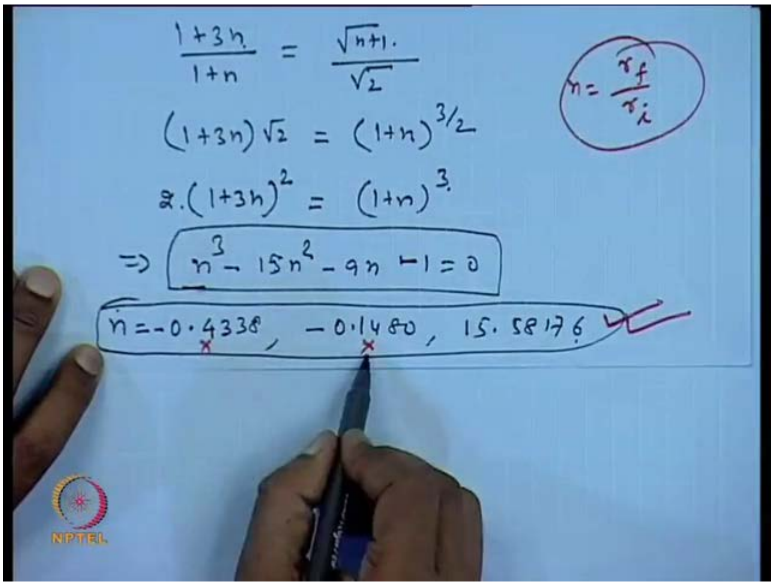

This implies, this we are using as n. So, 2, we can cancel from both sides and this under square root 2, we can take on the right hand side. So, what we get then…And moreover, few more simplifications we can do; this 1 plus n square root, this term, we can take from, because it is present here, in both the terms here, here and also here, in this place. So, we can take it on the right hand side, and the square, square root 2, this also we take on the right hand side; then, we get here…And what else we can do? And also, we can take under root n also, we can take on the right hand side. So, this will give us 1 minus under root n 3 by 2 times under root; root 2 root 2 gets cancelled out and it will appear here in denominator. So, here, we cancel out whatever possibilities are there; this term can be further simplified. So, this will yield us n plus 1 minus n.

(Refer Slide Time: 09:32)



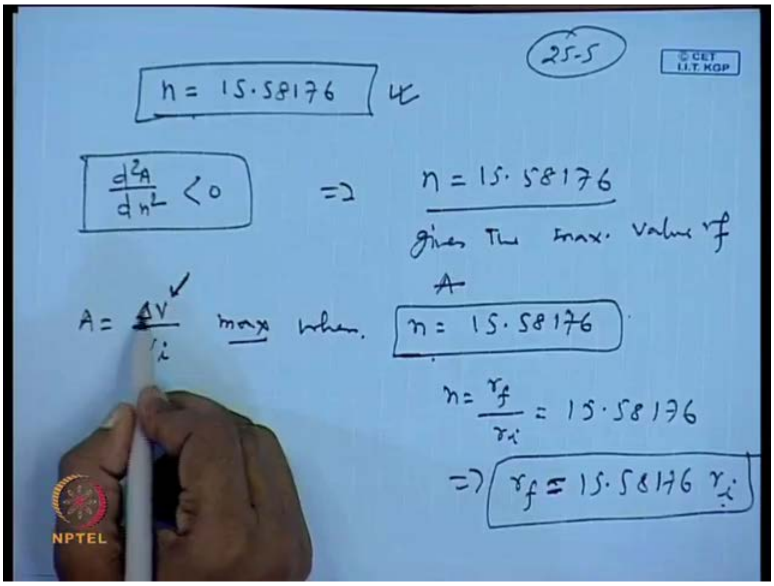

So, this is 1 by 1 plus n. This can be further simplified. This equation, we can further simplify it and this will yield us, n plus one plus 2 n divided by…This term, this term, this will cancel out and we get here 1 plus 3 n. So, further simplification of this…Now, this will give you certain polynomial. This can be reduced to a polynomial; so, after reducing that, then, we need to solve it. So, here, if we square on both the sides…So, what we get, 2 times 1 plus 3 n square is equal to 1 plus n whole cube. And, if you simplify it, so, this is going to give you a equation, which will appear as n cube minus 15 n square minus 9 n minus 1 equal to 0.

