Space Flight Mechanics
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Module No. 01
Lecture No. 24
Trajectory Transfer (Contd.)

Because, last time we had started with general to impulse transfer as part of trajectory transfer, so we continue with that.

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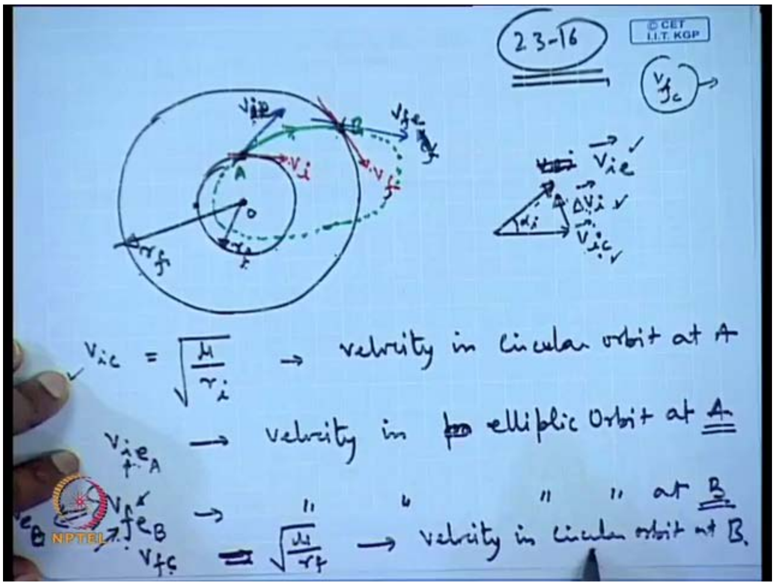

So, we took the case, where we had the inner orbit, whose radius is given by r I, and the final orbit was radius is r f, and a satellite is moving in the inner radius orbit, and it has to be transferred into the outer radius orbit, but not along the tangential path means, we are not a starting here in this point and then starting tangentially here in this point and going up to this point, rather at intermediate point anywhere we are passing from point a to point b. So, this is not Hohmann transfer what we have been discussing. So, for this kind of treatment, we need to generalized further how to find out the delta v required here in this place, and the delta v required here in this place. So, for that v i c here is given, which is the speed in the, or the velocity in the circular orbit, and similarly for the velocity in the outer orbit is also known to us, which is v f c, so this is the velocity in the outer orbit.

So, if v i e is the velocity in the electrical orbit, in which the satellite is to be put for the transfer purpose, and from there the orbit will be thrown into the, it will put into the final circular orbit. So, for sending it along this direction, so we need a velocity which is given by v i e. Now, v i e is the velocity required velocity, so how much change in velocity is given, how much change in velocity is required that is given by delta v i, so we are started working with this. So, we have the velocity in the inner circular orbit, which is given by mu by r i, v i e is the velocity in the electric orbit at point A will consider, and again at point B also will consider. So, we can put here subscript later on, but v i e indicates the velocity in the electric orbit, and we can put subscript here to indicate that this is at point A, and v f c is the velocity in the electric orbit at B.

So v f e, and here we have put the final, to indicate this is corresponding to this point, otherwise the same thing could have been indicated by. So, we can take it like v eccentric, v electrical A and v electrical B we can write, but here we have written v f means, this is the initial point here a starting point in the initial orbit, and this is the point in the final orbit. So, corresponding to that, this subscribe i and f they are appearing. And e is corresponding to the electrical orbit, and finally we can put a subscript here B to indicate that this is at point B. So, that completes our notation, and the v f c, this is the velocity in the final circular orbit. c is stands for the circular orbit, which is given by mu by r f, velocity in the circular orbit at B.

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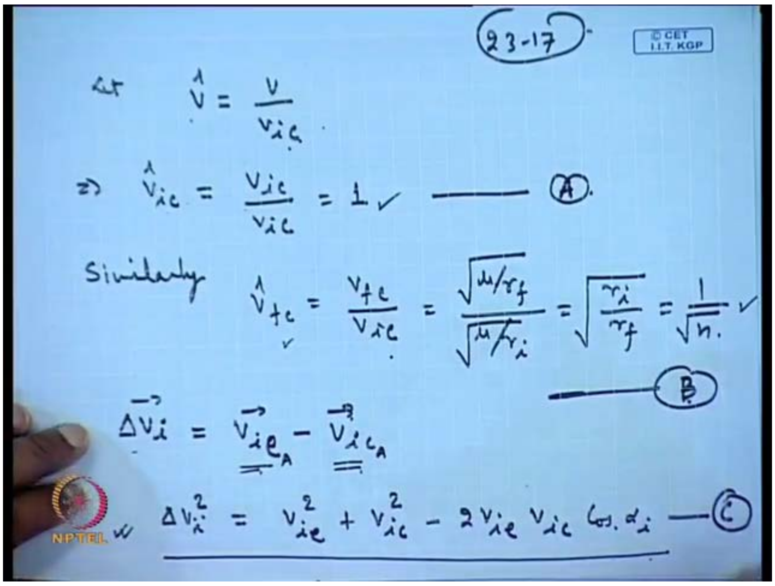

So, after completing this, we assumed the quantities like v cap is equal v by v i c, where v i c is the velocity in the initial circular orbit. So, this implies v i cap, v i c cap this will given by v i c divided by v i c is equal to 1, and similarly v f c will be v f c by v i c, so this is 1 by root n. Now, therefore delta v i this can be calculated, so how much this delta v i e is, that can be calculated. From here delta v i is equal to v i e minus v i c, so this is the velocity in the electric orbit at A, and velocity in the circular orbit at A. and this quantity in the vascular notation can be written like this.

