Space Flight Mechanics

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Lecture No. # 23

Trajectory Transfer (Contd.)

We have been working with trajectory transfer. So last time we were working with the minimum eccentricity transfer, which is also called Hohmann transfer. So, we continue with that and finish it.

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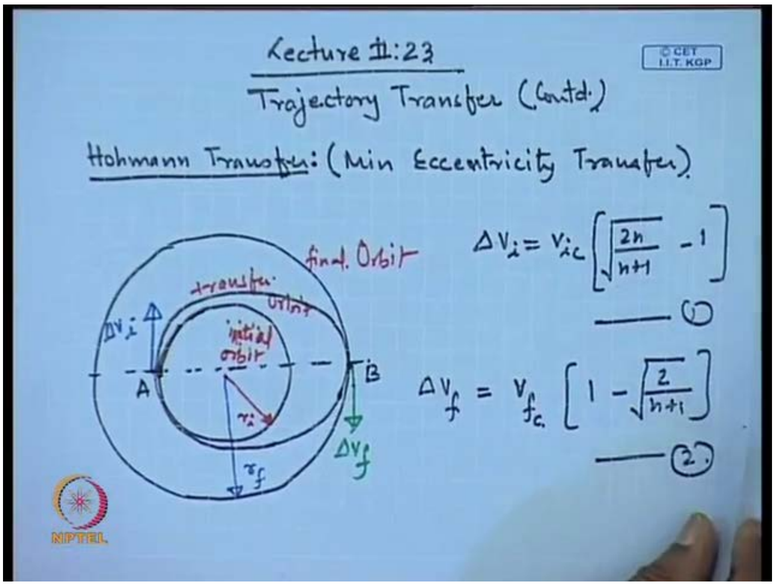

So, if given two orbits, this is the initial orbit, whose radius is r i, and then we have the final orbit, whose radius is r f. In a satellite is moving in this initial orbit and it is required to put in the final orbit here. So, one impulse is given here in this place and the satellite goes into this elliptical orbit. So, this is transfer orbit .This is final orbit and this one is initial orbit.

So, once we are given these two orbits, and we have to put the satellite in the final orbit and what we require, the impulse is in two places. One impulse is required here in this place, which we wrote as delta V i, another impulse is required here in this place and this we wrote as delta V f. So, we last time we worked out what is the expression for delta V i. So, delta V i we wrote as V i c times 2 n divided by n plus 1 under root minus 1 and this is our equation number 1, and delta V f, this we wrote as V f c times 1 minus 2 divided by n plus 1 under root; this is our equation number 2.

So, these are the two impulses required here in place A and this is the place B. So, this is the perigee of the elliptical orbit and this is apogee of the elliptical orbit. So, if we give these two impulses, it will put the satellite from the initial circular orbit into the final circular orbit.

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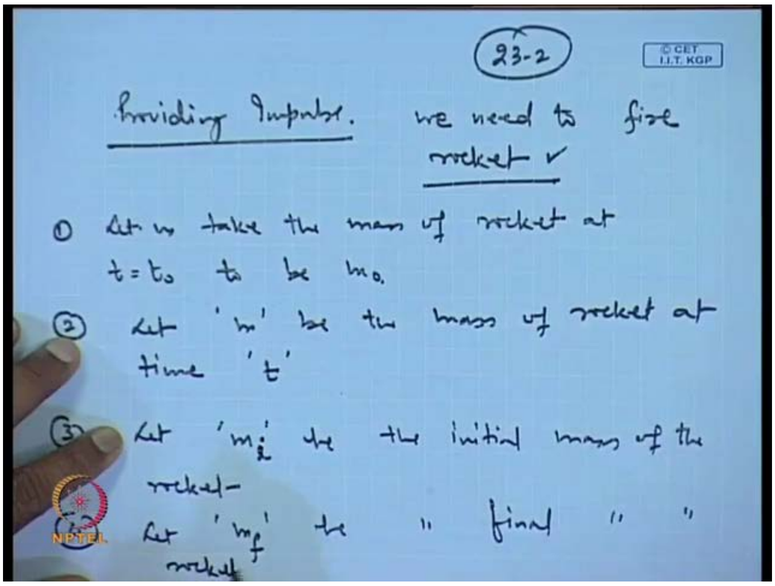

Now, for providing this impulse what we need, we have to fire the rockets. So, for providing impulse we need to fire rocket. So, the rocket, provide impulse by burning the propellant. So we are going to develop the equation for the rocket and it is a very simple equation. So let us take the mass of rocket at t equal to t 0 to be m 0. Let m be the mass of rocket at time t. Let m 0 or let us say m i be the initial mass of the rocket. Let m f be the final mass of the rocket.

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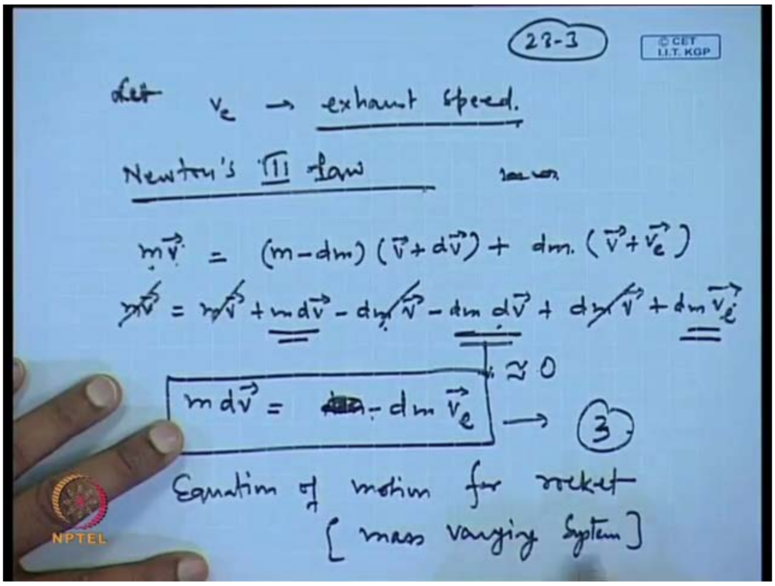

And let V e, exhaust speed: this is the speed at which the propellants are being thrown out. So, propellants are burnt and then they are thrown out at this speed, so, this is our V e. So we start using, start deriving the equation for the rocket using Newton's Third law. Newton's Third law. So, from that m times V, where m is the say the initial mass of the rocket. So instead of putting here m i, let us assume that m is the mass of the rocket at any time t.

