Space Flight Mechanics

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Module No # 01

Lecture No.# 22

Trajectory Transfer (Contd.)

We have been discussing about the trajectory transfer in the last lecture, so, we continue with that. So, this time we are going to discuss about the transfer between two coplanar circular orbits.

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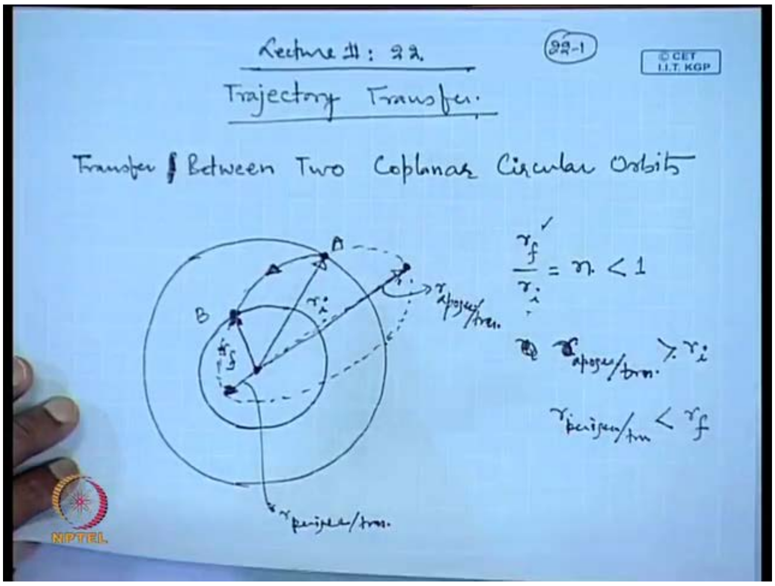

So, let us assume, we are given one orbit ,in which the satellite is currently moving, which whose radius is r i and we have to transfer the satellite from this orbit to another orbit in which the radius is, whose radius is r f. And let us assume, that the satellite is here in this place and this is to be moved to the point v, along this trajectory. So, we have to find out, what will be the eccentricity of this orbit in which the satellite will be transferred. This is the transfer trajectory and we have find out the eccentricity of this orbit.

So, if we extend, this orbit it, it looks like this, as shown shown by the dotted line. So, we can write r f divided by r i, this is equal to n,is the ratio of the final radius of the final orbit and ratio of the initial orbit, radius of the initial orbit. And this quantity; obviously, it is obvious that it will be less than one. And perigee we can see here, from this place. So, the apogee of the transfer orbit; apogee is the farthest point from the center of the force. So, you can see this apogee lies here. So, the apogee distance is of the transfer orbit is larger than the initial orbit. So, we can write, r apogee of transfer orbit, this will be greater than r i initial. Similarly r perigee of the transfer orbit, this will be less than r f.

So, this is the perigee point of the transfer orbit, from here to here and this line extends from here to here. This is r apogee of transfer orbit and this is r perigee of transfer orbit. So, if basically what we are interested in, we have to show what will be the minimum eccentricity of this transfer orbit and we have to find out under which condition that is going to happen.

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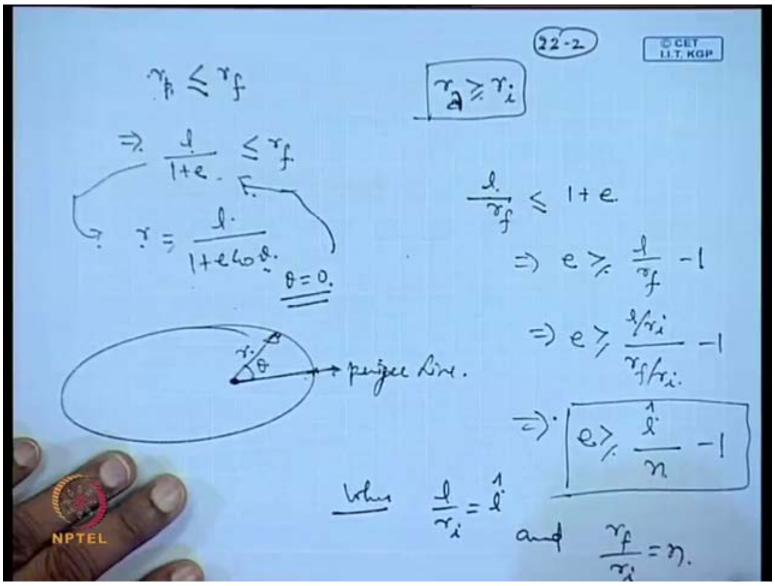

So, we write here. So, we will write in short, r perigee we will write as r p. So, this is going to be less than equal to r f, and similarly r apogee this will be greater than equal to r i. So, r p less than r f, we can write now. This implies and you know that, from where we are getting it. Already we know that r can be written as r equal to l by 1 plus e cos theta, where theta is measured from the caps line. So, if we have the center of force here, and this is the perigee line, which is also called the absline, perigee line and this is the angle theta here measured. So, and this is the radius r, so, r can be written in terms of semi latus rectum which go r is equal to l divided by 1 plus e cos theta. So, if theta is equal to 0 that gives the length of the perigee. So, this r p we have replaced by putting theta equal to 0. So, put theta equal to 0 and you you will get the quantity which is written here.

So, lby r f we can write this as, this will be less than 1 plus e. And this implies e will be greater than l by r f minus 1. And this can be written as l by r i divided by r f by r i minus 1. And this implies e is greater than, now l by r i we will write as l cap and r f divided by r i we write as n. So, this quantity can be written in this way, where l bar r i this is equal to l cap and r f by r i this is equal to n.