(Refer Slide Time: 09:32)



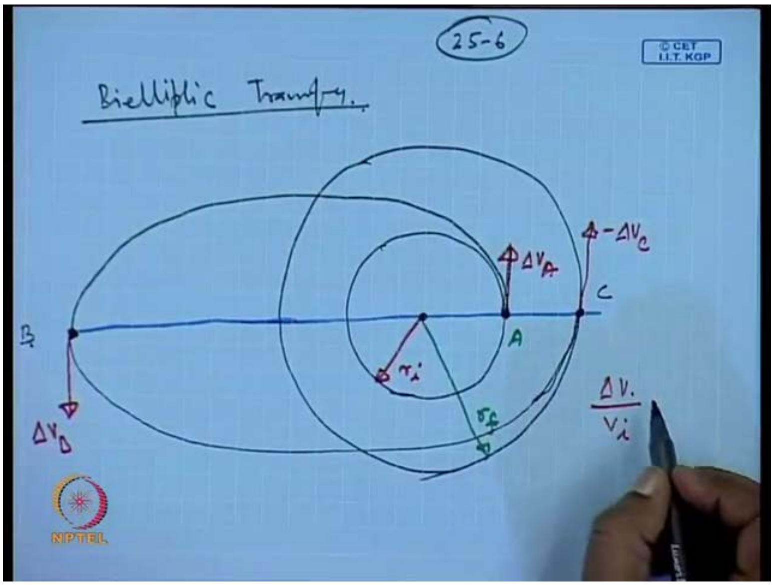
Now, in MatLab we can solve this equation; so, the solution to this n will appear as 0.4338; another root will be minus 0.1480, and another root will appear us 15.58176. So, now, in this solution, this two solutions are discarded, because it contains the negative sign; n is equal to, we have written as r f by r i. So, this cannot be negative. So, this two, we discard and we accept only this kind of solution here. So, 15.58176.

(Refer Slide Time: 12:17)



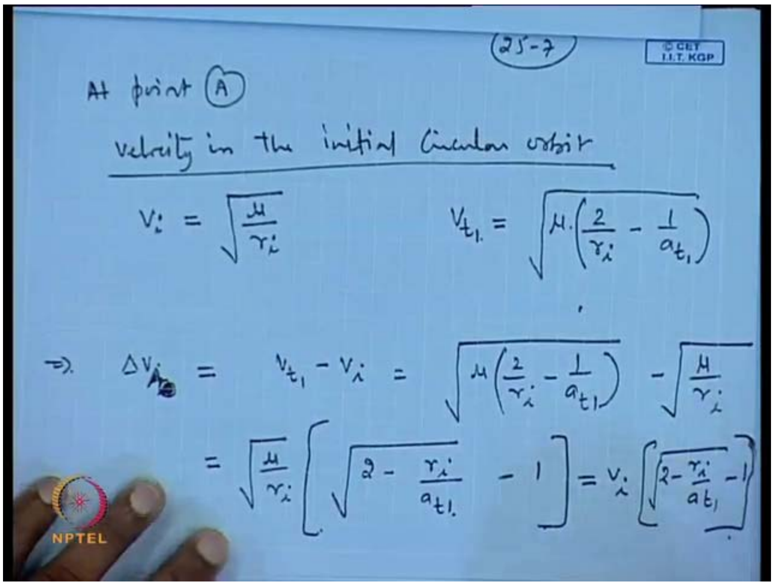

So, now, what we have got, that n equal to 15.58176. Now, if you take the second derivative of this, the quantity A, so, by taking it and inserting the value of n in there, this will turn out to be less than 0. So, this implies that, n corresponding to 15.58176 gives the maximum value of A. So, A equal to, we have written as delta V By V i and so, this becomes maximum, when n equal is to 15.58176. So, we can see that, how the efficiency changes. So, if you are taking n equal to r f by r i equal to 15.58176, and this implies r f equal to 15.58176 times r i. So, if your final radius is 15.58176 times r i, then, the efficiency of your Hohmann transfer is worse; means, you (( )) that time delta V, which will be delta V By V i, which will be maximum in that case; means, delta V is larger, then you have to spend more and more energy; the larger the value of delta V, the more and more mass of the propellant will be required.

(Refer Slide Time: 14:28)