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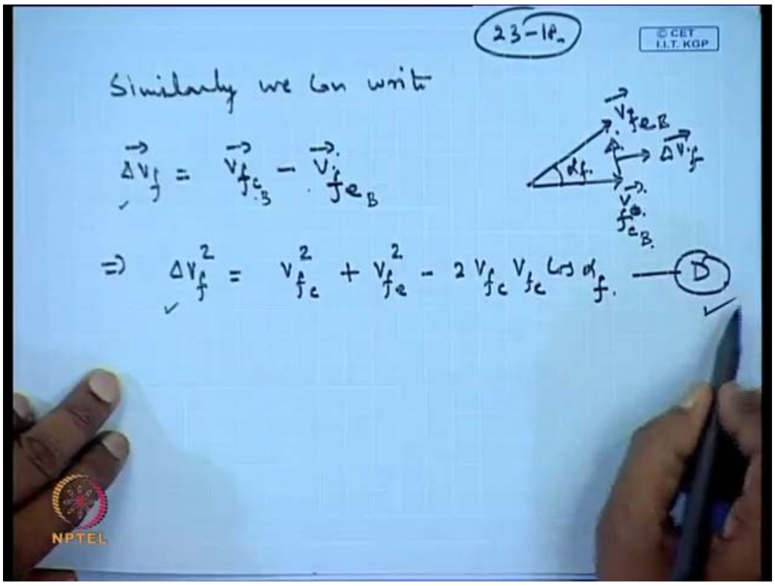

Similarly, we have written last time, that delta v f is equal to v f c, which is the velocity in the final circular orbit minus v f e. So, for that we have the case here. This is the velocity in the circular orbit, we put here v f c, and this is the velocity in the electrical orbit, which is shown by here the blue color arrow, so this is v f e, and these are at point B, so we can put a subscript here B. So, we can insert subscribe here B, to indicate that this is the velocity in the circular angle electric orbit at B. therefore, delta v f this will be given by this particular equation. So, we worked out to this extended last time.

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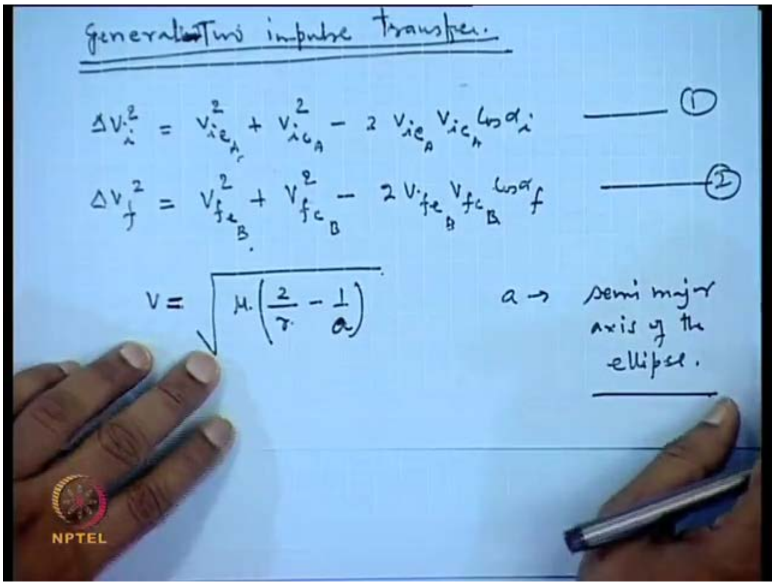

Now, we are start working further, so what we have got delta v i square, this equal to v i e square, plus v i c square minus 2 v i e and v i c to cos alpha i. This is equation 1, and the equation 2 is delta v f square, this is equal to. So, here the notation A and B they are not appearing, so if you want we can put here as A, and here we can put as B, but it is convenient to drop this subscript, because we convert this i with initial and f with the final, so automatically this will give you the meaningful expression. So, further while carrying out the work, so we are going to drop this subscript A and subscript B. Now, we know that v in a electrical orbit it is given by mu times, 2 by r minus 1 by a under root. So, here a is the semi major axis of the ellipse.

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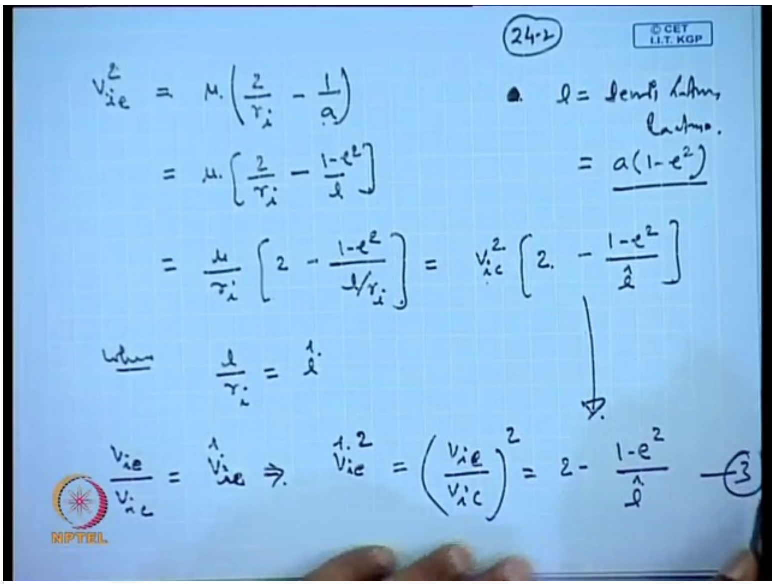

So therefore, we will have now v i e, so we are dropping the subscript A. So, v i e can be written as v i e mu square is equal to mu times 2 by r i minus 1 by a. Now a, we can express as, or other say l we can write as, which is the semi latus rectum. This is equal to a times 1 minus e square. So, we can put here, insert here in this place, 2 by r i and a will be, l by 1 minus e square. r i we can take it outside, and we can express it like this. So, mu by r i is nothing, but v i c, which is the velocity in the circular orbit square. So, this become minus 2 times, 2 minus 1 minus e square, and l by r i we write as l cap, so here l by r i this is equal to l cap. So, we have got very simple expression for v i e, written in term of v i c. So, going further into it is, now v i e divided by v i c, this can be written as v i e cap. According to our notation that we have developed earlier, so this become sv i e cap square, this will be equal to, this from here we can find it out. So, just we have to divided both side by v i c square, so we get this quantitative here. This is our expression number 3.