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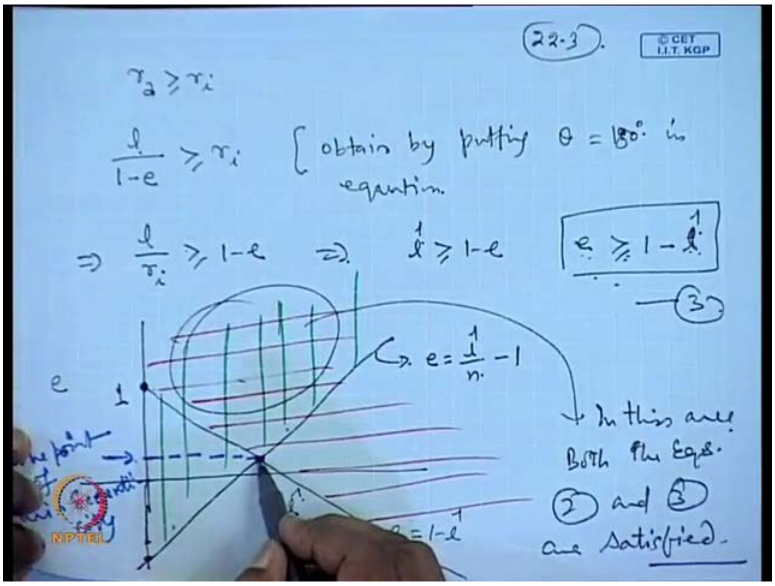

Similarly if r a is greater than r i, which we have already written. So, r a can be written as l by1 minus e and this we obtain by obtained by putting theta equal to 180 degree in equation. Thus number this equation, this equation we can write as equation number 1. And this equation we can write as equation number two. So, if we put theta equal to180 degree. So, we get here minus sign in this place. So, for theta equal to 180 degree; means you are measuring from this position to this position and for theta equal to 0 this is just coinciding with this line. Therefore, for 180 degree you get the perigee position. So, for apogee position this is r a, and this is your r p. Hence, in the similar way you can write l by r i, this is greater than 1 minus e, this implies and l by r i we have written as l cap. So, l cap becomes greater than 1 minus e or e can be written to be greater than 1 minus l cap. This we write as our equation number 3.

So, now, we can plot equation number 2 and 3. So, l cap verses e, if we try to plot, in the equation number 1, we have e is greater than l by n minus one. So, if we plotting do the plotting. So, you can see there, that if l equal to 0, if we put the equality sign here, e equal to l cap divided by n minus 1, so, you get here putting e equal to; l equal to 0, so, at that time this vanishes and what you get is e equal to minus one. So, by putting l cap equal to 0 means we are in this position and then at that time the value of e equal to is minus 1. So, this is minus one.

And slope of this curve is obviously, positive. So, this slope, it will look like, this curve will look like this. So, this curve is your e equal to l cap divided by n minus 1. And e is greater than and the quantity: the eccentricity is greater than l by n minus one on the upper side of this. So, we have this region in which the inequality sign is satisfied. So, inequalities is satisfied in this region.

Now the next curve we take e greater than 1 minus l. Again if we put l equal to 0, so, the for the the equality sign we can see that e equal to 1. So, this gives you the curve with cos. At l equal to 0 it start here in this place. So, this is again e equal to 1 here in this place and the slope of this curve will be minus one. So, this curve will appear as somewhat in this way. And again the region in which, so, if, let us first tag it here this is e equal to 1 minus l cap. So, the region in which e is greater than 1 minus l, so, that region will be shown by this line. So, the area in which both these conditions are satisfied; the equation number 2 and 3 both are satisfied. So, this is the area; in this area both the equations 2 and 3 are satisfied. So, this implies that you can choose eccentricity in this region to satisfy the condition that we are started with, that we are sending the satellite from point A to point Balong this trajectory, so, such that the perigee is less than the radius of the final or a if this final orbit and the apogee of the transfer orbit is greater than radius of the initial orbit.

So, from here what we get? The point where the two line intersects this gives you the point of minimum eccentricity, the point of minimum eccentricity eccentricity we can. So, if the transfer orbit cannot have less than eccentricity less than this value. So, we need to find out this value, how much this will be.

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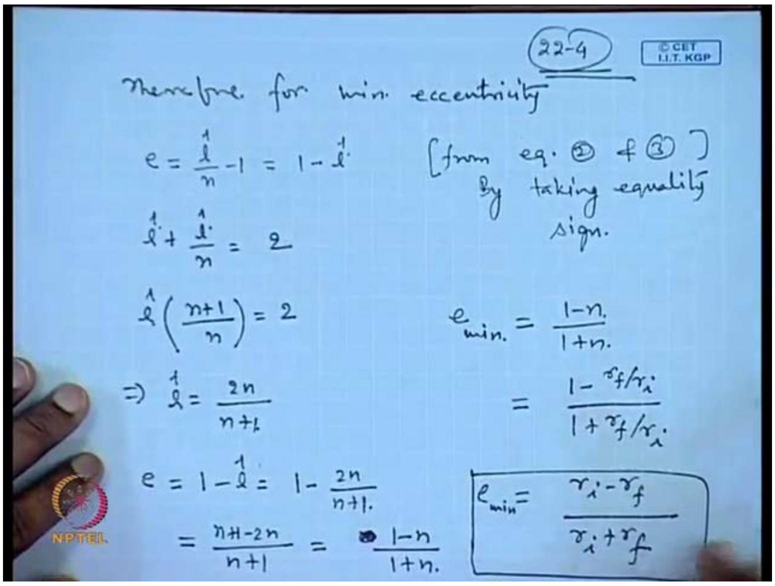

So, therefore, for minimum eccentricity we can write e is equal to l cap n minus 1, this must be equal to 1. This is from equation equation 2 and 3 by taking equality sign. Solving this this will give you l cap plus l cap divided by n equal to 2. And this gives you n plus 1 by n this equal to 2, and this implies l cap is equal to 2 n by n plus 1. And therefore, e can be written as 1 minus l cap equal o 1 minus 2 n divided by n plus 1. So, this becomes n plus 1 minus 2 n. So, 1 minus n divided by 1 plus n. So, the minimum eccentricity because this the minimum eccentricity we get for this point, so, this point is obtained from getting the from the intersection of this two lines. So, that this lines are defined by this equations which they are obtained; from the equation number 2 and 3 by taking the equality signs.