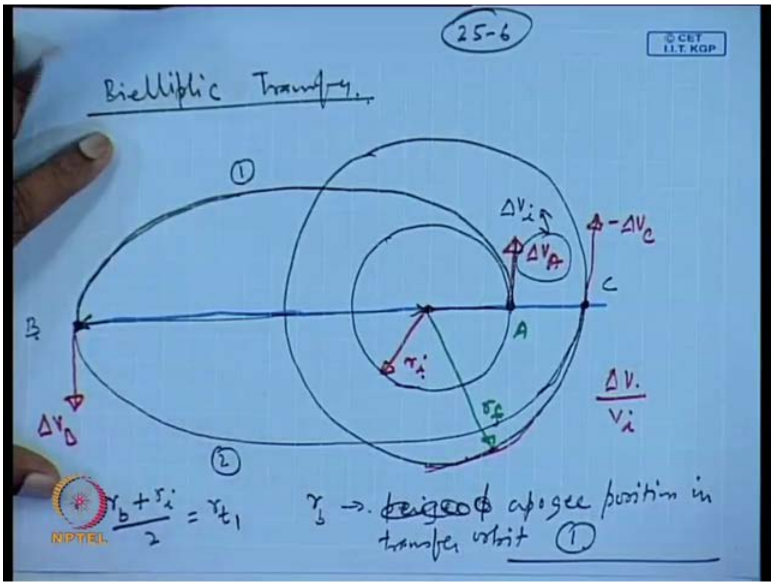

Now, before we discuss further, anything about the Hohmann transfer, let us go into the Bielliptic transfer and look, how the results appear for that. So, we do the (( )); we try to find out delta V By V i in the bielliptic transfer also, and then, we will compare both of them, how do they compare; and when the bielliptic transfer will be efficient, and when the Hohmann transfer will be efficient, then will be able to deduce. So, right now, the scenario is that, for n equal to 15.58, the Hohmann transfer is going to be the worst. So, in the bielliptic transfer, we have started with one initial orbit of radius r i, and we had a final orbit of radius r f. So, we were giving one impulse here, in this point. So, this was our starting point A and in a, in an elliptical orbit, it was transferred here; then, another impulse in required here, in this place and it is taken to this point, which is C; this is point B, here. So, here, we require delta V A; in this point, we require delta V B and here, we require a negative impulse, delta V C. So, this will point in this direction. Now, with this arrangement, find out how much this delta V By V i will be. So, V i is nothing, but the velocity in this initial circular orbit, whose radius is r i; and similarly, V f, we can write for the radius in this final orbit, as velocity in this final orbit.

(Refer Slide Time: 17:41)



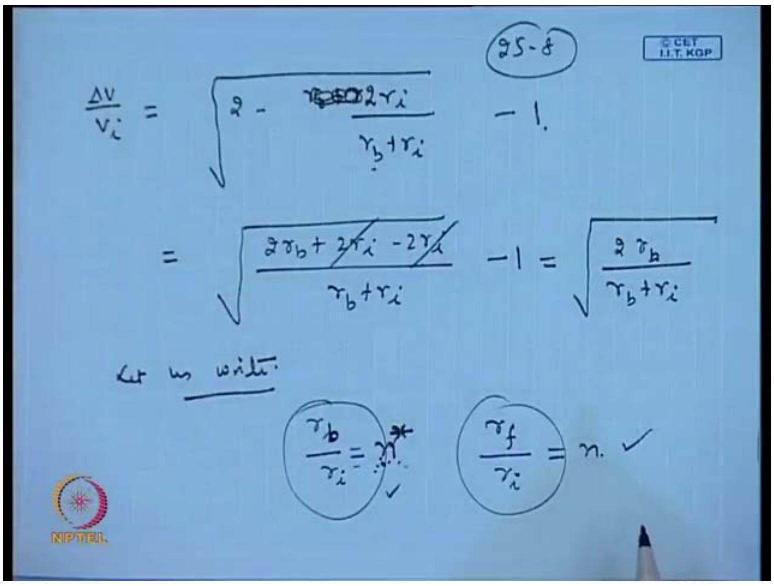

So, considering at point A, velocity in the initial circular orbit, this will be given by V i equal to mu by r i under root; and V in the transfer orbit…So, we can, from this trajectory as number 1, and this we can term as number 2; so, this is one transfer orbit and this is another transfer orbit. So, V t 1, velocity in the transfer orbit; similarly, we can write and mu times 2 by r i minus 1 by a t 1. So, from here, then, we can have this delta this V i, for trajectory 1 and we need not put here 1, subscript, because delta V i itself, connotes the meeting, meaning that, we are right now, transferring here, along this line. So, how much this delta V A will be, this you can write as delta V i, or as delta V A, whichever notation you choose. So, let us put this as delta V A. So, delta V A then will be equal to, V t 1 minus V i…And mu by r i we can take as common; so, this leaves us with 2 minus r i by a t 1 under root minus 1. And, mu by r i is nothing, but your V i. So, this quantity times, 2 minus r i by a t 1 under root minus 1. Now, a t is the quantity which we can look from the figure.

(Refer Slide Time: 20:28)



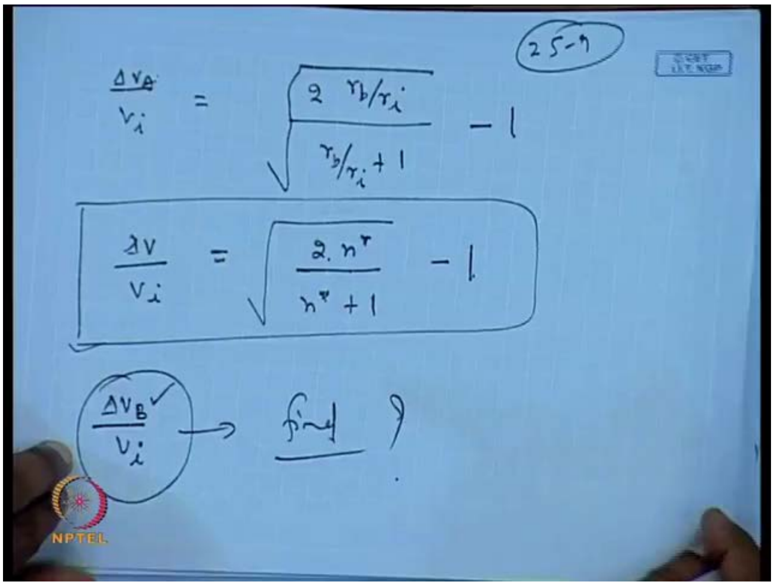

So, a t is the semi major axis of this transfer orbit 1 and this will be equal to r i, this quantity, plus r B; r B is from here to here; so, this quantity is your, r B is the perigee, apogee position of, in transfer orbit 1**.** So, from here, we can say, r b plus r i divided 2, this is equal to r t 1. So, thus, this is r t 1. This, we will write as, not as r t 1, but rather a t 1; this is the semi major axis. This quantity plus this quantity; this added together divided by 2, this gives you the semi major axis of this transfer orbit. Now, inserting this, a t 1 in this equation here…