So, m is the mass of the rocket as we have written also. So, no need of writing here and V is the speed at time t and then d m mass is ejected and the consequently the speed increases, changes by delta V and we have the ejected mass. So, ejected mass is d m and its speed is given by V plus V e, where V e is the exhaust speed with respect to, and it is always taken with respect to the rocket. So, therefore, from this place we can get m v equal to m V plus m d V minus d m V minus d m d V plus d m V plus d m V i. These two will cancel out, and minus d m V and plus d m V, they will cancel out, leaving you with these three terms. Now this term, this is very small term. So, d m is itself infinitesimal and d V is also an infinitesimal.

So, if we multiply them, so, the this is almost a second this is a second order term basically. So, we neglect it and put it as 0. So, m times d V you can write it as, m times d V is equal to d m d V, we are neglecting this one, so we will simply write as d m times V e with a negative sign. So, this is the equation number 3.This is the equation of motion for rocket, which is a mass varying system. Remember, Newton's Second law is not directly applicable to the mass varying system.

Once we have done this, then it is easy now to calculate how much mass will be required to provide the necessary impulse that we have computed.

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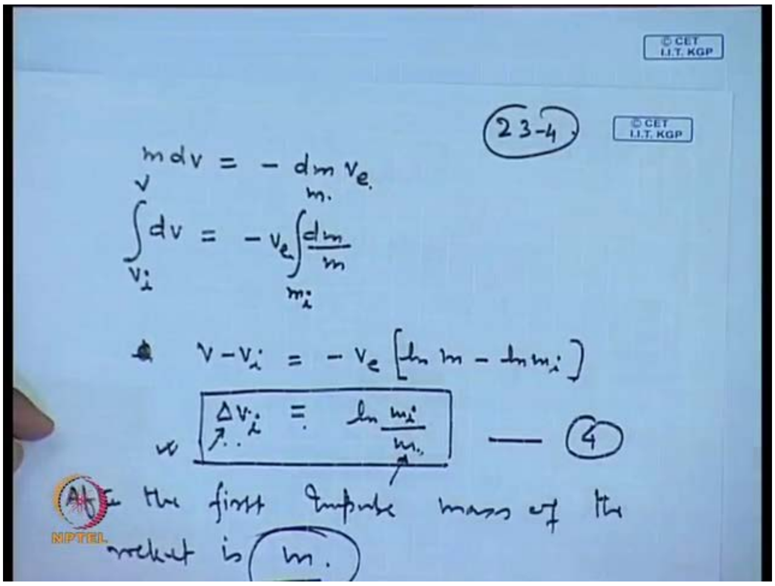

So, we have m d V, we can drop the vector sign now and simply write this as m d V equal to minus d m times V e. So, this becomes d V is equal to minus d m by m and V e. We take V e is a constant, integrate it and integrate between the initial time V i to V and m i to m. So, this becomes V minus V i equal to minus V e times l n m minus l n m i. So, this we can write as … and this becomes delta basically v i. So, at the initial point what is the impulse required, and that can be equated with the initial mass and the changed mass. So, if I need to give this much of impulse, accordingly our mass will be, the changed mass will appear as m here in this place.

Now, this mass, now we are having the mass of the rocket, which is m right now. So, after the first impulse after the first impulse mass of the rocket is m.

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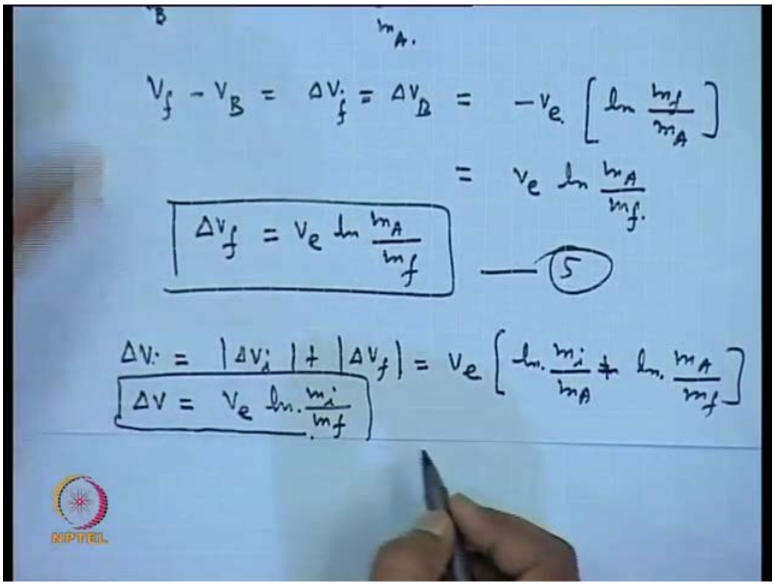

So when we need to give the second impulse, so for the second impulse similarly we can write here So, we have the starting, speed will be given and the say in the elliptical orbit, so we put here some different notation. So, the starting speed here we, let us assume this is V i, so, we write it as V 1 and the final speed we write as V f, or we can put here it as V B. Similarly here, we can put here this as V A. So, V A indicates the speed in the elliptical orbit at point A. And similarly this mass can be put here as m A. So, this is mass in the elliptical at point A. So, after the first impulse mass of the rocket is, we can put here as m A and this also becomes m A, and here we can replace this with m A. So, m A is the mass of the rocket after first impulse at point A. Now with this changed notations, so again we can write d V … minus V e times d m by m and now final mass is m f and the initial mass is m A. So integrating, we get V f minus V B is equal to delta V f, which is nothing, but the impulse required at B . So, this becomes equal to minus V e times l n m f by m A, or the same thing can be written as V e times l n m A by m f. And this is thus we what we get delta V f equal to V e times l n m A by m f. This is our equation number 5.

So, from here the total delta V, the total impulse required will be delta V i plus delta V f, the magnitude of both of them, so this becomes V e times l n m i by m A times we we can write here plus and then later on multiply, so, l n m A by m f which is equal to V e times l n m i by m f.