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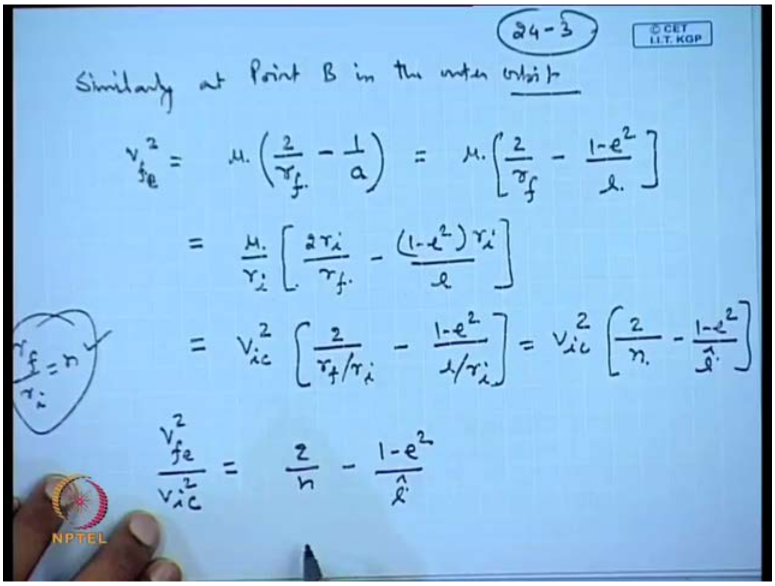

Similarly, at point B in the outer orbit, we can write v f e square, this is equal to mu times 2 by r f minus 1 by a, where a is the semi major axis of the transfer orbit, which is the electrical orbit. Therefore, this is 2 by r f minus,1 minus e s square by l. And we can take it out, r i we can take as a common, so this can be then expressed as 2 r i divided by r f minus. Now, we have the known quantities here, so mu by r i is nothing, but v i c square, and this we can write as 2 r f by r i minus 1 minus e square l by r i, this is v i c square. Hence, 2 by r f by r i is nothing, but n. You get from the beginning we have been assuming this quantity. So, this is 2 divided by n, l by r is nothing, but l cap. So, thus what we get here v f e square divide by v i c square, this is going to 2 by n minus 1 minus e square by l cap.

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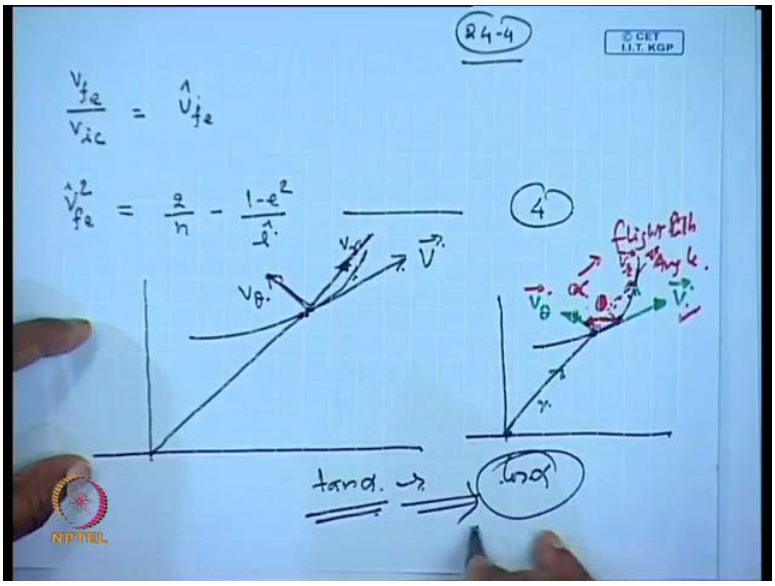

So, from here we can write following the earlier notations, so v f e by v i c, this is nothing, but v f e cap. This is the notation that we will follow, and therefore v f e square v f e cap square, this becomes equal to 2 by n minus 1 minus e s square by l cap. This is our equation number 4. Now, going back into our old figure, so here if you look into this figure, so what are the quantities to be found further is, this alpha i, which is the angle from here to here, this is angle alpha i, alpha i, and the angle between this and this alpha f, so this two need to be worked out. Now, say this is our transfer trajectory, this is v theta, and here we can show this as v r, this is the velocity vector v.

So, we will do in new figure for this, this figure is blurred. v theta is perpendicular to the radius vector, this is the radius vector here. And suppose from the velocity will be tangential to the trajectory, so velocity can be shown along this direction as v, and this is v theta. So if you look into this, the flight path angle is defined as the angle between the local horizontal. So, in case of this trajectory transfer, or the any particle moving in a gravity field, so this particular angle we can write here as alpha and this is call the flight path angle. So, alpha is flight path angle, and v r will be just in the radial direction, so if this is the direction of r, so here we will have v r in this direction. So, we have v theta here, v here and v r along this direction. So, resultant of this two will be the velocity vector v.

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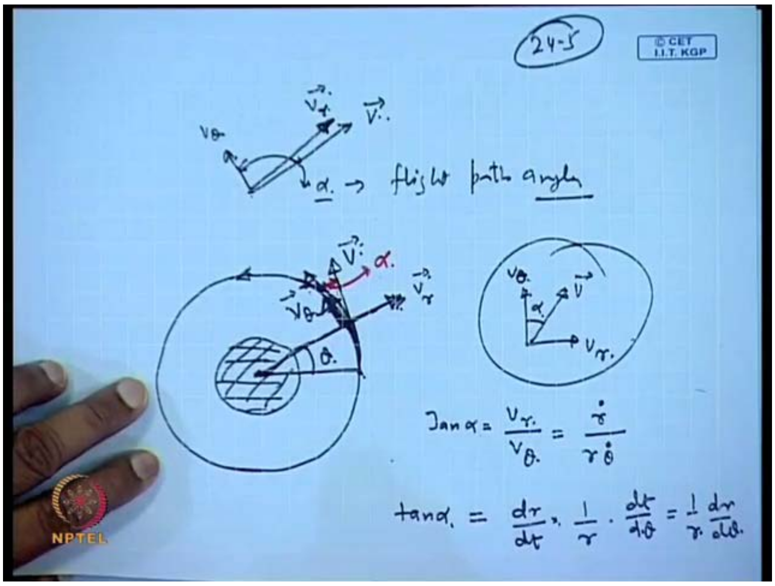

So, what we have done here, say this is the velocity vector, and this is v and this is our v theta, and this is our v r. so, where v is the resultant of v r and v theta. And the angle from here to here, this we have shown as angle alpha, which is the flight path angle. So, taking the same in the perceptive of shape around the earth, this is the earth, in some orbit. Here we are showing this as a circular orbit, but suppose this little bit electrical, so we have this angle as theta. So, we are measuring the true anomaly from this place, and then v theta will be perpendicular to this radius vector r. So, v theta will be along this direction, and suppose this is electrical. Then v direction will be tended to the path, so let us make it little bit electrical, so that we can show here, v and this is not v r. So, this depicts the real picture in the case our, may be any heavenly body. In this case we are taking this as the earth, and around that in a circular, in a electrical orbit or in any trajectory, some particle is moving under gravity.