(Refer Slide Time: 21:20)



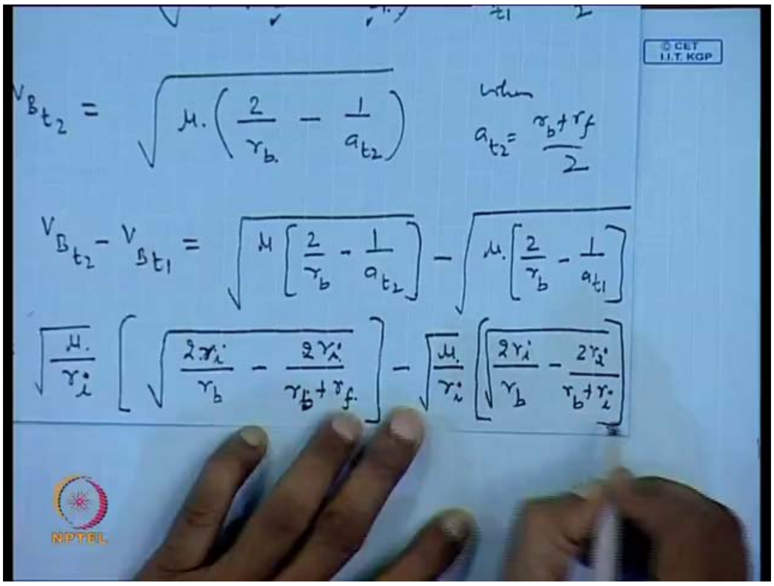
So, your delta V By V i then becomes equal to 2 minus r i is equal to r b plus, r i is given; and this is, r i is already, this is given quantity; this we have to insert. So, this r b plus r i divided by 2. So, 2 goes in the numerator, and this is minus 1. So, this becomes 2 r b plus 2 r i and this will get reduced to 2 r i, 2 r i cancel out; 2 r b, 2 r b plus…Now, after calculating this, we need certain more work out. So, let us write…Now, we have two quantities, say, in this figure, the ratio of the r b divided by r i, we have one quantity this, r b by r i; another quantity is, when we go into the final thing, which is, r f is the radius of this; so, r f by r i. These are the two quantities involved in our case. So, based on this two quantities, we can write certain expression. So, let us, in this expression, put r b by r i is equal to n star and r f by r i s n. So, depending on how do we term this quantity, so, accordingly, the equation will appear. If we term this as n, so, here also, instead of n star, we can put here n star; but it is, it will be more logical, because n, we are using as the quantity, which is the ratio of the final orbit to the initial orbit radius. So, it is better that, r f by r i, we term as n, and the quantity which is appearing as r b here, this r b by r i, we term it as n star.

(Refer Slide Time: 25:34)




So, if we choose that notation, then, delta V by, delta V by V i, this will become 2 r b by r i divided by, and this is, minus 1 is missing here; we put minus 1. Now, after doing this, so, the next quantity that we can compute, is delta V by, delta V B, this is our delta V A. So, delta V B by V i. So, first, we have to compute delta V B, and thereafter, we will be able to…So, find this. Now, delta V is the quantity, delta V B is the quantity that we are producing here. Here, you have the velocity; once you give the impulse here, the satellite will come here, in this place, and after, next you give impulse here in this place, so, it will go in this place, and the third impulse is required here. So, this is called bielliptic, because we are putting in two different elliptical trajectory. So, one is here, and the other one is here; and that is why, this is called bielliptic transfer.

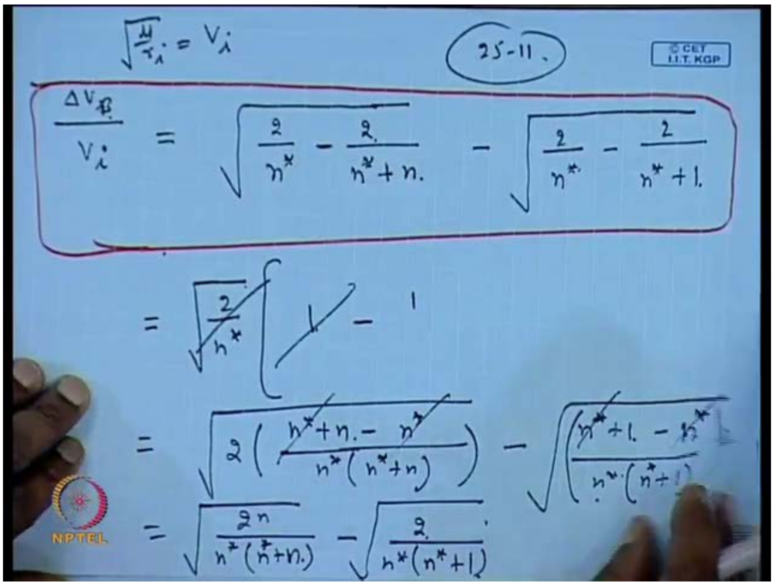
(Refer Slide Time: 28:00)