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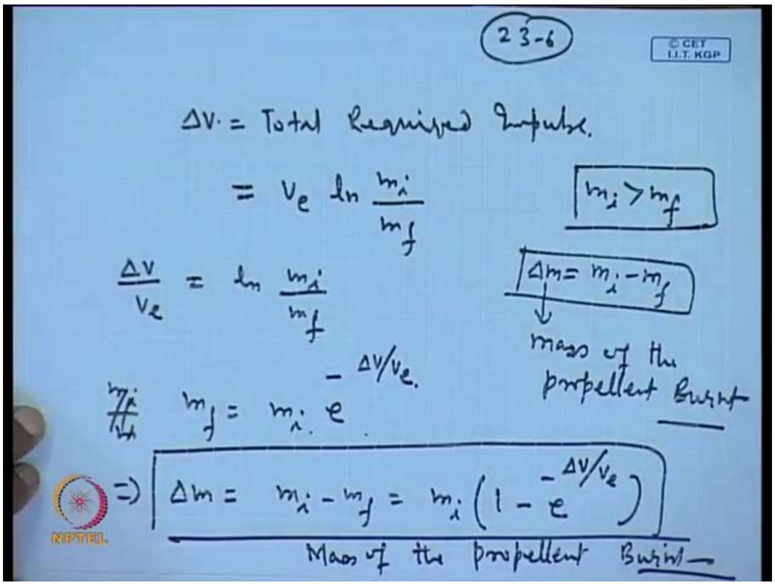

So, what we have got, the total impulse required delta V, which is the total required impulse and this is equal to V e times l n m i by m f. Obviously, m i is greater than m f because as the fuel is burnt, so, the mass of this propellant is decreasing. So, m f is the final mass of the propellant or the rocket.

So, here we are taking the mass of the rocket which includes the body. So, whatever the decrease in the mass of the rocket is there, so, m i minus m f, so, this is your delta m, this gives the mass of the propellant burnt, mass of the propellant burnt. So, from here you can calculate how much this m f will be. So, delta V by V e this becomes equal to l n m i by m f. The same thing we can write as m i by or, m f we can simply write as m i times e to the power minus delta V by V e. Therefore, this implies delta m will be equal to m i minus m f equal to m i times 1 minus e to the power minus delta V vs V e. So, this is the mass of the propellant burnt.

So, you can see that how simplified this equation is. So, if you need to do some manoeuvre, you just calculate the impulses required in various places, and then using this equation, that we have developed for the rocket, the total amount of the propellant required can be computed.

Once we have done this, now we go back again into the Hohmann transfer. So, the Hohmann transfer earlier we worked, that derivation was little long. So, a shorter way of writing the same thing is, not to go through the energy method, but to use the velocity equations, which are derived from the energy equations itself. So, we are at the need of using the energy equation and go in a longer procedure. So, the alternative way for deriving the Hohmann transfer, we are going to do now. We will carry this out in the future also, for any other purpose. So, we will follow the same method for any other computation.

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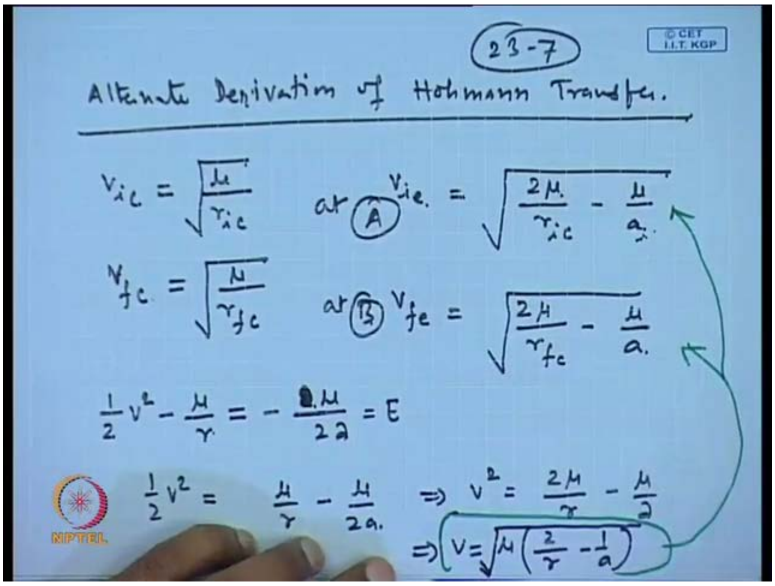

So, here we have alternate derivation of Hohmann transfer. So we take the same problem, in which the satellite is moving in the initial orbit and then we need to put it in the final orbit. So the satellite is here at A and this is to be put in B. So, the procedure is the same, and we get the exactly the same equation 1 and 2, for this alternative route also so. In fact, the alternative route we are taking, it is a just a compressed form of the energy equation that we have used. So, we go into this right now.

So, let us say the V i c is the velocity of the satellite in the initial orbit, initial circular orbit. So, this will be written as mu by r i c under root and V f c, velocity in the final circular orbit, this will mu divided by r f c under root. Now, in the elliptical orbit at the initial point A, so, there the velocity can be written as 2 mu by r i c minus mu by a, where a is the semi major axis of the transfer orbit, which is the elliptical orbit here in this case and, V f e at the final point which is the point B. So, this is at A at A and this is at B. Same equation is valid. This quantity, a, is constant of the orbit, of the elliptical or transfer orbit, so, it is not changing here in this place. So, how we have got it, this equation we have done this earlier also and it is easy to work out. So, we know that the energy per unit mass can we written in terms of the kinetic energy per unit mass minus mu by r, which is a potential energy per unit mass of the satellite and this will be equal to minus E by, E is the total energy here. So, sorry this will be minus mu by 2 a, which is the total energy of the satellite. So, from here we can write now 1 by 2 V square is equal to mu by r minus mu by 2 a. And this implies V square is equals to 2 mu by r minus mu by a and therefore, this implies V equal to mu times 2 by r minus 1 by a under root. So, this is the equation that we are using in both these places.