So, this is the real situation, which is shown here. So, this is our v vector and this is the resultant vector, so v theta and v r. So, we are taking a in a proper way v, v theta and this is v r. And, the flight path angle obviously, we have told that in that case this will be indicted by the angle from here to here, which is nothing, but alpha. So, whatever we have shown here this is a generalized representation. With this representation what we want to work out, what will be the value of tan alpha, and then if you find out tan alpha, so from here we will be able to find out the value of cos alpha. So, this is the generalized representation, and the representation that we follow here. This is for our gravitational case. So, this angle obviously we have written as alpha, so we can write tan alpha, this equal to v r by v theta, and v r is nothing, but r dot. And v theta is nothing, but r times theta dot. Therefore, this becomes d r by d t, into d theta. Write it in this way, so this becomes 1 by r into d r by d theta.

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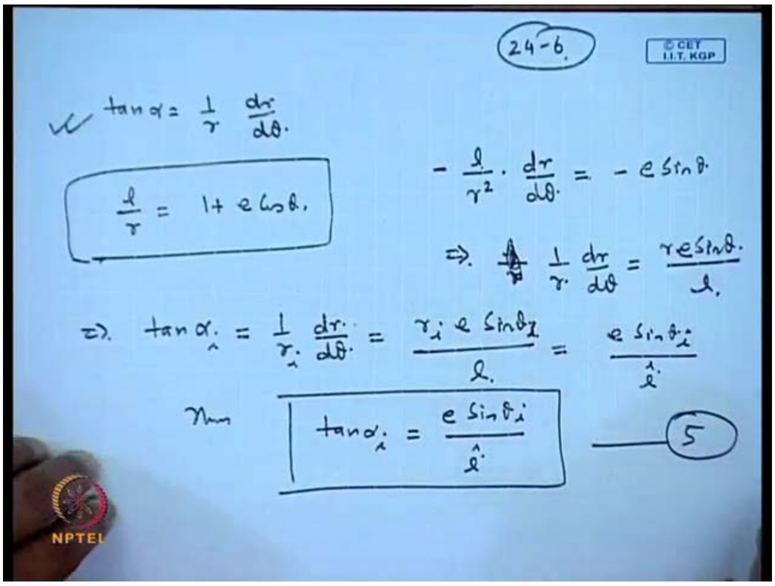

So, now given tan alpha is equal to 1 by r d r by d t theta. We can utilize the relationship for the ellipse developed earlier, which is l by r equal to 1 plus e cos theta, to work out the value of tan alpha. So, from here we can find out the value of d r by d t. So, differentiating this, this gives us l by r s square and d r by d theta. So, thus we will have l yr, because what we need is 1 by r times d r by d theta, so we put on the left hand side 1 by r times d r by d theta, and l and r we can tip on the right hand side, so we will have r times e sin theta divided by l. So, this implies tan alpha a, this will be nothing, but 1 by r i times d r by d theta, equal to r i times c sin theta, divided by l. And we will put a subscript i to indicate that true anomaly, at the initial point, theta i. And l divided by r i, this will becomes l cap. So, thus tan alpha iis equal to es in theta i divide by alpha. This is our equation number five.

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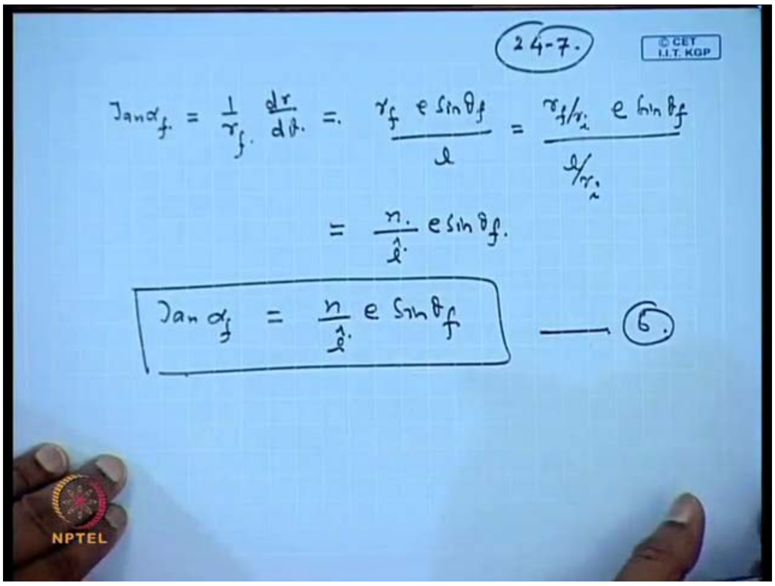

Similarly, we can work out for the tan alpha f, so again we will have for tan alpha f is equal to 1 by r f times, d r by d theta. This we will be nothing, but r f times e sin theta f divided by l. this we can express as r f by r i, l by r i, and r f by r i nothing, but n, and l y r i is nothing, but l cap, and this is e sin theta f. And this tan alpha f, this becomes equal to n by l cap. This is our equation number 6. Once we have got this, so from here now it is easy to work out. Now, the quantity which is present there, which is theta f and theta i, this two also need to be eliminated.

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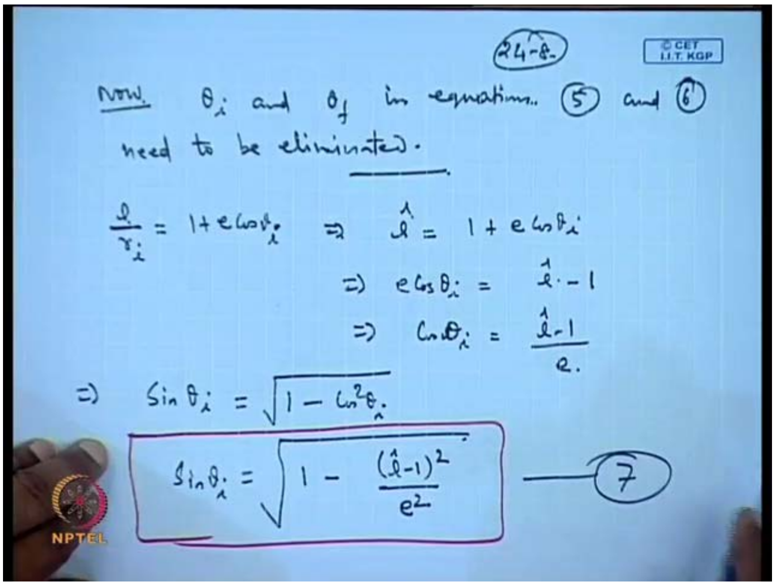

Now, theta i and theta f is equations five and six, need to be eliminated, and this we can do using the original equation which is given to us, which is l by r is equal to 1 by e cos theta. So from here we can replace theta in terms of l and r and e. So, this implies, now l by r i, so if we take here as theta i, so we will have l y r is equal to 1 plus e cos theta i. So, this implies l cap equal to 1 plus e cos theta i, and from here we get l cap minus 1, cos theta i, and this implies. So, this is our sin theta i.

