Space Flight Mechanics

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Lecture No. # 18

Three Body Problem (Contd.)

We in the last lecture, we have been discussing about the three body problem and for that, once the z is equal to 0. So, we consider we were to trying find out x and y. So, we in that course.

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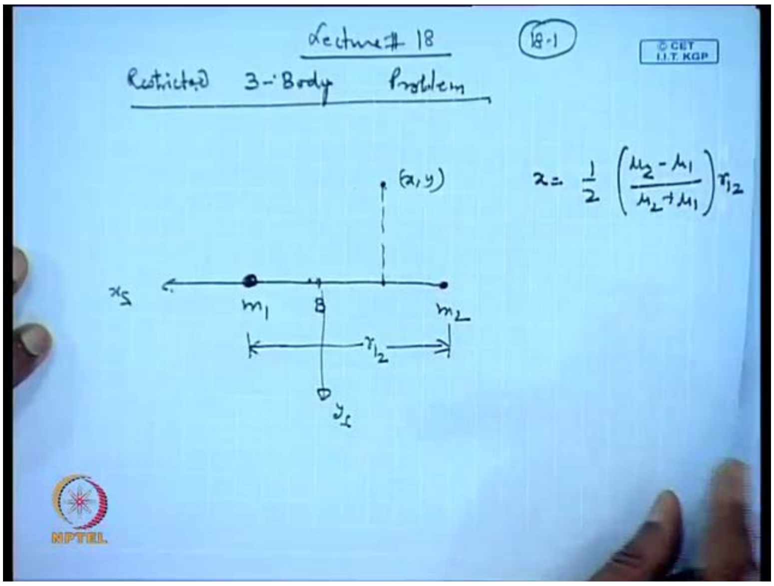

So, we have the body here which we wrote as m 1, another body as m 2 and we define the bary center somewhere here in this point as B and then in the x, y plane once z is equal to 0. So, in the x, y plane, we are trying to solve for the x and y. So, this was the point x and y in synodic reference frame. So, in the synodic reference frame this is the x direction of the synodic reference frame and this is the y direction of the synodic reference frame. So, for x and y length, we wrote for x we solved for x and we wrote as 1 by 2 times, mu 2 minus mu 1 divided by mu 2 plus mu 1 times r 12, here, r 12 is nothing, but the distance between the mass m 1 and m 2. So, this is r 12.

And we were working out for y so, the expression for the y dot, we that we got.

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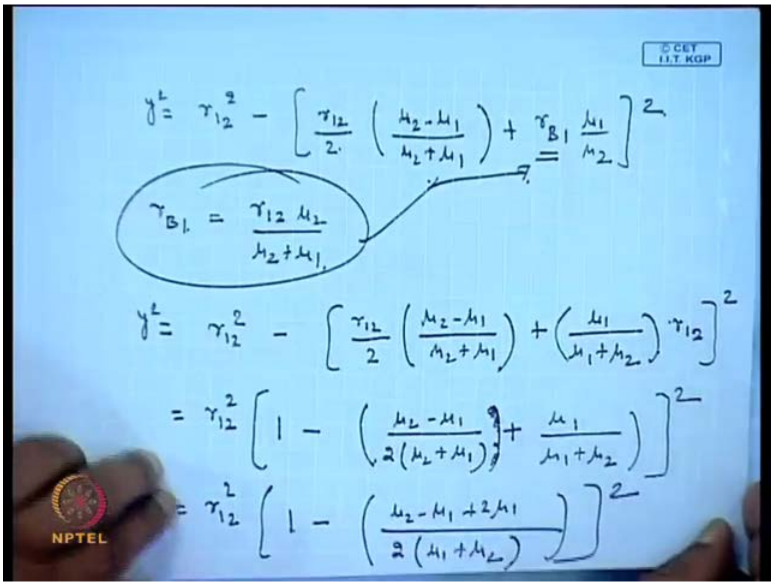

So, we wrote y square as r 12 s square minus r 12 by 2 times mu 2 minus mu 1 by mu 2 plus mu 1 and we have already worked out for the r B 1. So, r B 1 this can be written as r 12 times mu 2 divided by mu 2 plus mu 1. So, substituting this into this place. So, y square then becomes this, we have already done in the previous lecture. So, r 12 whole square minus r 12 divided by 2 times, this becomes mu 1 by mu 1 plus mu 2 times r 12. So, r 12 can be taken out of the bracket throughout. So, this gets reduced to 1 minus mu 2 minus mu 1 divided by 2 times mu 2 plus mu 1 plus, this is a whole square. So, this can be further simplified as.