So, now, we have V B in the trajectory t 1. So, this will be given by, mu times 2 by r b minus 1 by…This is the initial one, initial velocity at this point. So, here, we, in this ellipse, the semi major axis is known to us. So, that we are using here, and r b is the radius here, in this point; that is known to us. So, this is the radius from here to here. So, that we have put in this place and we use this equation for the velocity to compute this. Similarly, once we give the impulse…So, we are putting it in a different trajectory. So, V B at, in the trajectory t 2; so, at this point, in this trajectory, how much impulse is required; so, that will be given by mu by 2 by…Now, for this particular trajectory, r b does not change. So, r b remains same; but here, the semi major axis will change. So, semi major axis will be summation of this and this divided by 2. So, we can write, a t 2, this is equal to r b plus r f divided by 2. So, this is 1 by a t 2, under root, where a t 2 is r b plus r f divided by 2, and a t 1 is r i plus r b divided by 2.

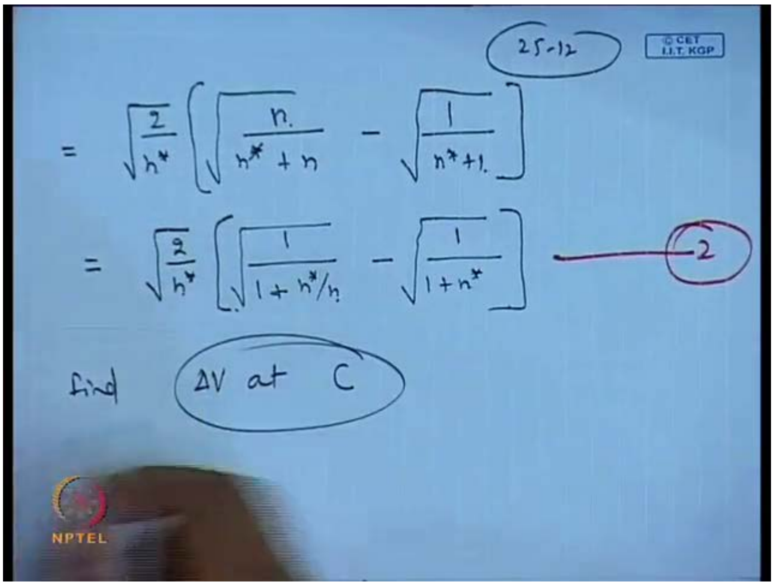
So, now, from this, we will have delta V B. Now, inserting the value for a t 2 and a t 1, so, this will get reduced to…Let us take again the mu by r i term outside. So, if you take mu by r i under root outside, so, this will become 2 r i by r b and a t 2 is nothing, but r i plus, r b plus r f divided by 2. So, 2 will come upside. Similarly, we take from here mu by r i outside; 2 r i by r b minus 2 r i by r b plus r i under root.

(Refer Slide Time: 31:52)



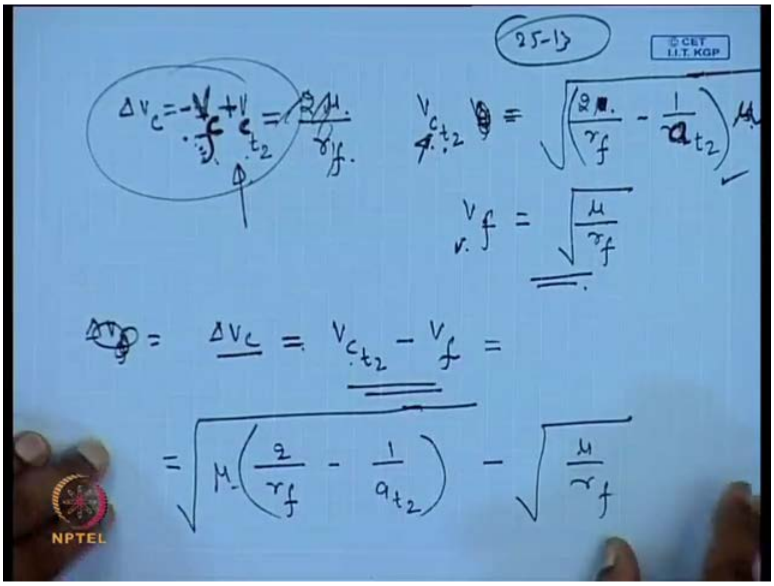

So, then, delta V B…Now, mu by r i under root, this quantity is nothing, but V i. So, we can take it outside and write here, in this place. Now, r b by r i, we have written as r star. So, this becomes 2 by, n star we have written, 2 by n star under root minus…This quantity, again, we divide the up and down, numerator and denominator by r i. So, this becomes, r b by r i is nothing, but n star and r f by r i is nothing, but n. Similarly, this quantity has to be inserted here. So, this is 2 by, r b by r i is n star, minus 2 divided by, r b by r i will be here n star, plus 1. So, this is the quantity we are getting here. Now, let us term different equations. This is a equation number 1; here, we can do a little bit more simplification, in that, we can take n star, 2 by n star outside.