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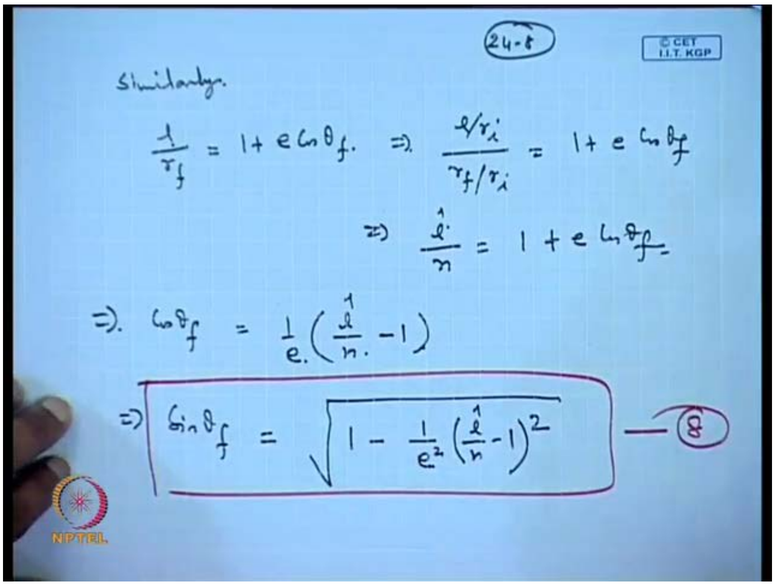

Similarly, we will have l by r f is equal to 1 by e cos theta f, so this implies l by r i divided by r f by r i, plus e cos theta f, and this implies l cap by n is equal to 1 plus e cos theta f. and, this implies cos theta f is equal to l cap divided by n minus 1 and shown by. And, therefore sin theta f then becomes 1 minus 1 by e square times l cap n divided by n minus 1 whole square under root. This is our equation number eight.

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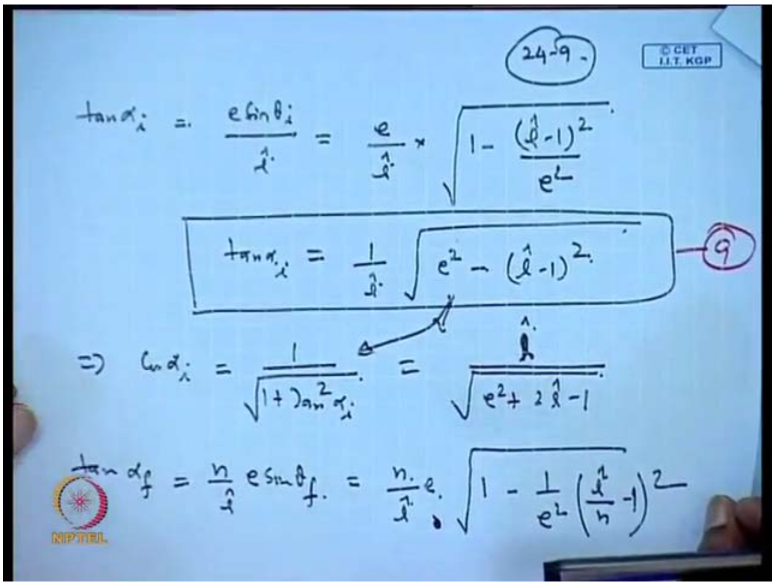

Once we have got this, now we can write an alpha i equal to e sin theta i by l cap, so this will be e by l cap, and sin theta is our 1 minus l cap minus 1 whole square, divided by e square under root, so taking e inside. So, this implies cos alpha i equal to, and if you do the processing, you can prove that this quantity will be equal to, by inserting this quantity here in this place. This will turn out to be 1 by l cap divided by e square plus 2 l cap minus 1 under root. Similarly, you have tan alpha, tan alpha f equal ton by l cap, e sin theta f, and the sin theta f just now we have worked out. So the sin theta value is given here, so we can insert here in this place, this l cap time e sin theta f, 1 minus 1 by e square, and if you simplify it.

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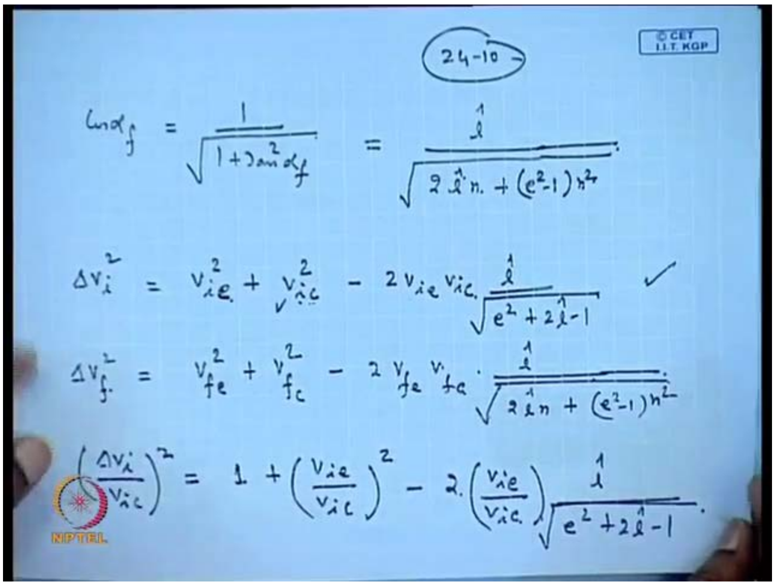

So, from here we can find out cos alpha f, inserting this values and simplifying it, finally you will get l cap, divided by 2. So, we have got finally, the alpha e and alpha f, so our job is done, and therefore delta v i, this can be written as v i e square plus v i c square, minus 2 v i e times to v i c. This is the equation that we wrote, so these are the two equations. So, insert the value of the cos alpha i and cops alpha f in this two equation; equation 1 and 2, so after inserting the value. So, these are the two equations which are available to us, and this two will give us the amount of impulse required, but we need to further simplify these two equations. So, the notation that we have developed earlier, that if we divide the whole thing by v i c in this equation, so delta v i square this can written as delta v i by v i c square, and v i c divided by v i c, this will become 1. So this particular thing we will write here as 1, and plus v i e by v i c square minus, and here v i c, v i c cancels out and we get v i e by v i c, times l cap by e square plus 2 l cap minus 1 under root.