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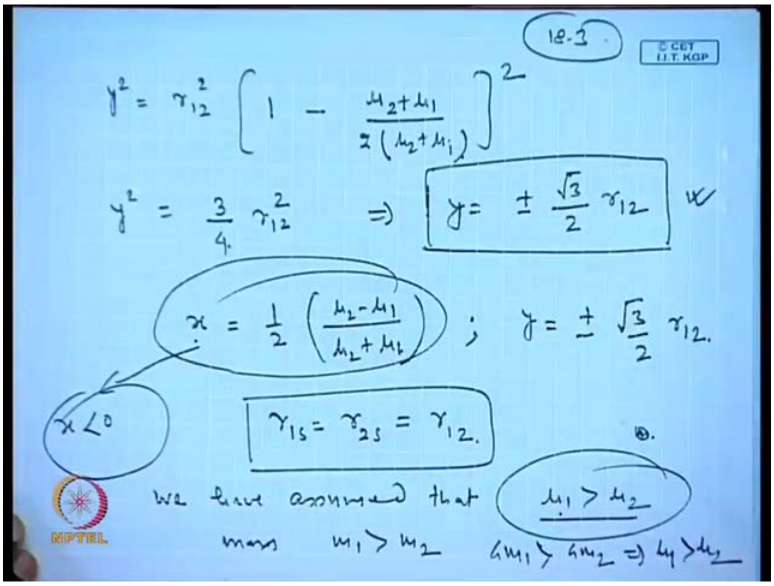

Therefore, y square I can write as r 12 whole square and 1 minus mu 2 plus, mu 1 divided by 2 times mu 2 plus mu 1 whole square. So, y square becomes 3 by 4 times r 12 whole square, this implies y equal to plus minus under root 3 and this is what I stated last time, that we are going to find out this value of y. So, ultimately we have got x equal to 1 by 2, mu 2 minus mu 1 by mu 2 plus mu 1 and y equal to plus minus root 3 by 2 r 12 and we have already proved that r 1 is equal to r 2 s equal to r 12. That is in this figure this was the distance r 1 s and this was the distance r 2 s and this is the distance r 1 s. So, these distances are equal, hence the point x y along with mass m 1 and m 2 this makes a equilateral triangle.

Moreover, we can see from this place if we are assuming that, because we have already assumed, we have assumed that mu 1 is greater than the mu 2, that is the mass of mass m 1, this is greater than m 2. So, if we multiply both sides by G. So, this will be written as this, this implies mu, mu 1 is greater than mu 2. So, if this we have already discuss all this things. So, mu 1 is greater than mu 2. So, this implies that x is, from this equation we can directly see that x is less than 0 that is x is negative.

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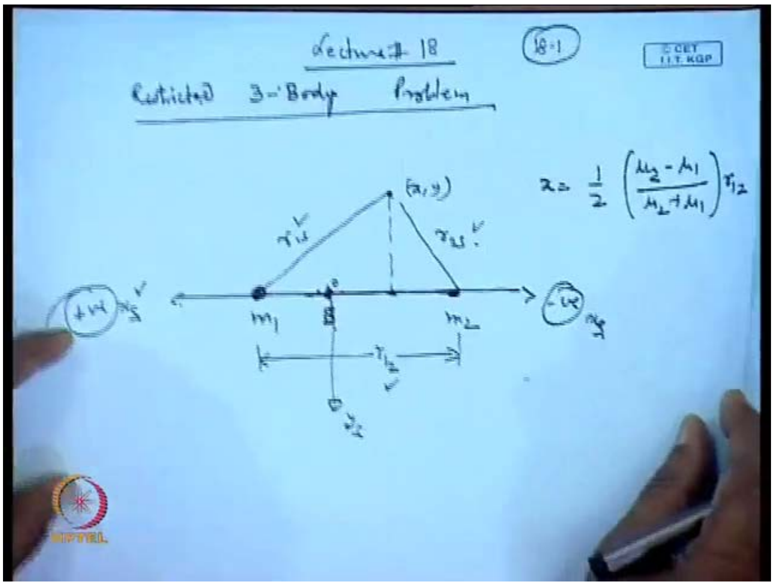