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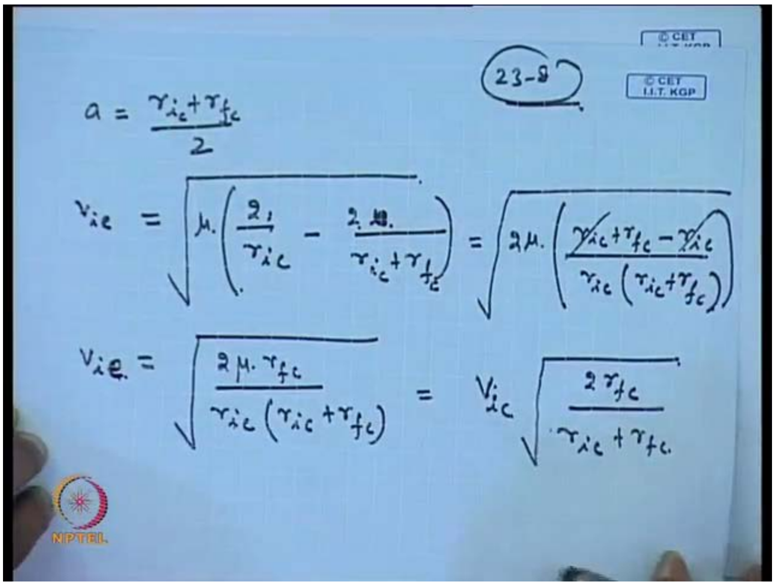

And a, is given as r i plus r f by 2, this we have done earlier also. So, the semi major axis of this elliptical orbit will be nothing, but the distance from here to here, which is r i, this distance is r i and distance from here to here, which is nothing but, this distance r f, from the center to point B, this is r f. So, semi major axis will be obtained by summing this r i plus r f and dividing it by 2.

So, a, is equals to r i plus r f by 2 and therefore, the V i e, this can be written as 2 mu, we can take it outside and 2 by r i c minus mu by i r i plus r f, mu we have taken outside, so, mu will go outside and this is a square root. Now this becomes mu times r i c if we have put the to indicate the radius of the circular orbit, so, we can put here c and here also we can put c. This is r i c plus r f c and 2, we can take it outside the whole thing, now this becomes 2 r i c plus r f c minus r i c this whole under root. So, this is 2 mu times r i c, r i c cancel out and we get r f c divided by r i c times r i c plus r f c. So, mu by r i c is nothing, but velocity in the circular orbit. This is e and mu by r i c is V i c, so, we take it outside and inside we can write as 2 r f c by...

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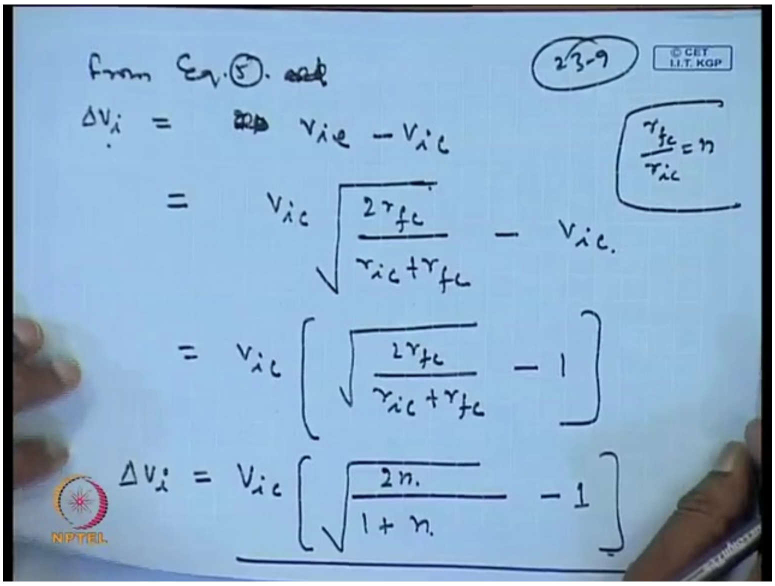

Therefore, so, the change in velocity required at the initial point delta V i, this can be written as 2 mu, V i c from this equation, this equation also we can number. This is equation number 5. So, from equation 5 and from equation 5 delta V i we can write as V i e minus V i c this will be the impulse required at the point A. So, V i e we have written as V i c times 2 r f c divided by r i c plus r f c under root minus V i c. Taking V i c outside, and if we remember what we have done last time, we put the r f by r i as n, it is this ratio we have written as n. So, in this case, the final circular orbit and r i c this equal to n. So, we divide the numerator and denominator here with r i c. So, this becomes V i c equal to 2 n divided by 1 plus n under root and then minus 1. So, this is the impulse required at the initial point and if you remember that this is the equation we have derived and this was the equation number 1 that we have written.

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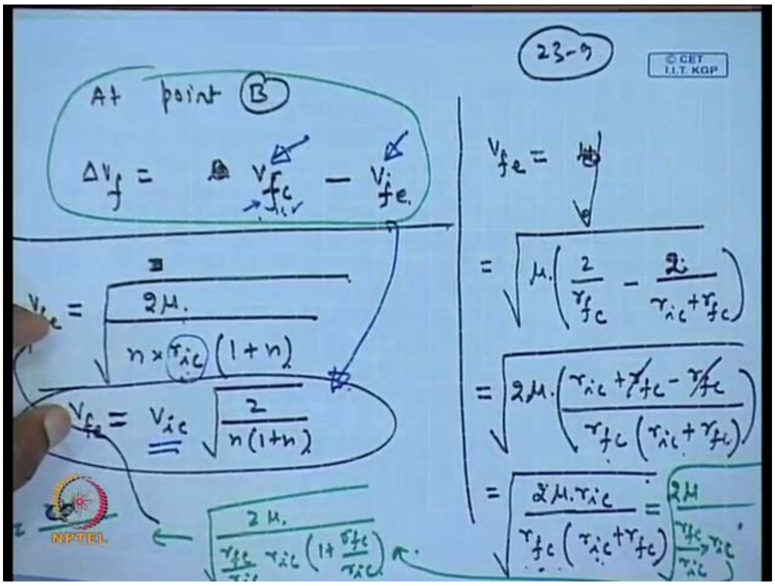

Similarly at point B, we can write delta V f equal to V f c, this is the velocity in the final orbit circular orbit minus V f e at the final point in the elliptical orbit.