So, the same thing we have done mathematically here in this place. So, this e minimum eccentricity will be one minus ndivided by 1 plus n and in of course, we have written it as r f by r i. So, this becomes. So, the minimum eccentricity depends on the radius of the circular orbit in which the satellite is moving the initial and the final one. So, we can see from this place, and also it will be very visible from this point, that how the eccentricity will look like from this figures, let me show it.

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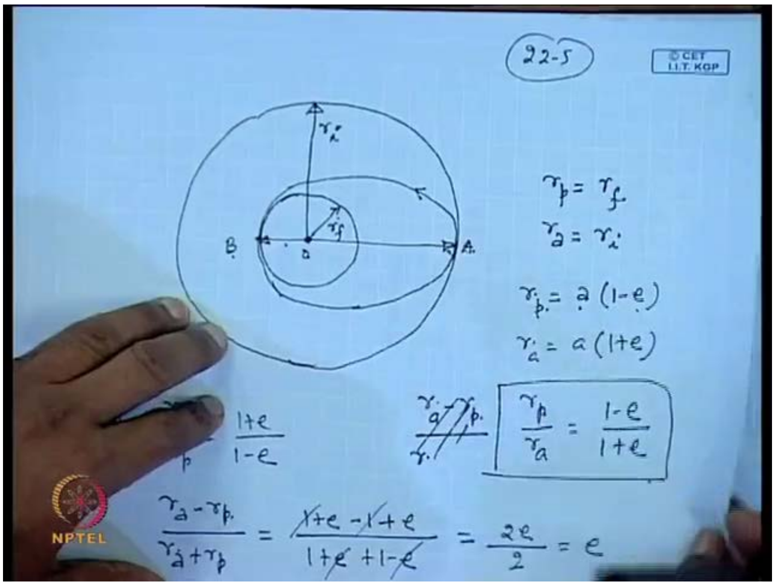

So, this is the suppose transfer orbit that we are choosing. So, we have started point aand this is the point b, now for this can we find out the eccentricity from this figure? So, if you try to do it, so, you can write this r perigee here, this is equal to r f and r apogee you can write as r i . Now we know that, r perigee, this can be written as a times one minus e; where e is the eccentricity; a is the semi major axis of the this ellipse, which is the transfer orbit in this case and also we know that r apogee equal to a times one plus e. So, from here, if we take the ratio, r apogee minus r perigee divided by, or simply you take the ratio here r perigee by r apogee, so, this becomes 1 minus e plus 1 plus e. Now you use component and dividend. So, from there you can write as, now we can take the in inverse of this first. So, let us write it in this way r p r a by r p equal to 1 plus e by 1 minus e and now apply component o dividend o. So, this will give you're a minus r p divided by r a plus r p, this will be equal to 1 plus e minus 1 plus edivided by 1 plus e plus 1 minus e. You can see this this cancels out and this becomes 2 e divided by this this cancels out to this is equal to e and the. So, the equation that we have got here r a minus r p divided by r a plus r p equal to e, r a minus r p divided by r a plus r p this quantity equal to e now already we wrote r a equal to r i and r p is equal to r f.

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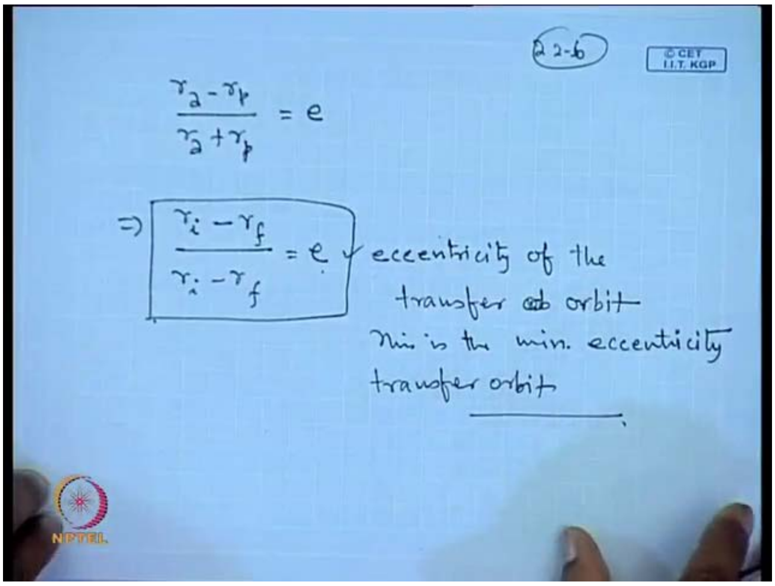

So, we have written r a equal to r I and r p is equal to r f. So, this is what we get from this figure, that the ellipse in which the earth satellite is being sent, which is the transfer trajectory here. So, its eccentricity will be given by this. So, this is eccentricity of the the transfer orbit. Now pick up the equations that we obtained. So, this equation we can number it as equation number 4. So, this is the minimum eccentricity possible eccentricity we found out, that if the initial radius is given the and the final radius is given the of the circular orbit then the transfer orbit will have minimum eccentricity which will be given by this equation.

And now you compare this equation and this equation. So, both the equations are same. So, this implies that the minimum eccentricity orbit that you are choosing, it will start at a tangentially and it end at a tangentially; means this the transfer trajectory which is an part of a ellipse from this place to this place, it will touch here in this point and touch here in this point. So, this is cotangential here in this point and in this point. So, this is the minimum eccentricity transfer orbit.