So, if we look on into this figure. So, we are measuring the distances from this point fixing at a synodic reference frame here in this place where the origin of the synodic reference frame and B is the bary center in this place. So, x is coming out to be negative means the, if this is a smaller mass and this is a bigger mass. So, x is going to lie on this place, because on this side we have the negative x and x s and this side this is positive on this side, this is positive and here this is negative and y is both positive and negative. So, which according to the solution that we have got plus minus root by 3, 1 by 2, if r 12 so, from directly we can see from a figure here.

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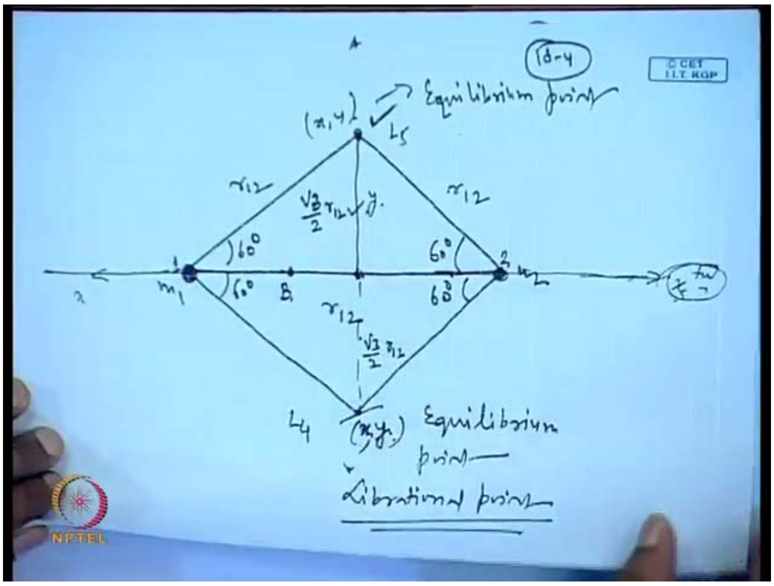

This is r 12 and this is also equal to r 12 and this is also r 12. So, this forms a equilateral triangle and obviously and then this distance which is y another point that we are getting down side. So, this point is minus on this side, in our page we have taken on this side, this is the positive side of the y and this is the negative side of the y. So, both the points they will appear as this is m 1 and m 2, this is point 1 and this point 2 and bary center is lying somewhere here.

So, because it is equilateral triangle so, this is 60 degree and this will turn out be a root 3 by 2, r 12 and this will also turn out to be root 3 by 2 r 12 from the equilateral triangle properties. Thus, we have got we have got a total solution of 2 solutions of 1 equilibrium point which is existing here, another equilibrium point which is existing here, in this place. This is also called like Librational point.

If moreover from, In this figure you can see that, because these points are lying just on the same distance from both sides. So, if we name this point as, if we named this as L 4 so, this will be naturally called L 5 and we have chosen x in this direction and y positive, in this direction and vice versa, if we choose x positive in this direction, if we choose x positive in this direction. So, y will be positive in this direction.

So, if anyone of them can be called L 4 and L 5. So, for this notation is just arbitrary and we initially restricted that, there are total 5 equilibrium points. So, are the rest of the 3 they lie on this x. So, next we are going to find out where those 3 rest of the points are lying.