And again V f e we can evaluate here. So, V f e is nothing but, mu times is equals under root mu times 2 by r f c minus 1 by a and a is nothing but r i c plus r f c divided by 2, so 2 goes into the numerator. So, this gives us mu times 2 and then r i c plus r f c minus r f c divided by r f c times r i c plus r f c. This cancels out and we get here, 2 mu 2 mu r i c divided by r f c times … under root. So, let us develop here further. So, this can be written as, we can divide r f c by r i c. So, if we do that, this will become 2 mu times r f c by r i c and then r i c is, here r i c we can take it outside the bracket and the whole thing then will become 1 plus r f c by… So, we write here in this place.

So, this becomes 2 mu divided by r f c divided by r i c times r i c plus 1 plus r f c by r i c under root, and this becomes equal to 2 mu. Now r f c by r i c is equal to n, so, replace we replace it in that terms. So, again rewriting, here let us make this space here available. So, we will carry out on the next space. So, this becomes V f e equal to 2 mu; this quantity is n times r i c times 1 plus r f c by r i c is equals to n under root. So, we haVe started with this equation mu times 2 divided by this, this is fine. So, 2 mu r i c,2 r f c and, mu by r i c is nothing, but our, we have already written this this is our V mu by r i c under root, this we can write as V i c under root and then we will have 2 times n times 1 plus n. Now if we try to write it in this format, then you can see the difficulty that, here V f c is the velocity in the final circular orbit and V f e just now we have computed here which appears here, but it appears in terms of V i c and therefore, we cannot take common the V f c. So, what we will do instead of taking r i c, what the r i c is appearing here we should have r f c appear in this place. So, if that happens then the mu by r f c then becomes V f c and then, we can take common and write in the, will be able to write in the previous format that we have developed.

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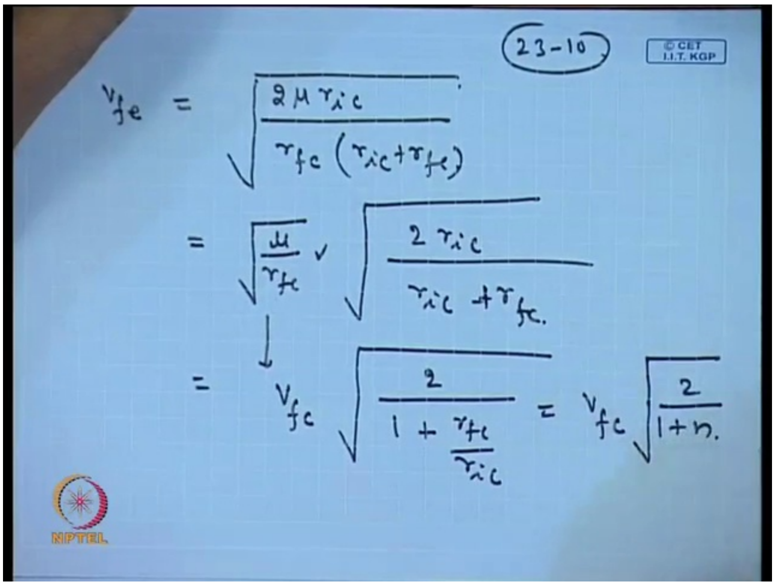

So, we will be able to write it in this format. So, we rework it. So, what we do here, now V f e we can simply write this as mu 2 mu times r i c divided by r f c times r i c plus r f c under root and mu by r f c we will take it outside. So, this is mu by r f c, let us write it in this space and this becomes 2 r i c divided by r i c plus r f c. Now if we look into this way, so, here this quantity which is appearing here this is nothing, but your velocity in the final circular orbit and then you can divide here r r by r i c, so, this becomes 2 divided by 1 plus r f c by r i c. So, this is your V f c times 2 divided by 1 plus n.

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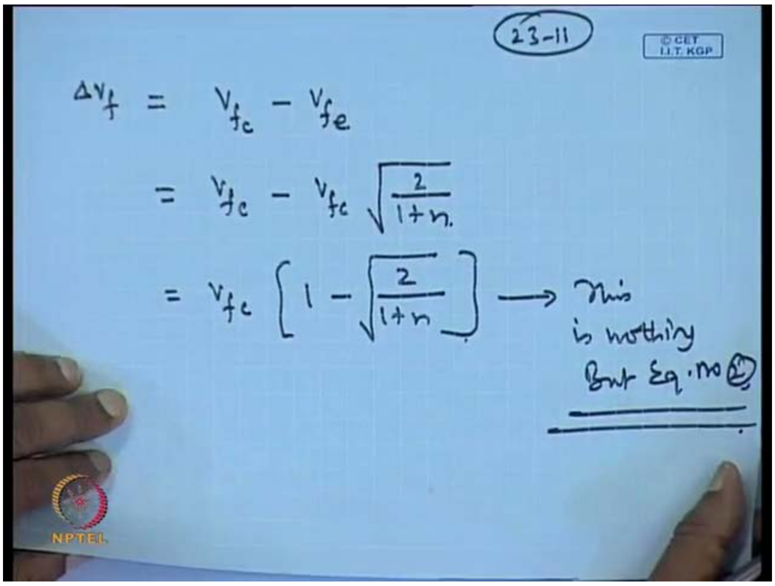

Once we have got this, now delta V f delta V f will be equal to V final in circular orbit minus V final in elliptical orbit. So, V final in circular orbit minus V final in elliptical orbit, just now we have written, this is f c times 2 divided by 1 plus n. So, this is V f c times 2 divided by 1 plus n, taking out V f c outside and this becomes 1 minus 2 divided by 1 plus n under root. And this is nothing, but your equation number 2 that we have written earlier. So, this is nothing, but equation number 2. Thus we see that, the Hohmann transfer, for the Hohmann transfer the delta V i and delta V f that we computed using the energy method, the same thing can be worked out in a very short way using the equation for the velocity. So, we directly and of course, there we have used the, we have equated the energy equation only, but in the while choosing this way we have shortened the whole derivation a lot, otherwise it was little more complicated and little difficult to handle also.