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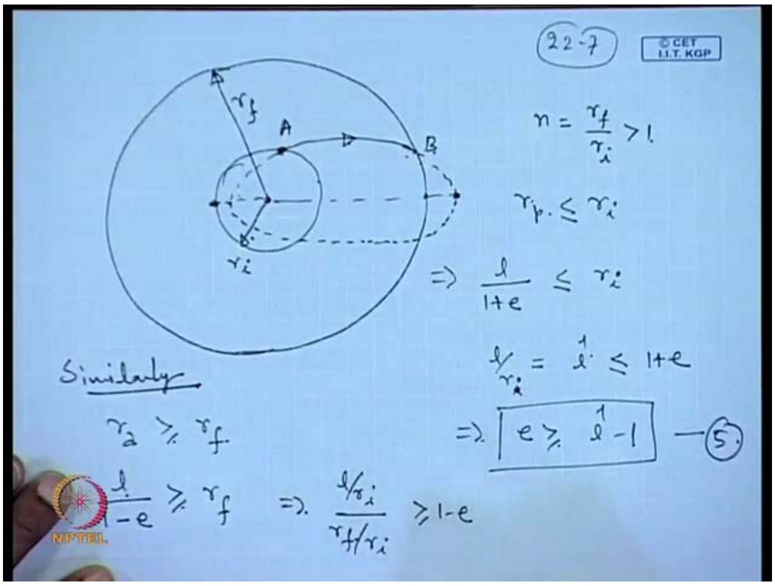

Now the way we have treated this problem; in the same way we can start with the inner orbit, if the satellite is moving in ain an orbit of radius r i and it is to be sent into an orbit of higher radius which is given by r f. So, we can start at any point say a and send the satellite into this orbit, and we want the satellite to move along certain trajectory. So, this is the point B. So, proceeding in the same way we can prove that the result that we have obtained at, if the satellite starts in this place instead of this place, and it hence appear tangentially in this place, then the eccentricity of the transfer orbit will be minimum. So, now, proceeding in the same way, n is equal to, we define as r f by r i, in this case this is greater than one ,which is very obvious from the figure and r p is r perigee r perigee is less than r i.

So, from there we can write l by one plus e, this is less than r i and l bar l by r i equal to l cap,this is less than one plus eand this implies e is greater than l cap minus one. And this is our equation number write from this as 5. Similarly r apogee, this is the apogee of transfer orbit, so, this is the r a is greater thanr f. And r a,we have written as 1 l by 1 minus e. This implies l by r I divided by r f divided by r i is greater than 1 minus e.

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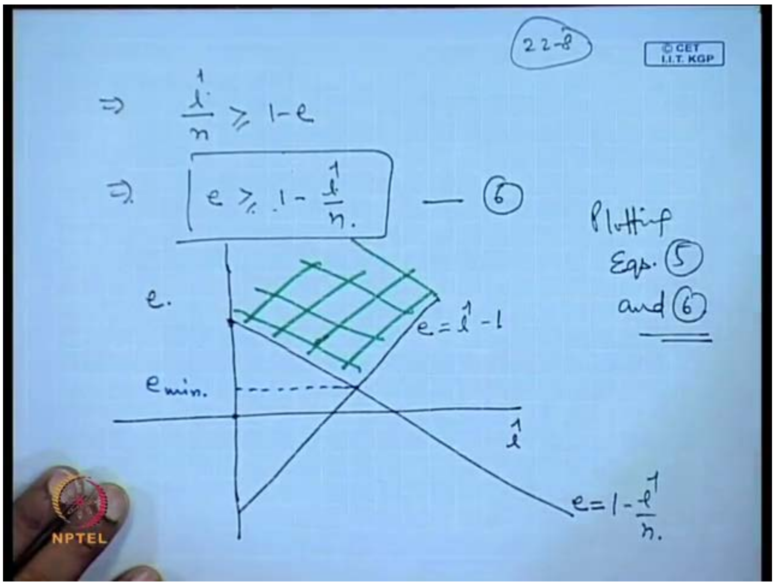

And this implies l cap by n, this is greater than 1 minus e. This implies e is greater than 1 minus l cap by e. Again we plot them l cap verses e. So, this is equation number 5 and this is our equation number 6. So, plotting equations 5 and 6 so, e is greater than 1 minus l, here we can see that when l equal to 0, e equal to 1 for the equality sign. So, this will come like this and the here the slope is negative. So, this curve will appear like this and so, this is curve belongs to l cap and this curve will become then equal to l cap minus 1. And the common area, this area then where the eccentricity can be chosen for the transfer orbit and this is the point of minimum eccentricity. So, this is e min.

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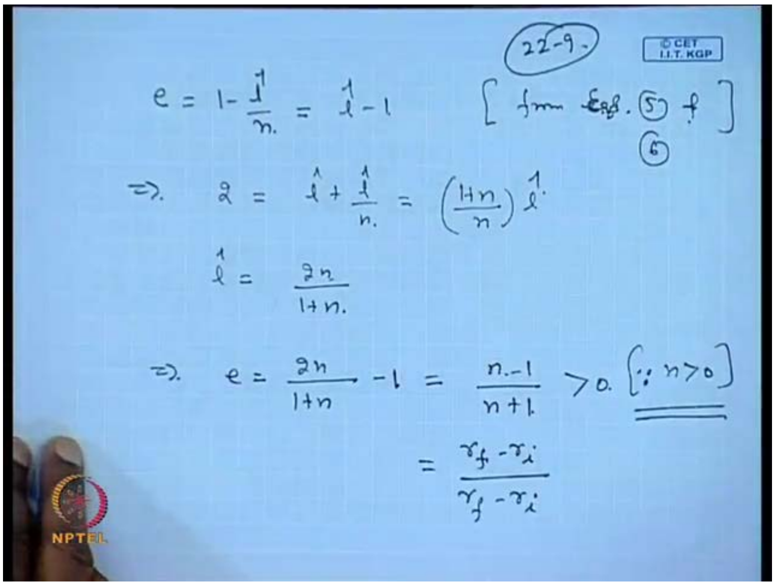

So; obviously, these are the intersection of this two equations. So, e minimum is given by intersection of this two equation therefore, we writee equal to one minus l cap by n equal to l cap minus n, this is from equation 5 and 6 . And this is greater than 0 because n is greater than 0. So, n is, we have written r f by r i.