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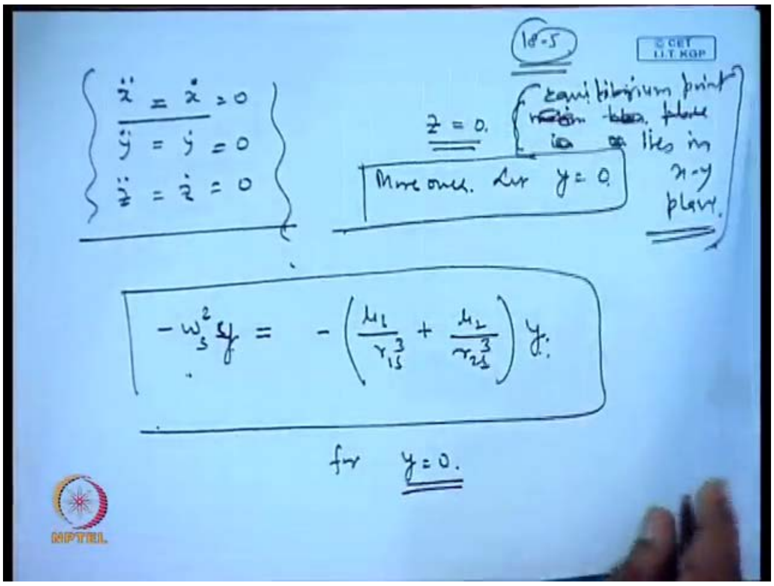

So, earlier we have the written the equation of motion in a synodic reference frame and then we set there x double dot equal to 0 and x dot equal to 0. Similarly, we set y double dot equal to 0, y dot equal to 0 and z double dot equal to 0 and z dot equal to 0 to find out the equilibrium points and if you remember.

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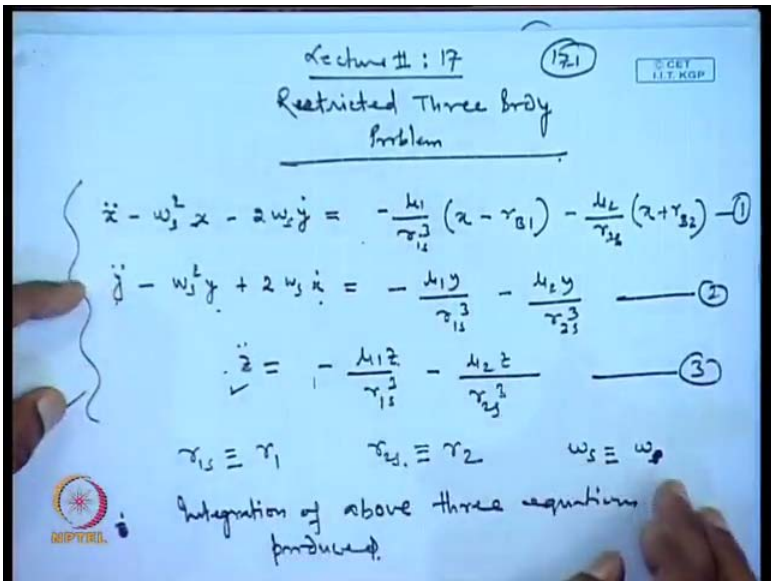

So, if the equations in our lecture number 17th these were the 3 equations, that we were using and here we set x double dot equal to 0, y double dot equal to 0 and y dot equal to 0, x dot equal to 0 and z dot double dot equal to 0. So, here the same thing is available to us. So, with this assumption then we further work and we now assume that let us say z equal to 0 and moreover, let y equal to 0. So, y z equal to 0.

So, in this equation what we did? We put z double dot equal to 0. So, if we set equal to this. So, the quantity which is present here so, we can separate out z from this place and the quantity mu 1 by r 1 s and mu 2 by r 2 s, this quantities are non 0 and therefore, we put z equal to 0. That we, from there we found out that the motion take place in this the equilibrium point. The equilibrium point lies in x y plane. 1 z equal to 0 so, this implies that the equilibrium point lies in the x y plane.

If we look into the next equation so, this equation number 2, when the lecture number 17 that we have used. So, y double dot equal to 0 and x dot equal to 0. So, we are left with minus omega s square y and on the right hand side we have this term. So, from here we can write minus omega s square y equal to minus mu 1 by r 1 s whole cube plus, mu 2 by r 2 s omega, omega square s, y this is. Now, this equation will be satisfied for y equal to 0, this can be satisfied for this equation get satisfied for y is equal to 0. We have already worked for the non. When y was not equal to 0 and we solved the equation number 1 and 2 by putting x double dot, y double dot, y dot and x dot equal to 0, then from there we got the librational points 4 and 5. Now, our effort is here to get the librational point's L 1, L 2 and L 3. So, here we set y equal to 0, because this equation get satisfied and then we use this in the equation number 1 which is available here.