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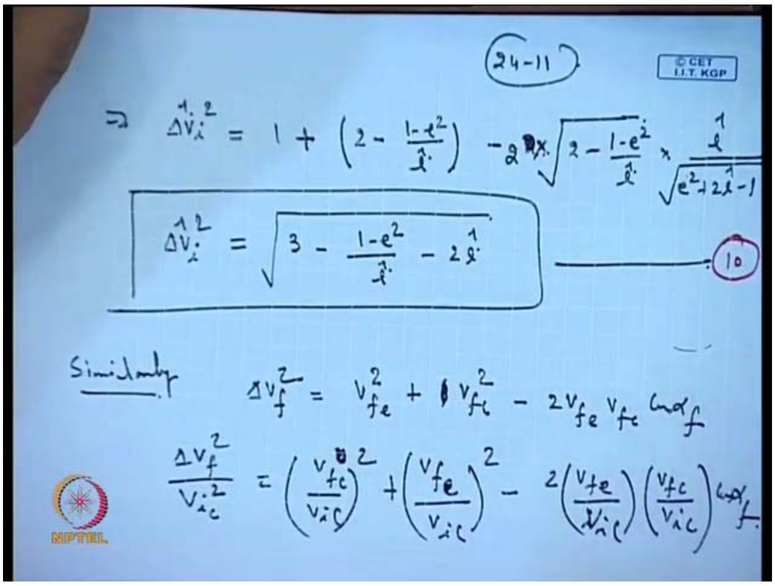

So, this implies delta v i cap square equal to, do the simplification while putting the values of v i e by v i c, so this will be 1 plus v i e by v i c, earlier we have derived. this quantity is nothing, but 2 minus 1 minus e square l cap, so 2 times, this is 2 times, v i e by v i c is 2 times, 2 minus 1 minus e square by l cap, and this multiplied by l cap divided by e square plus 2 l cap minus 1 under root. And then simplifying this will give you 3 minus 1 e square. This is our equation number 10. Similarly, delta v f is our v f e square, plus v f c square minus 2 cos alpha f, dividing both side by v i c, v f e divided by v i c square, cos alpha f.

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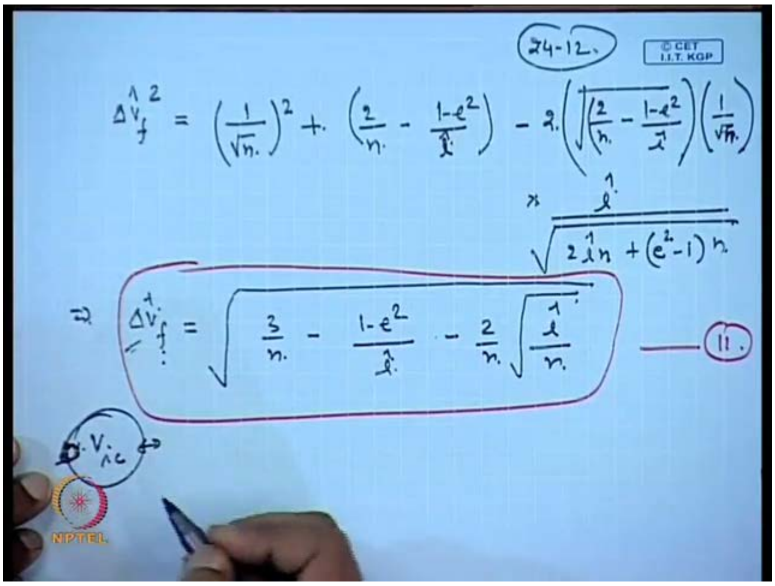

So therefore, we can write as delta v f cap square. Here we have taken the under root, so we remove the s square term here, this is fine. So, delta v f by v i c, this will be nothing, but delta v f cap. So, delta v f cap square, then this becomes. Now, we need to insert this values here v f c by v i c and v f e by v i c, and all this quantities we have developed earlier. So, v f c by v i c is nothing, but 1 by n under root. And v f e by v i c, this is nothing, but 2 divided by n minus by l cap, then minus 2 times, and this is one quantity, another quantity we have 1 by n under root, n times cos alpha e, which is nothing, but l cap divided by 2 l cap n, plus c square minus 1. So, if we do the simplification what we get, delta v f cap equal to 3 by n, minus 1 by e square, divided by l cap minus 2 by n, times l cap divided by n under root. And this is our equation number 11. So, these two equations give you the impulses, in terms of delta v i cap and v f cap at the point A and B. So, delta v i cap is at the point A, and delta v f cap this is at the point B. And this quantity is nothing, but if you multiply this quantity by delta v, this multiply this by v i c, so this gives you the actual amount of impulse. So, this equation looks very elegant, and this is simply to work with, and therefore this representation has been taken up.

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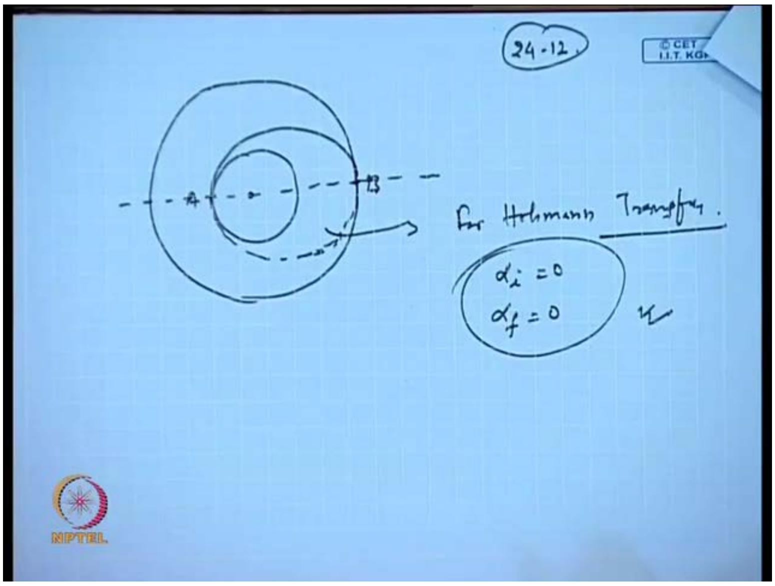

So, this is the generalized transfer, and the generalized transfer just now we have worked, and the earlier the particular transfer for the, what we have turn as the Hohmann transfer which we have taken, in which we transfer from point A to point B, in an electrical orbit. This is minimum eccentric electrical orbit and we proved this also in the beginning of the trajectory transfer lecture. So, whatever we have worked out which is representing the generalized transfer. So, from here this you can reduce to this particular case, because in this particular case alpha i and alpha f, both will be equal to 0. So for as Hohmann transfer alpha i equal to 0, and alpha f equal to 0. So, if we take these two, so it is easy to work out and what ultimately you get the same result, so those it is dwells we can show as.