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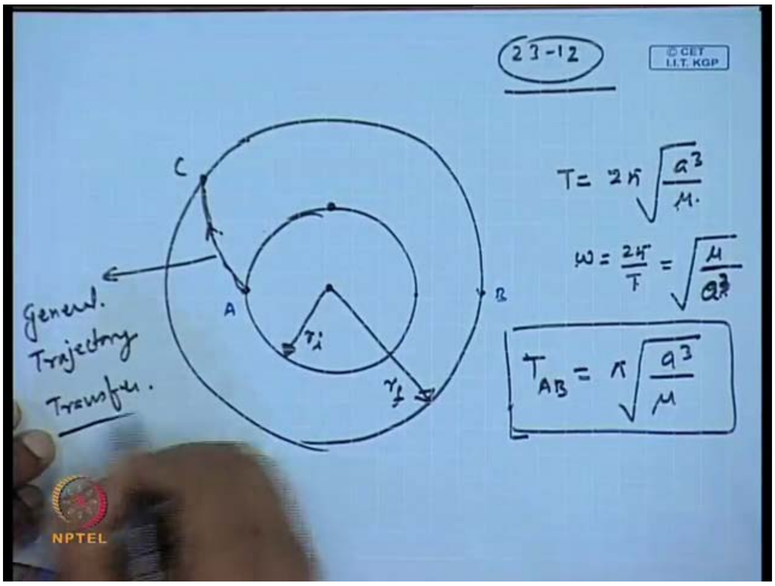

So, till now whatever we have done, this is all about the transferring transferring from one circular orbit to another circular orbit in an elliptical orbit, where the transfer orbit is an elliptical orbit. So, we have the starting point was here and the ending point was here in this place. Now suppose I do not want to do this, the reason is very simple, the time period of the transfer, the time taken for the transfer will be time is given is 2 pi by, see the if angular velocity how do we write ? So, if angular velocity, once we write the expression for this, so from there we can write the equation for the time period. So, here the time period if you, see it is going to take a lot of time in this orbit. So, look in this figure, so, here point from point A to point B, it is a going in an elliptical orbit. So, from going to point A to point B and coming from this place to this place, it will complete one orbit, and for one orbit the time period if we know. So, we can find out how much the time will be taken to go from point A to point B and that is the time the satellite requires to, go from point A to point B. But this kind of calculation, it poses certain problem. What is the exact problem here, because if we go for this calculation? So, for going to point B, always I have to go from this point to point. Only thing I am consuming little lesser energy, as we know that if the if the velocity vector at point A and the impulse given at point A both are in the same direction, then the kinetic energy, the change in the kinetic energy will be maximum. So, from that point of view, we are going to spend little less energy, but the time of transition from point A to point B will will be large. And in many cases it may not be acceptable. We want that the time period should be small. So, the time period is given by T equal to 2 pi times a cube divided by mu. So, from where we have derived it it? Actually we have derived it from omega is equal to 2 pi by T. You can write it in this way.

So, this is what earlier we have used frequently, this equation mu by a cube under root . So, your time to go from here to here, so, T A B, this will be equal to pi times a cube by mu under root. So, you can assume that, if the radius of the initial orbit, this is r i, this is much less than the final orbit, then how much time it is going to consume? and it is a really pathological. So, in that situation you would like to not to do like that, but rather transfer from some point say A to some point C here, along this orbit. So, time of transition from this point to this point; obviously, it will be very small and thereby you save a lot of time, though of course, we know that the energy to be given at this point and this point will be much larger.

So, we have to pay in terms of the propellant mass, there is no other options. So, we cannot get the 2 advantage simultaneously, either we can have lesser transfer time or either we save the energy or the mass of the propellant.

So, going to the next step, before going to the next step which we were discussing here. So, this is basically a general trajectory transfer, and before we carry out this a step, we will just have a few steps about the energy difference between the circular and the parabolic orbit, parabolic and the hyperbolic orbit to complete the topic.

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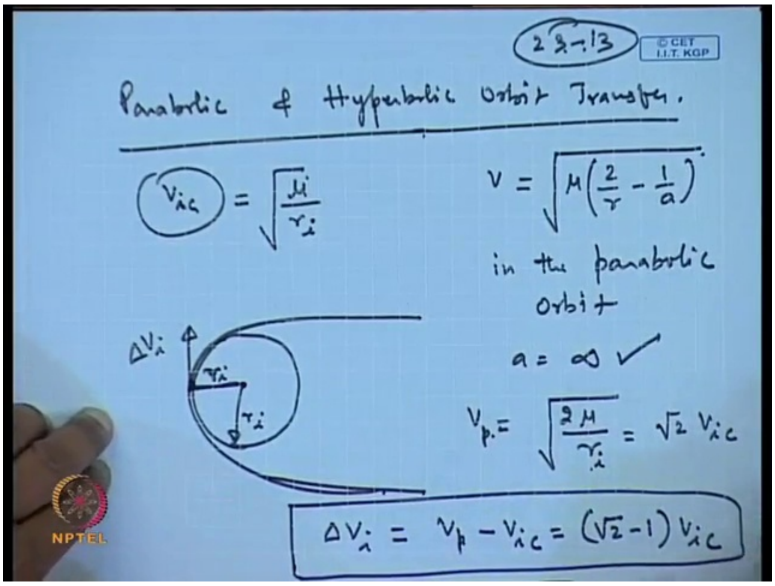

To say we have one circular orbit given, in which the satellite is moving. So, V i c this will be equal to mu by r i under root. Now if you need to put the satellite in a parabolic orbit, so, you have the circular orbit here, whose radius is r i and, you are showing it in certain parabolic orbit. So, we use the equation for the velocity, so, V equal to mu times 2 by r minus 1 by a. In the parabolic orbit a equal to infinity, semi major axis of the parabolic orbit is infinity and therefore, V parabolic we can write here as 2 mu by r under root and we put here r i. So, if we are here in this place, so, this is the distance r i. So, this is nothing, but under root 2 times and mu by r i, is nothing, but V i c. Therefore, the delta V, the impulse, that you need here in this place, delta V i that will be, V p minus V i c is equals to under root 2 minus 1 times V i c.