(Refer Slide Time: 35:53)



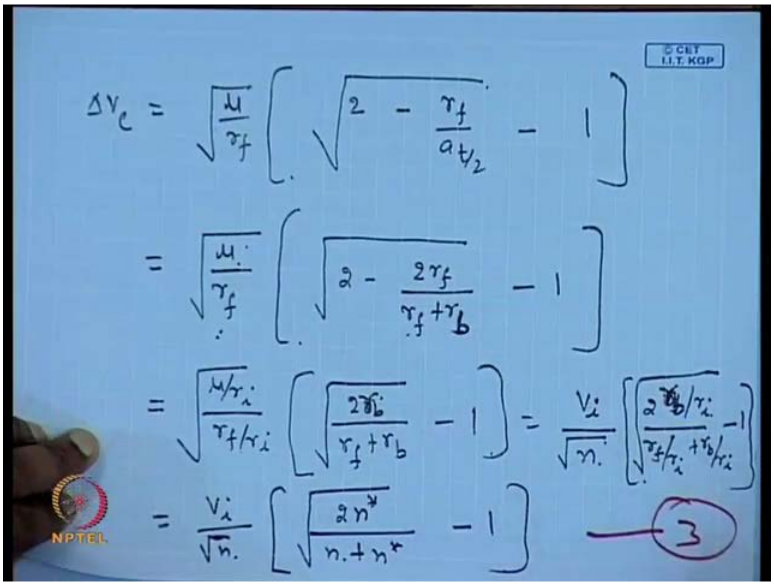

So, 2 by n star, we are taking it outside; this becomes 1 minus, 2 goes outside and this will become 1 by…And here, further simplification is possible here, in this place. So, we can do the first simplification and rather, later on, we will take it outside. So, we will have 2 times n star plus n divided by…Similarly, here, in this place. Now, we will take outside 2 by n star under root, and this is our equation number 2. Now, after working this, we go into the next level. So, now, we have to find out, find delta V at C. So, we proceed in the same way.

(Refer Slide Time: 37:15)



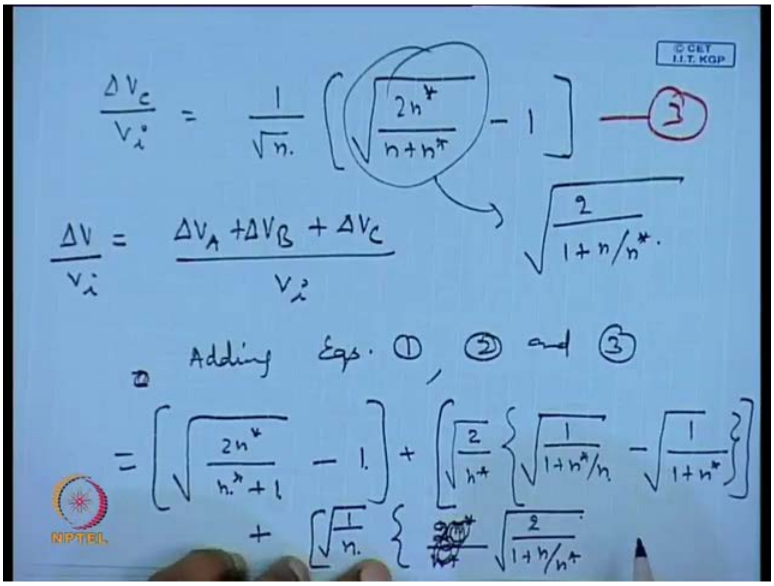
So, delta V C, we can write as, 2 mu by r f; this is the velocity at, delta V C equal to velocity at C minus V C in the transfer orbit 2; or rather v f minus V C 2 or V t 2 also, you can write. So, V f , let us write separately. V f, now becomes equal to 2 mu by r f minus 1 by, 1 by a t 2 under root; mu is outside and similarly, the V, this is the velocity in the transfer orbit. So, we write here V C t 2 and v f, we write as mu by r f under root. So, this is at position C, the velocity in the transfer orbit, which is an elliptic orbit. So, for this, we have got this equation and in the circular orbit, this is the final velocity. So, delta V C, now, this will be equal to delta V C by…delta V C, first we write; this will be V C t 2 minus…Here, the quantity that we have written, we can, we can write either us v f minus C t 2 or V C t 2 minus v f. This will turn out to be negative, because finally, in that circular orbit, your velocity will be, is smaller than that in the elliptical orbit. So, rather writing in this way, we could have put here the sign minus here, and the plus here, in this place. So, it will get into this shape. So, now, in this equation, this quantity will turn out to be positive.