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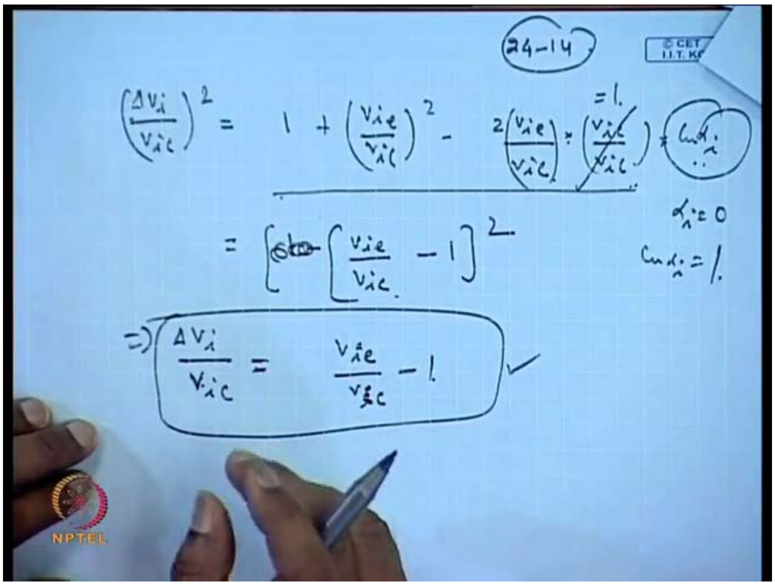

So, what you need to do for that, we had the equation where delta v i by v i c square, we wrote as 1 plus v i e by v i c square, minus 2 v i e by v i c, and v i c by v i c times cos alpha i. So the cos alpha i, once alpha i is equal to 0, so cos alpha i becomes equal to 1, and therefore what you get the remaining quantity which is present here. So, this can be written as 1 minus v i e by. This quantity becomes equal to 1, so this becomes v i e by v i c, we will write this as v i e by v i c minus 1 square. So, this implies v i delta v i by v i c, why we have written in this format, because v i e is greater than v i c. Therefore, it is appropriate represented like this. Similarly, you can work out for delta v f by v i c, and if you try to equate and you will see that the results are same, for both this case and in this case, there is no difference. So, this you can take up as an exercise, and we can go to the next topic.

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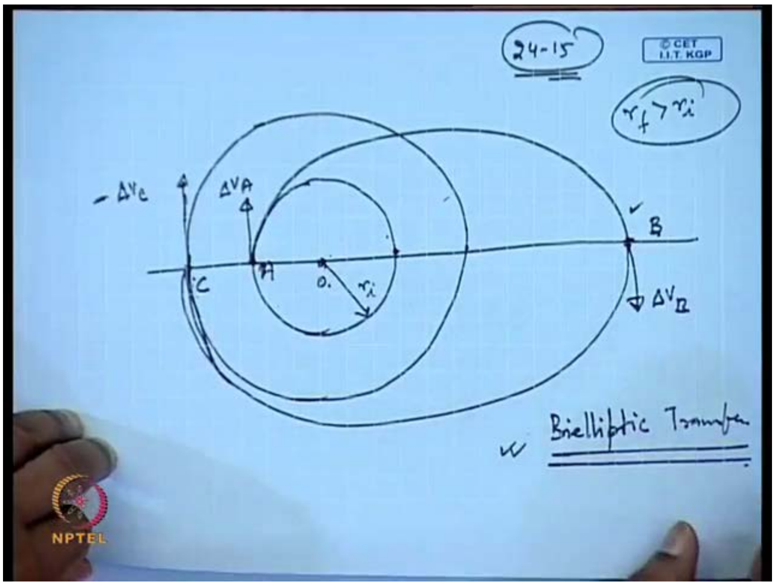

So, once we have done this, now the obviously, while starting in the beginning we discuss that Hohmann transfer is the minimum eccentric transfer. But how efficient this is, that was not solved. So, that we can solve further, by taking what is call the by electric case. So, by electric case, in that case we have one circular orbit like this, and another circular orbit like this, so instead of transferring from point A to point B here. The, transfer is starts like at point A, the impulse is given and it is, the satellite goes into and an electrical orbit, reaches from like this. So, this is point A and the point B is shown here in this point, and again at Ban impulse is given, so that is satellite comes and becomes tangential at this point, so at that point then, we give an impulse of delta v c in the negative direction of course. So, this is the point C and that we will put it into the circular orbit. So, here the impulse required is delta v A, and here the impulse required is delta v B, and here a negative impulse is given, to put it in this circular orbit and this is the center.

So this is called Bielliptic transfer. So, Hohmann transfer this is not always cheaper, so as the radius, inner and the outer radius difference increases, so say this is the inner radius r i here and r f tends to infinity, so we keep out keep it increasing the outer radius, and then we want to take it satellite from point A to point C in the outer circle. Then which one will be more effective; Hohmann transfer or Bielliptic transfer. So, the question here can be posed as given two orbits of radius r i and r f, where r f is greater than r i, than find out either Hohmann transfer will be more efficient or Bielliptic transfer will be more efficient. So, before we go into the Bielliptic transfer, we need to work for the, how much energy is required in the Hohmann transfer, as we increase the radius of the circle r f, and this we have done earlier also. If you remember in our lecture number three, in the lecture number 23. This was the, we have started from this point at point A and we give the impulse delta v I and finally at point B, we give the impulse delta v f, and this where the results obtained.