Similarly, if you want to find out the what will be impulse required if the satellite is moving in a parabolic orbit and if you need to send into a hyperbolic orbit, so, how much impulse you require that can be calculated. Or either the satellite is in the circular orbit and you need to send it into the hyperbolic orbit, so, how much impulse is to be given? So, what is the benefit of the parabolic and the hyperbolic orbit? So, Benefit is that, these orbits are faster because you can see that V p, in the velocity in the parabolic orbit in this point is much larger, this is root 2 times larger than the circular orbit and therefore, it will cover certain trajectory up to certain distance in a shorter time. Similarly the same thing is true for the hyperbolic orbit also.

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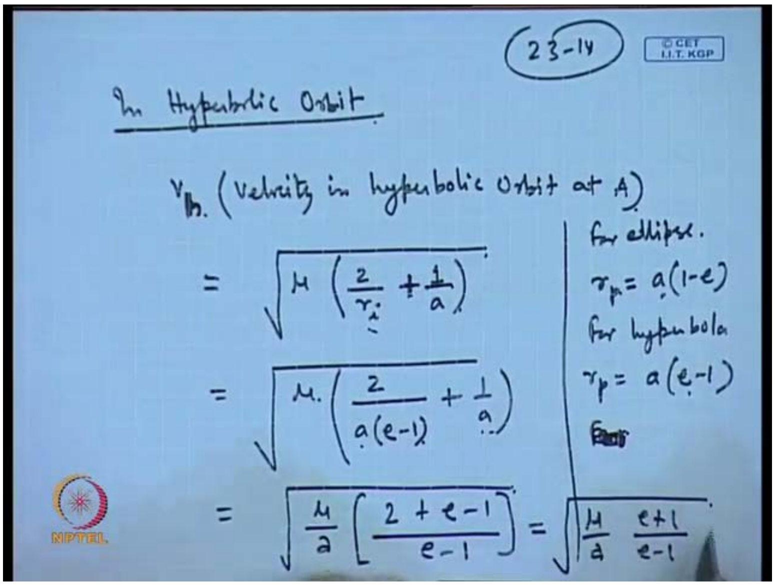

So, for the hyperbolic orbit and this is the perigee point here. So, this is your perigee point. In hyperbolic orbit, we can write the velocity as p V, here we have written for the parabolic orbit, so, we will write here as V h, this is the velocity in hyperbolic orbit hyperbolic orbit at the initial point A. This is the initial point, let us say A, and this will be equal to mu times 2 by r i plus 1 by a. This we have discussed already, this minus sign will change to plus plus sign for the hyperbolic orbit. So, now, you have given this equation. So, r i is the perigee position, so, therefore, you can write this as 2 times r i will be nothing but a times e minus 1,for hyperbolic orbit.

For ellipse, r perigee, we can write as a times 1 minus e. For hyperbola, r perigee is written as a times e minus one, because, for hyperbola e is greater than 1. For parabolic orbit, obviously, we have a equal to infinity, so, here in this case now we can separate out a and write it in a way, mu by a times 2 plus e minus 1 divided by e minus 1 under root. So, this quantity becomes equal to e plus 1 divided by e minus 1 under root.

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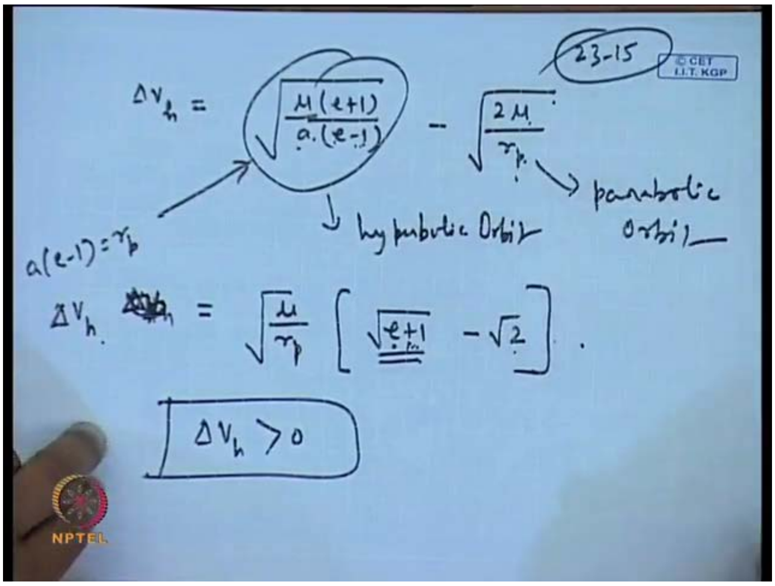

So, delta V h in this case required will be, mu by a times e plus 1 divided by e minus 1 under root minus 2 mu by r p under root, where r p is the perigee distance. This is for parabolic orbit, this is for hyperbolic orbit and a times e minus 1 is nothing but the r p. So, the whole thing can be simplified and written in the way, now a times e minus 1 is nothing, but r p here in this case, so, this will become mu by r p under root times e plus one under root minus under root 2, inversely because e is greater than 1 and therefore, this quantity is greater than 2, the under root here. So, this is; obviously, a positive quantity. So, delta V h is greater than 0. So, you need more energy to put in the put the satellite in hyperbolic orbit from parabolic orbit.

So now we have completed these topics and we can go for generalized transfer. So, for generalized transfer we worked out the figure earlier, this figure again we can repeat here.