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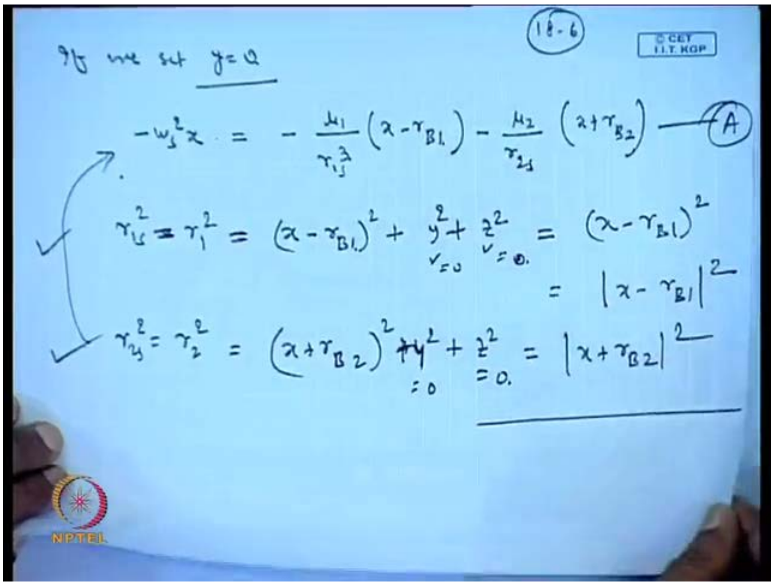

So, if we set y equal to 0, then we have minus omega s square, s x. So, this is our starting point here. Now, r 1 s this is nothing, but r 1 s is equal to we write this as in the simplified notation dropping the s, r 1 square. So, this is x minus r B 1 square plus y square and plus z square. So, here y and z both are 0, this equal to 0. This is also equal to 0. So, this gets reduced to x minus r B 1 square, in this we write in this way. Similarly, r 2 s, y square plus z square, this is 0, this equal to 0. So, this gets reduced to x plus r B 2 its magnitude square. Now, we utilize this 2 results in this equation. So, let us suppose we write this equation as here as equation number A.

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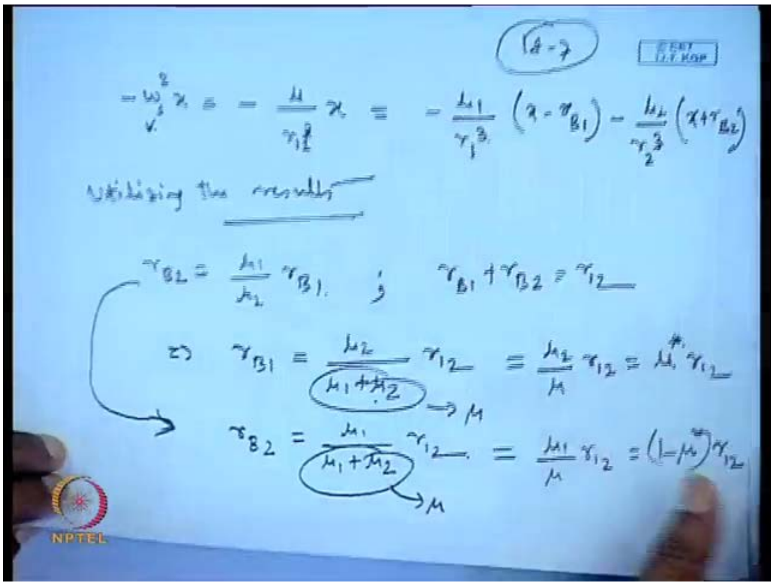

Therefore, we can write omega square s, x is equal to, now omega s is nothing, but the angular velocity of the 2 masses due to primary masses and this is nothing, but equal to mu by r 12 whole cube and we have proved it many times, this omega s square is equal to this quantity and therefore, we can write this equal to this equal to minus mu 1 by r 1 cube times x minus r B 1 minus mu 2 by r 2 cube times x plus r B 2.