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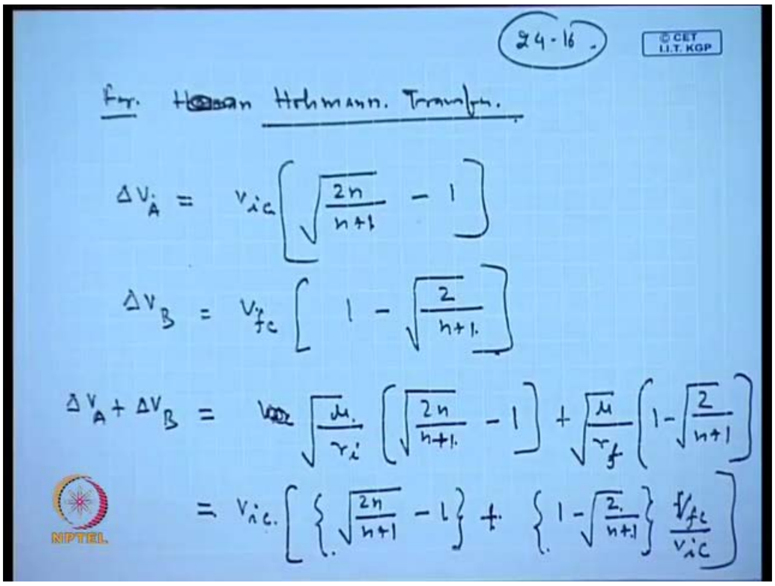

So, we can utilize this result, so for Hohmann transfer, we can write here as delta v A, this equal to v i c times 2 n by n plus 1 under root minus 1, and delta v B equal to v f c 1 minus 2 by n plus 1 under the root. Starting with this 2 equations, therefore delta v A plus delta v B, so v i c we can write as, mu by r i under root. Similarly, v f c we can write mu by r f under root, times 1 minus 2 by n plus 1 under root. So, if we take mur i outside means we are virtually taking v i c outside, so if we take v i c as the common, so what we get here. 2 n by n plus 1 under root minus 1, into v f c by v i c. Now, v f c by v i c can be written in terms of r f and r i.

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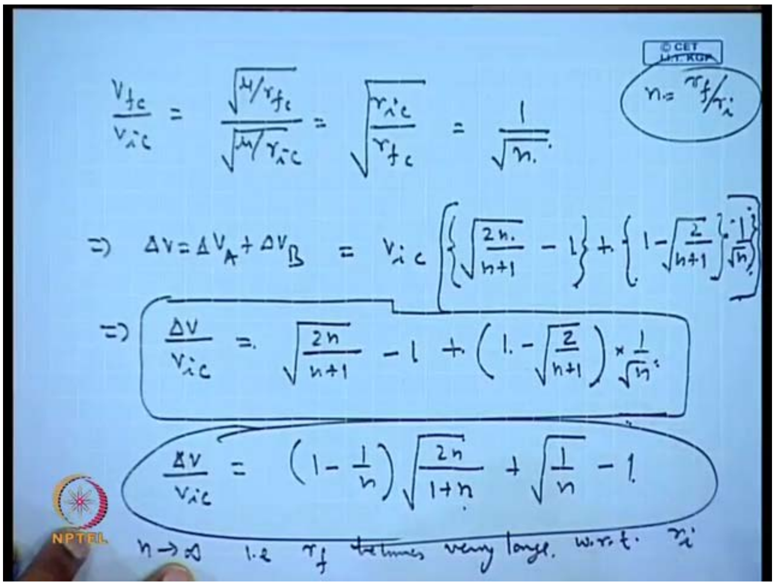

So, v f c by v i c this is nothing, but mu by r f c under root, divided by mu by r i c under root. So thus, we have delta v equal to delta v A plus delta v B, is equal to v i c times 2 n divided by n plus 1 under root, minus 1 plus, and v f c by v i c this we can write as here 1 by n under root and close the bracket, so this implies delta v by v i c. So, in this representation, some simplification can be done, if you try to write for the simplify it, so this can be written as 1 minus 1 by n times 2 n by 1 plus under root. So, whatever we get here in this place, n is a quantity which we have written as r f by r i, so if n tends to infinity, that is r f becomes very large with respect to r i.

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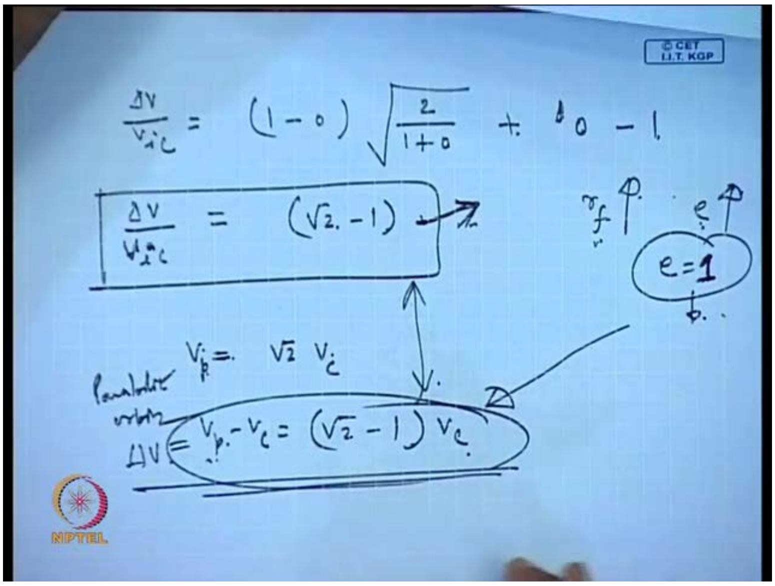

So, once n becomes very large, so this particular quantity can be written as delta v by v i c, this will becomes 1 minus 0, and this 2, here n we can divided by n, so this will become 2 divided by 1 plus 0, plus 1, 1 by large quantity n is equal to infinity, so this becomes 1 is to 0 n minus 1. So, what we get here, this is under root 2 minus 1. So, this is the limiting case, so if you remember in the earlier we have written that in the parabolic orbit, velocity is will be given as, root 2 times v circular orbit. And therefore, the difference in the velocity in the parabolic orbit and the circular orbit, v p minus v c this becomes root 2 minus 1 times v c, this is parabolic orbit.

So, in this case what is happening, we have been working with the electrical orbit, but as the value of r f increases, r f goes up, so similarly e goes up. So, for the electrical orbit, so finally once r f becomes infinity, so e will be equal to 1 at that time, and once e equal to 1 so this is referent nothing, but to the parabolic orbit, for which the difference in the velocity is given here by this equation, and this equation and this equation both are same there is no difference. So, in the limiting case electrical orbit also this acts as the parabolic orbit. So, we will continue with this, and we will try to find out what will be the efficiency of the Hohmann orbit, the Hohmann transfer and the Bielliptic transfer, and we will try to do some comparison for that in the next lecture. Thank you very much.