(Refer Slide Time: 40:24)



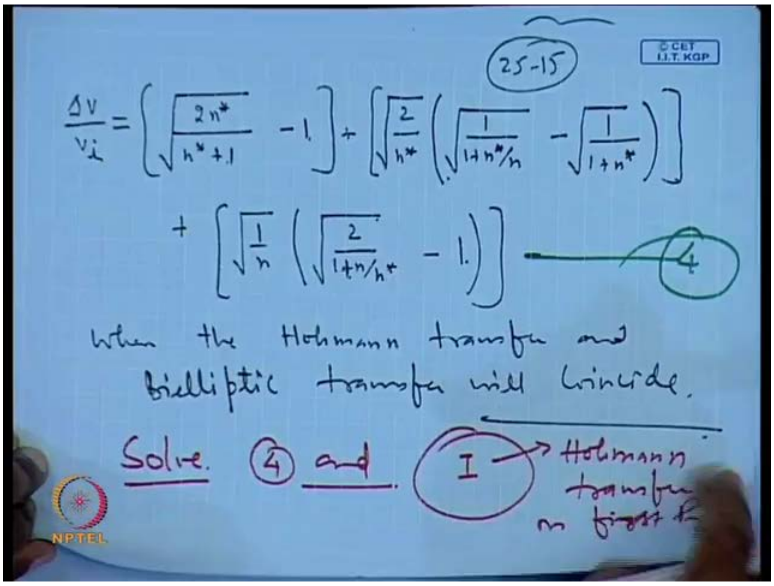

So, delta V C, that we indicated with a negative sign in our figure. So, here, this delta V C quantity will be negative and within the minus sign, this will turn out to be positive. So, that is, you have to apply the impulse in this direction, and reduce the velocity, so, this will get into this orbit. So, from here, then, this results in 2 by r f minus 1 by a t 2 into mu under root minus mu by r f. So, take mu by r f outside; so, delta V C becomes, mu by r f under root, we are taking outside, and this becomes, 2 minus r f by a t 2 under root minus 1. Now, the quantity which is present here, this is nothing, but your, the velocity in the final circular orbit. So, here, this becomes 2 minus 2 r f by, 2 r f by r f plus r b; a t 2 is r f plus r b by 2; so, this will appear in this way. Here, we can divide numerator denominator by r i, and this will become, 2 r b divided by r f plus r b; and this quantity is nothing, but mu by r i is V i and r f by r i is nothing, but n. And here, similarly, we can divide numerator and denominator by r i. So, this will become 2 by r b divided by r i, and r f by r i plus r b by r i under root minus 1. So, this is V i by under root n and 2 r b by r i is nothing, but, r b by r i, we have taken as n star. Similarly, r f by r i, we have taken as n; this is n star. Now, you have got the velocity change in that C position. So, this is our equation number 3. One more step we will write here.

(Refer Slide Time: 43:04)




So, we have delta V C by V i equal to 1 by under root n times 2 n star divided by n plus n star under root minus 1. So, from here, total delta V will be equal to delta V A plus delta V B plus delta V C and this, we divide by V i. So, this will be the quantity. Now, from all the three equations we add. So, this is our equation number 3. So, adding equation 1, 2 and 3. So, the very first equation, we had 2 n star divided by n star plus 1 under root minus 1; and then, second equation we got, this is 2 by n star under root times, under root and plus the third equation, which is present here, 1 by n under root 2 n star by n plus…So, this n star, we can divide here, in this place. So, this quantity can be written as, 2 divided by 1 plus n divided by n star under root; so, we write in that way. This will be 2 divided by 1 plus n star under root minus 1.

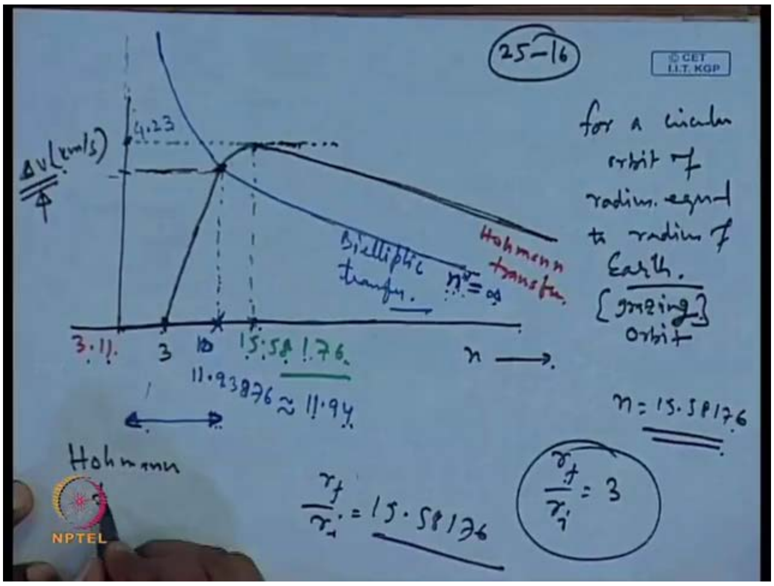
(Refer Slide Time: 46:11)