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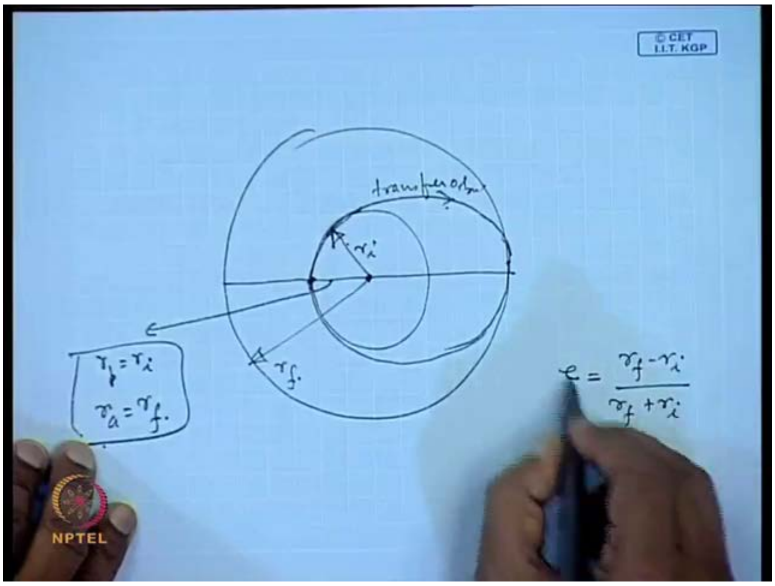

So, proceeding in the same way, we can show that, as earlier we have shown, that this is the minimum eccentricity eccentricity transfer orbit. So, this is the transfer orbit and this is our r i and this is r f. So, from the inner orbit if the satellite is sent into the outer circular orbit, so, this has to be done along this orbit. So, proceed in the same way as we have earlier done, there is nothing new in this. So, here in this case r perigee will be equal to r i and r apogee will be equal to r f. And ultimately you will get the same equation: e equal to r f minus r i divided by r f plus r i. So, once we have finished, that how the satellite will be transferred from one orbit to the another orbit, now we can find out what will be the energy required to transfer from one orbit to another orbit. So, if this is a natural sequence that should be followed. So, first we decide along which orbit,then how much energy is required,and from the energy requiredwe can find out how much fuel is required for that particular transfer. So, going into the next stage.

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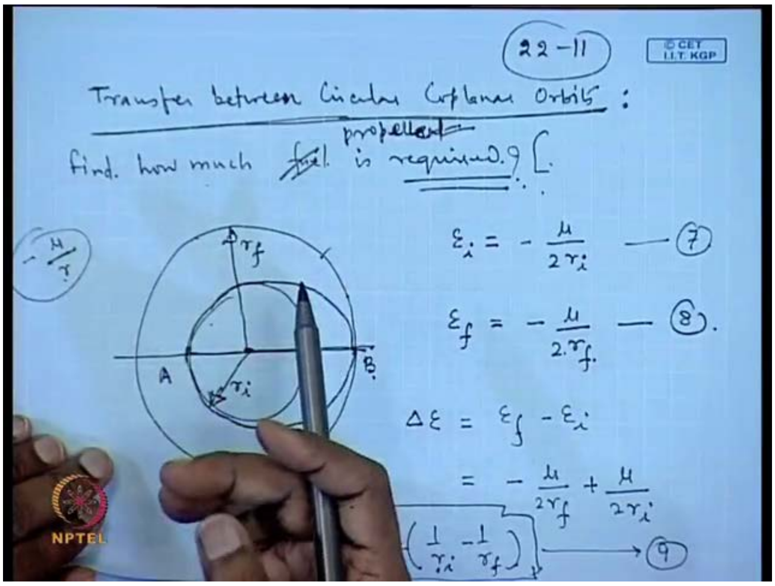
, 

So, transfer between circular coplanar orbits: So, what is our ultimate objective is to point, how much fuel is required. How much fuel, in this case we write this as propellant, how much find out how much propellant is required. So, we start with two circular orbits and now we know that this is the minimum eccentricity transfer.

We are not still proved that minimum eccentricity transfer will give consume minimum energy, or the some other eccentricity transfer will consume minimum energy. Those things are matter of discussion, which will go into in details later on. But right now we concentrate that once the minimum eccentricity transfer is fixed, then how much energy is required to go from point A to point B. Not only going from point A to B, but from the in inner orbit, which whose radius is r i to the final orbit, whose radius is r f. So, in the inner orbit we can write E initial is equal to mu minus mu by 2 r i ,these things we have already worked out,because this orbit is circular, so, these mi major axis in this case refers to the radius of this circle. So, this is our equation number 7. Similarly we can write E final is equal to minus mu by 2 r f. So, this is the total energy in of the satellite in the initial orbit here, and this is the total energy of the satellite in the final orbit. So, the change in energy that takes place, that will be given by E f minusE i and E f is minus mu by 2 r f. So, this becomes, taking mu outside and 2 outside, we can write this as 1 by r i minus 1 by r f; this is delta E. This is equation number 9. From here you can see that, because r i is less than r f therefore, this quantity in the bracket is positive.

So, this implies, that, there is positive change in the energy,and that energy is supplied by giving the impulse. So, if that is telling that the increasing the kinetic energy of the satellite. And also, as the satellite moves from this orbit to this orbit, the potential energy also goes up, because, the potential energy we are writing as minus mu by r per unit. This is the energy per unit mass. So, as the r increases, so, becomes this becomes less negative, therefore, it implies, that the potential energy is increasing. For the smaller orbit, this is more negative, means the potential energy is less.