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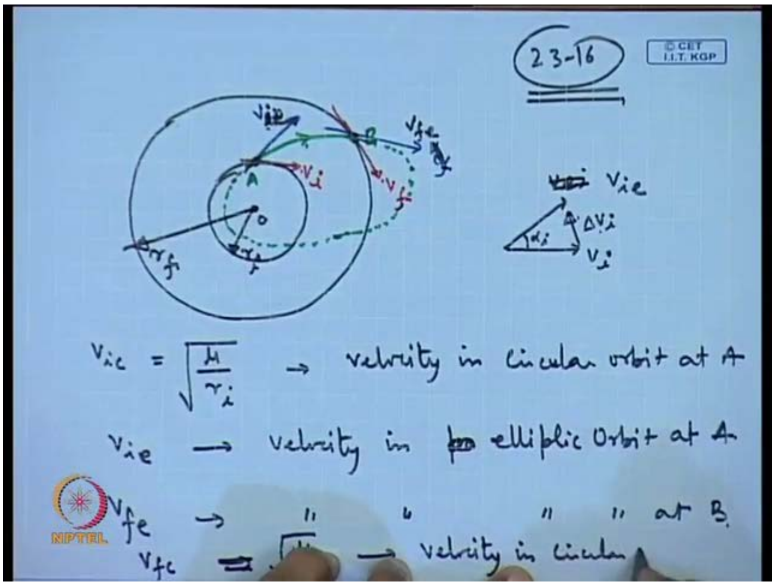

We want to send the satellite from point A to point B along this trajectory. Velocity in this circular orbit will be in the tangent direction, so, this is say V i and here, the velocity will be tangent to this circle, so, this V f. And if we are sending the satellite along this orbit, so, here the orbit, again the velocity in this elliptical orbit will be written as V e i and here in this place it will be tangent to this orbit, and this, is let us say V e f. So, your elliptical orbit will be somewhat looking like. So, now, the perigee distance of the transfer orbit is less and apogee distance is larger, and going from this point to this point, the time taken will be small. So, this we need to work out.

This is your center a here. So, for working out this we take this initial velocity as V i and this is V e i and, the required impulse will be delta V i and let us say this angle is alpha i. So, we need to work out what is the impulse required at the point A, which is given by delta V i. So, again V i c this is mu by r i under root. This is your r i. This is your r f , and this is nothing, but velocity in circular orbit at A, and V i e , and we can put here this, as

V i e, to indicate this is the initial point and this the initial orbit and similarly here we can write this as V f e, this notation is better. So, V i e is the velocity in elliptical orbit at A, V f is the velocity in elliptical orbit at B and V f c this is equal to mu by r f under root. This is the velocity in circular orbit at B.

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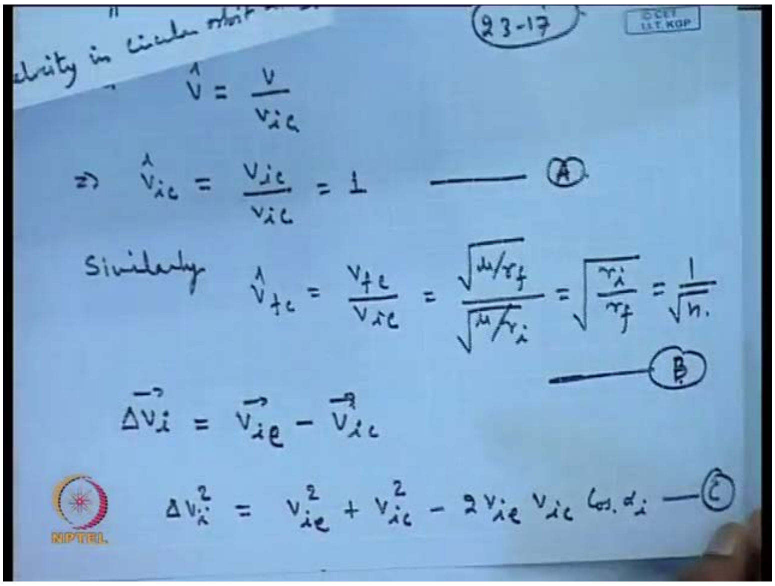

Let V cap equal to V by V i c. So, this implies, V i c cap will be equal to V i c by V i c, this is equal to 1. Similarly we can write V f c cap is equal to V f c by V i c, is equal to mu by r f under root divided by mu by r i under root. So, this becomes r i by r f under root is equal to 1 by n under root. This is our equation number, you write this as equation number A and this as equation number B. Therefore, delta v i, this will be equal to V i e… . So, from this figure, this is the velocity in the circular orbit V i c, and this is the velocity in the elliptical orbit and this is the delta V i, alpha i is the angle between them. So, we can write here delta V i square equals V i e square plus V i c square minus 2 V i e times V i c times cos alpha i. This is equation number C.

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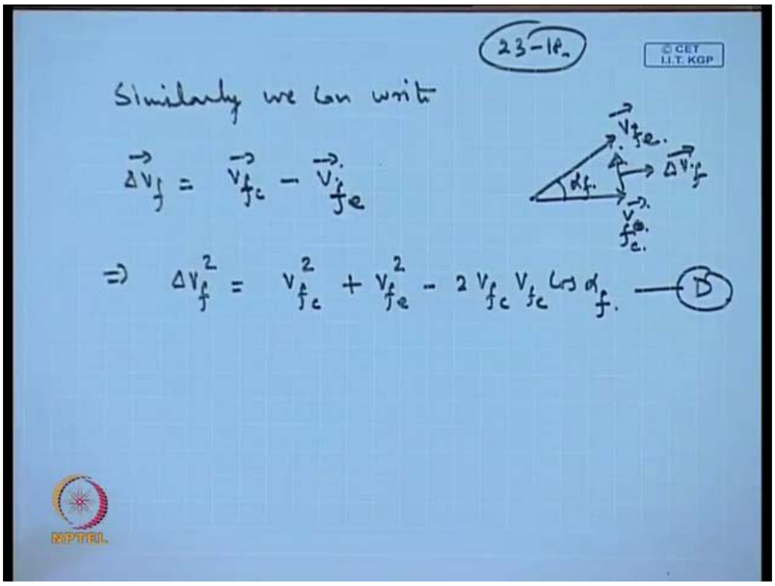

Similarly we can write delta V f is equal to V f c minus V f e. This implies delta V f square, this will be equal to V f c square plus V f e square minus 2 V f c times V f e times cos alpha f. So, in this case we have V c is the V f c is the velocity in the circular orbit in the largest orbit and largest circular orbit and this is this is this is V f e, this angle is alpha f and therefore, the quantity here this will be delta V.

So, these are the vectors here. So, this is delta V f. So, here also we can show them by vector. So, we put here alpha f. Now this we can number as equation number D. So, we have two equations C and D, which gives you the impulse required at the initial point and the final point, but you can see that this equation is not easy to work out with, we need further simplification of these two equations to get the total amount of impulse required. So, will work out this in the next class. So, we end here.

Thank you very much.