Now, we have got the, this final equation and from here, you can get a lot of insight into the equation. So, now, the only way for working out this, you can choose the different values of n star, this can be solved also and ultimately, it will look like, in the form of a polynomial; and, from there, then, we are start working out. But the best thing is, here in this case, that this delta V By V i that we have written, we can choose certain value of n star and then, start giving different values to n, and then, plot the result. So, by plotting, you get the graph for delta V By V i. So, if you plot that, so, here n star will appear as a parameter. So, this is basically a parametric; it will appear in the form of a parametric study. So, now, go back into our old thing. So, one of the study, what we can do, here, we have got this equation, this will, let us name this as equation number 4. So, write fresh here, in this place; this is, 2 n star divided by…And, this is our equation number 4.

Now, question arises, whether the Hohmann transfer and bielliptic transfer, they will coincide at certain place. So, for that, we need to solve this equation 4. So, when the Hohmann transfer and bielliptic transfer will coincide. So, for this purpose, we need to solve equation, solve equation 4 and for the Hohmann that we wrote, this is the equation that we wrote for Hohmann. So, let us term this equation as, and, this equation let us say, this maybe, we write in Roman notation, this as I. So, solve equation 4 and I, which is referring to Hohmann transfer, Hohmann transfer on first page.

(Refer Slide Time: 50:17)




So, by solving this, we will be able to tell, in what place they will coincide. But before we do any further exercise, let us look into the plot of the Hohmann and bielliptic transfer. Now, we have already calculated that maximum of a Hohmann transfer. It occurs for n equal to 15.58176. So, if we make a plot of this, plotting n on this axis, and delta V on this axis, which is in kilometer per second, this we do for a orbit of, for a circular orbit, circular orbit of radius equal to radius of earth. So, this is called the grazing orbit. This is being worked out for this grazing orbit. Now, if we do this for grazing orbit, and, take the final orbit to be 3 here, final orbit, final orbit r f by r i, they show to be equal to 3. So, we start with this value. So, corresponding to this value, here, we will have a value of 3.11.

And, this curve will, this is the Hohmann transfer curve and this will go like this, and then, for Hohmann transfer and the maximum of this occurs at 15.58176 and that maximum value is, here 4.23; this is 4.23. So, delta. So, delta V equal to 4.23. So, in the Hohmann transfer, you can see that, maximum change in velocity occurs for n equal 15.58176. On the other hand, if we take the bielliptic transfer, which is given by the equation that we have developed earlier, here, and in the next lecture, we are going to explore it. So, the bielliptic transfer, it will appear…And, this is for n star equal to infinity; n the star equal to infinity means, in the intermediate radius in the bielliptic transfer is taken to be infinity. So, for that case, we will see that, the Hohmann transfer and the bielliptic transfer, they intersect at a point, 11.93876, which is nearly equal to

11.94. So, they match at, only at this point. So, in the next lecture, once we explore this, we will see that, before, between 0 to 11, your Hohmann transfer is the best and thereafter, between, from 15.58176, this bielliptic transfer, this will turn out to be the best.

So, overall, what the picture we are getting here, that Hohmann transfer, the change in velocity required to do the changes for r f by r i equal to 3 is 3.11. And, till 15.58, Hohmann transfer, it remain, till 11.94, till this range, Hohmann transfer remains the best and thereafter, in this range, bielliptic transfer or Hohmann transfer, (( )) will be good, because this is for the n star equal to infinity. So, we have to further explore it, and we will discuss further on this in the next lecture. Thereafter, the Hohmann transfer, it becomes not so energy efficient; for this curve is more energy efficient, because lesser change in velocity is, for achieving the same value of n, you need lesser velocity change and that implies that, you need lesser amount of fuel. So, here, the maximum velocity change is required at this point, for the Hohmann transfer. So, this corresponds to 15.58176. So, as we increase the radius, r f by r i to 15.58176, this is the maximum amount of velocity change will be required.

That is, you have to spend more amount of fuel here, in this case. And, between this range, it is a little dubious, but before here, the Hohmann transfer is the best, because it requires less amount of delta V, that you have to give less impulse and therefore, less amount of fuel consumption. And therefore, the Hohmann transfer, it remains the best in this range. So, we can write here that, the Hohmann transfer is best, if n is less than 11.94. And, rest other, we have to explore further that, what will happen beyond this range. So, Hohmann is the best for, in this case. We will continue with this lecture further, deriving how to get this point 11.94, where this two, the bielliptic and the Hohmann transfer intersect and discuss more about the properties of this. We will continue in the next lecture. The time is getting over. Thank you very much.