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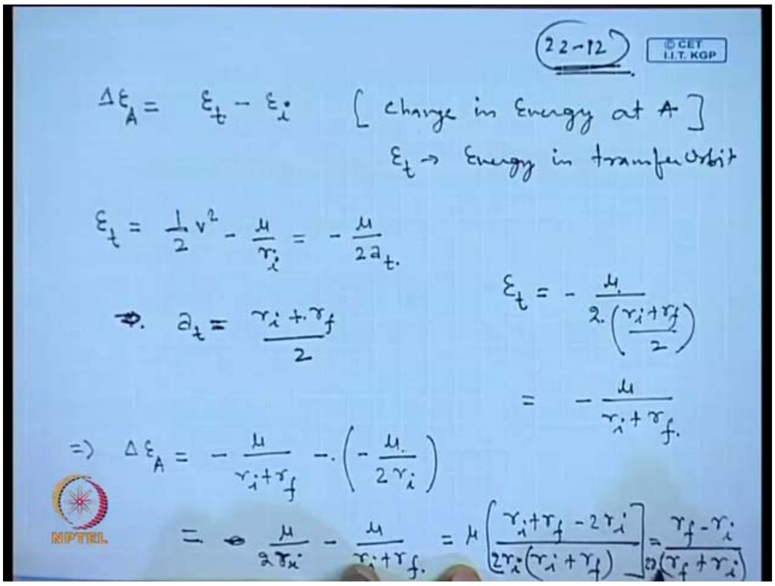

Therefore, we have delta EA will be equal toE transfer orbit minus E i. So, this is the change in energy at A. And E t is the energy in the transfer orbit.Now E t can be written as 1 by 2 v square minus mu by r, is equal to minus mu by 2 a t, and here r of course, we know that that point A the r is nothing, but equal to r i. And more over, we know that a t equal to the transfer orbit, now this will be r i plus r a by two, the semi major axis. So, therefore, E t becomes minus mu by 2, minus mu by r i plus r f. Therefore, delta E a, this will be given as E t equal to minus mu by r i plus r f minus E i, now E i we know that this is equal to minus mu by 2 r i. So, the quantity that will result, this will be mu by 2 r i minus mu by r i plus r f. This is 2 r i.

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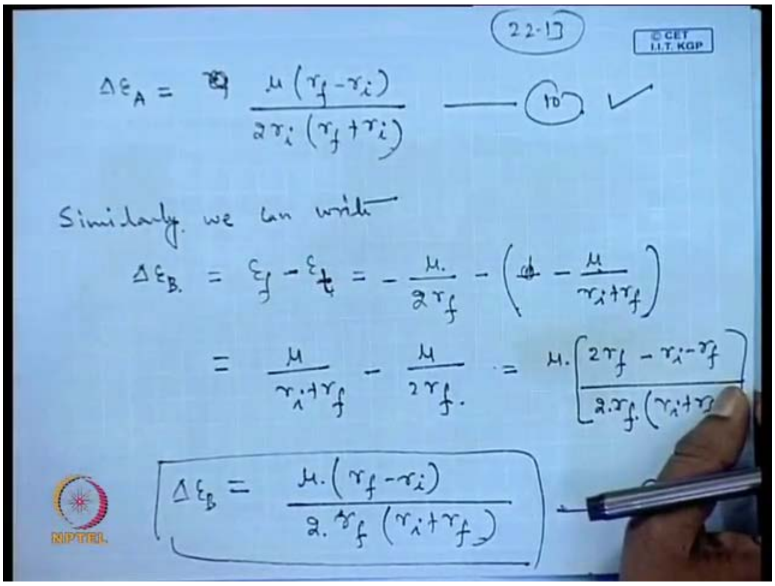

So, what we have got now delta EAis equal to r f mu times,then mu is missing hereand this is mu timesr f minus r idivided by 2 r i times r f plus r i. This is our equation number 10. Similarly we can write delta E B, the change in energy at B, this will be E f minus E t. So, E f is nothing, but minus mu by 2 r f and then minus E t is nothing, but 1 by, now in that E t is the energy in that transfer orbit. So, the total energy remains constant. So, wherever the particle be; wherever the satellite be in this orbit, either be satellite here, or either here, in this place the total energy does not vary. Kinetic energy will vary, potential energy will vary, but total energy will remain constant. Therefore, this can be written as minus mu by now the semi major axis, two times the semi major axis will be the total length.The semi major axis means the distance between A and B divided by 2. So, the two times the semi major axis will be distance A B ,which is nothing, but r i plus r f by 2. So, r i plus r f this is r i plus r f.

So, therefore, this becomes and this can be written as mu times 2 r f minus r i minus r f divided by 2 times r f times r i plus r f . So, the previous equation we have equation number 10 here and this we term as equation number 11. So, whatever the energy has being the energy change takes place at A and B. So, these are imparted by giving the impulse. So, from here the impulse arises, it impulse arises from firing of the rockets. So, as we fire the rocket, so, if it burns the mass, and by burning the mass the consequent change in the velocity takes place. So, next we are going to compute this.

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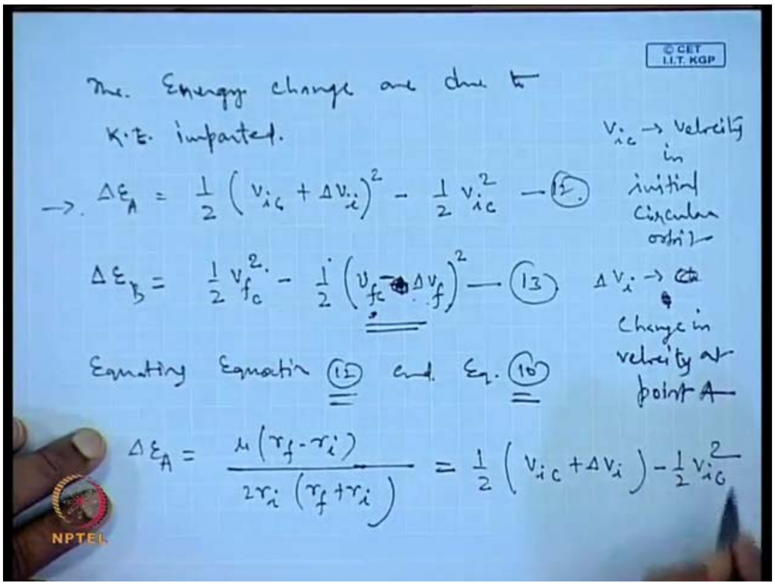

So, we have the energy change are due to kinetic energy imparted. So, we write delta E A…1 by 2 v i c plus delta v i, …,here v i c is the velocity in initial circular orbit and delta v i is the impulse imparted or the change in velocity at point A.. So, delta EB now this is our equation number 12. Similarly we can write delta e B equal to 1 by 2 v f c square minus… .We are writing in terms of the final orbit velocity in the final orbits therefore, we are subtracting here delta v f. So, so if how much the velocity is lacking over the final orbits final circular orbit velocity.

So, this is indicated by this and therefore, this minus sign appears here. Now equating equation 12 and equation 10. So, both are connected to the change in the energy. So, therefore, we have delta EA equal to mu times r f minus r i,this will be equal tothe quantity we have written here… .So, simplifying this equation.

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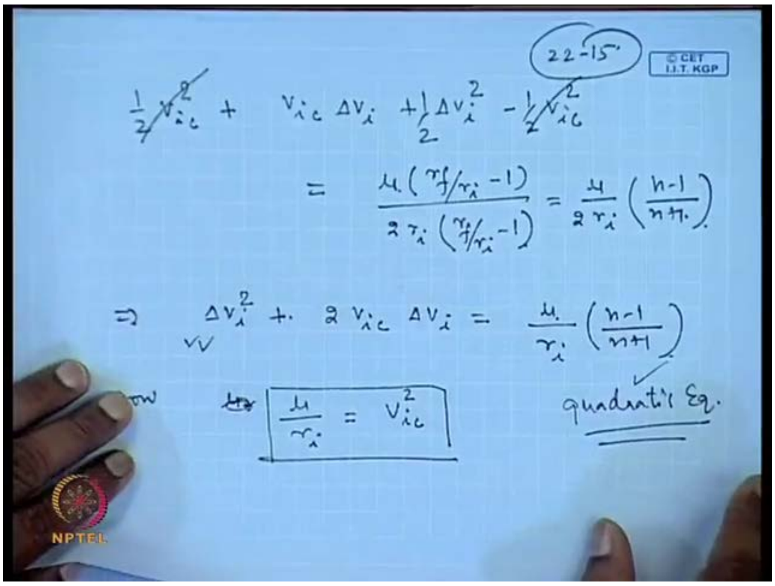

So, this gives us 1 by 2 v i c square plus v i c into delta v i plus delta v i 1 by 2 delta v i square minus 1 by 2v i c square,this is equal to mu timesr f minus r i. So, ifeither we can proceed in the same wayr f minus r i, or we can change it to when dividing r f by r i.

So, for simplicity we canfix this as r f by r i minus 1 dividedby2 r i times …is equal to mu by 2 r i times n minus 1 by n plus 1. So, this this cancels out and we are left with delta v i square plus2 v i c times delta v i equal to mu by r i times n minus 1 by n plus 1. Now we know that mu by r i, now this mu by r i , this is nothing, but velocity square in the initial orbit, so, this is v i circular square. Now that the what we have got, this is a quadratic equation, which can be solved for delta v i, and this delta v i will give the change in the velocity that has to be imparted at the pointA.

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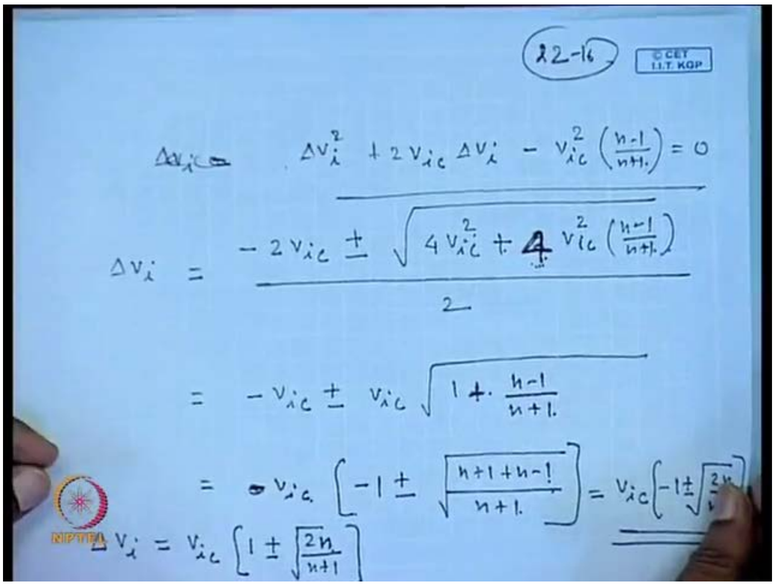

So, from here we can write, delta v i equal to, one more step we can go, delta v i square plus 2 v i c and delta v i minusv i c squaren minus 1 by n plus 1equal to 0. So, find out the value of delta v i. So, delta v i will be minus 2 vi cplus minus…2 times v i c square times n minus 1 byn plus 1 divided by 2….v i c can be taken out of the root sign. So, if this is minus 4 4a c. So, this quantity will be 4 here in this place. This is 1 plus n minus 1 by n plus 1 under root. v i c times, now v i c we can take it outside,…1 minus plus minus n plus 1 plus n minus 1 divided by n plus 1…..minus 1 plus minus 2 n divided by n plus 1 under root.

So, what we got is delta v i is equal to v i c times minus one plus minus two n under root divided by n plus 1 under root. And here minus sign is just missing out. So, naturally if you look into this equation.

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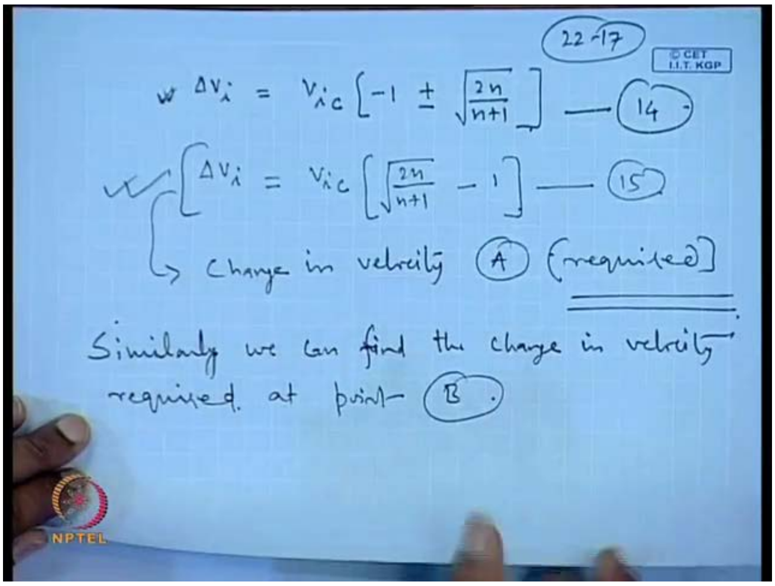

So, finally, we can write here this is our delta v I equal to v i c minus 1 plus minus 2 n divided by n plus 1 under root, and this we term as equation number 14. Now we can look in this place, the minus sign is not allowed because, in the elliptic orbit we have velocity higher than the circular orbit and therefore, the positive impulse must be given, that is the increase in velocity should takes place.

So, delta v i will be given as v i c times 2 n divided by n plus 1 under root minus 1. So, this is equation number 15. So, this is the change in velocity at A which is required. So, similarly we can find the change in velocity required at point B. So, for that we need to solve equation13 and 10, 13 and 11. So, these are the equations this is the equation 11 which describes delta EB and equation 13 also this describes delta EB. So, we equate these two equations. Equating equations 13 and 11.

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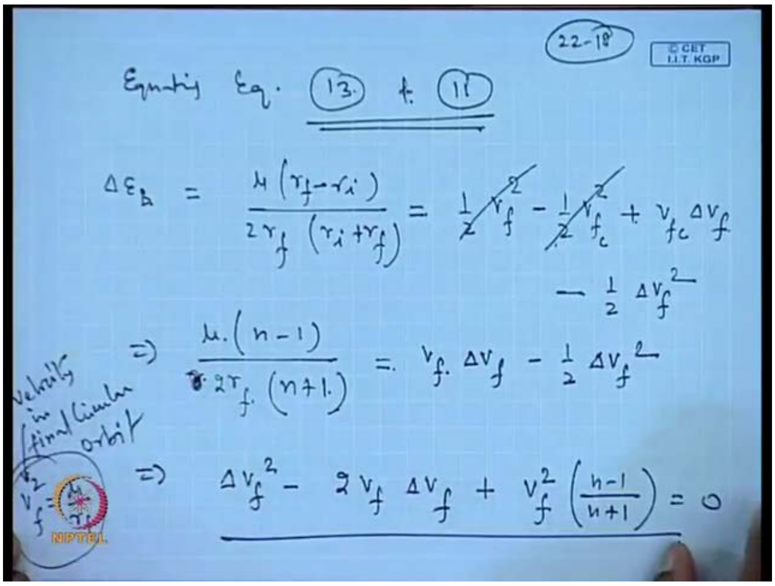

So, delta E B equal to mu times r f minus r i… So, this this cancels out and here, so, this implies mu times r f by r i, we have written as n, this is n minus 1 divided by 2 r f and here r f by r i, again this can be written as n plus 1. So, this is equal to v f c times delta v f and minus 1 by 2 delta v f square. This implies delta v f square minus 2 times v f delta v f plus v f square times n minus 1divided by n plus 1 equal to 0. So, here v f we have replaced as v f square equal to mu by r f because this is the v f is the velocity in circular orbit, in final circular orbit.

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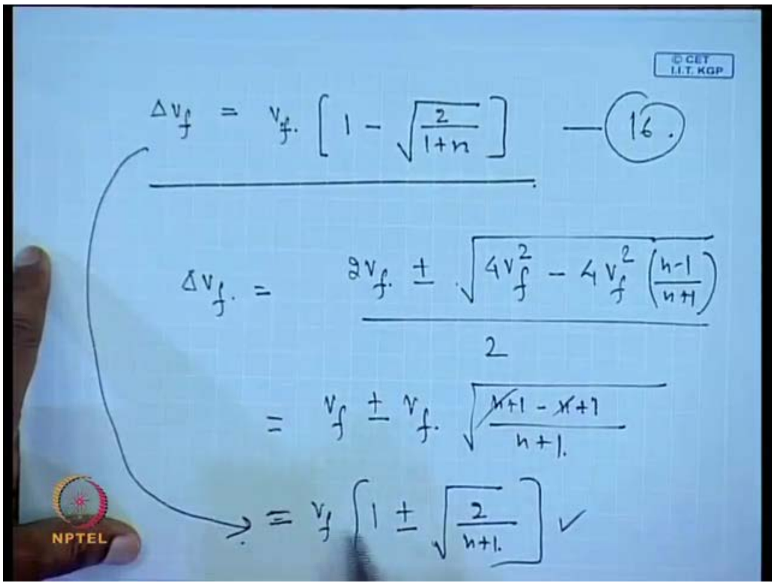

So, after we get into again we are getting a quadratic equation and this quadratic equation can be solved to give delta v fis equal tov final… So, this is our equation number 16. So, how much is the impulse required at point A and how much is the impulse required at point B is given by this equation number 15 and equation number 16. So, at least we can do the solution of delta v f here, and this will be 2 v f plus minus 4 v f square minus 4 v f square times n minus 1 by n plus 1 divided by 2… So, we we are getting this equation. So, equation has been obtained from this equation. So, we continue to discussin the next lecture.

Thank you very much.