Now, utilizing the results r B 2 equal to mu 1 by mu 2, r B 1 this we have already derived again repeating here. So, from here we get r B 1 equal to and r B 2 from here, we can get and this will be mu 1 by mu 1 plus mu 2 times r 12 and if we divide mu 2 by and mu 1 plus mu 2 is nothing, but mu this quantity is mu. So, we can write this as mu 2 by mu r 12 and this we write as mu 2, as mu star times r 12 and this we will write as mu 1 by mu times r 12, this will then become 1 minus mu star r 12, this is very easy to see.

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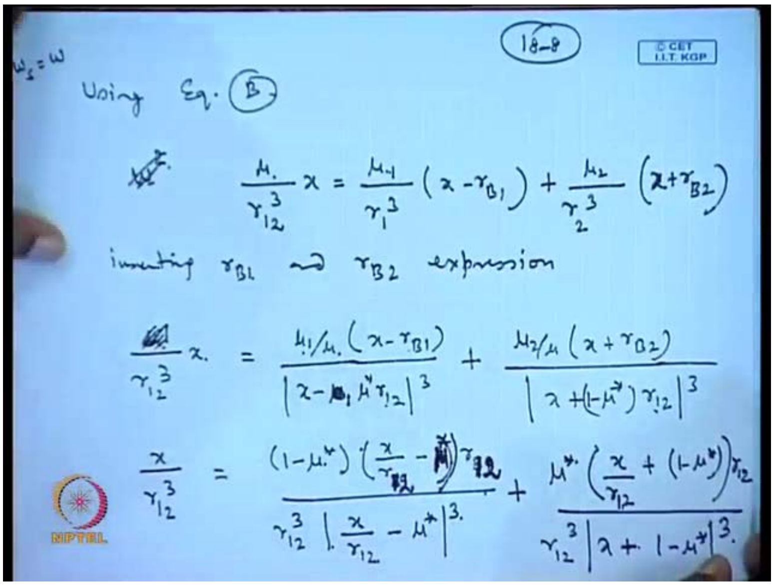

Now, with this results this equation can be further simplified. So, if this is the equation that we have written earlier. So, this can be further simplified and this can be written as minus, minus, minus can be canceled out on both the sides. So, let us term this as the equation number this is B. So, using equation B, then we get an s also we drop. So, omega s we simply write as omega, which we have replaced by mu by r 12 whole cube. So, again we remove all the things and simply write mu by r 12 whole cube times x is equal to mu 1 by r 1 cube x minus r B 1, r B 2. Now, r B 1 and r B 2 just now we have on wrote in this page, we can insert into this equation.

So, inserting r B 1 and r B 2 expression, here mu by r 12 whole cube times x, this is equal to now, mu 1 is available. So, mu 1 we can divide this mu, we can bring this mu from here on the right hand side and we can write this as mu 1 by mu and r 1 cube is already we have written in terms of x minus, x minus this plus r B 1. So, x minus r B 1. So, r B 1, we have written as mu star times r 12. So, this is mu star times r 12 whole cube times x minus r B 1 plus mu 2 by mu times x plus r B 2 divided by x plus 1 minus mu star times r 12 whole cube. Now, look into this. This we can replace as mu 1 by mu, we have written as 1 minus mu star.

So, this gets 1 minus mu star and r B 1 we can take common from this place. So, this will become x by r B 1 minus 1 and r B 1 we can write outside, from here also, because r 12 is the magnitude of the distance between the point 1 and 2 and therefore, this can be taken outside of the modulus sign. So, this becomes r 12 whole cube x divided by r 12 minus mu star whole cube. Here one more simplification is in this place itself we can do, this r B 1 we have already written as mu star r 12. So, here we directly replace this in terms of r 12.

So, instead of taking B 1 here out, we write here as r 12 and here we will take outside r 12 and therefore, the quantity which will remain here this is x times r B 1 was r 1 times mu star. So, this mu star will remain inside. So, this quantity mu 2 by mu is nothing, but mu star. So, this becomes mu star and here also we can do the same thing x and r B 2 is 1 minus mu star times r 12. So, we can replace this as 1 minus mu star this r 12 and outside r 12 divide by r 12 we can take out of the bracket here, x plus 1 minus mu star. This is whole cube. See, this can be further simplified now.

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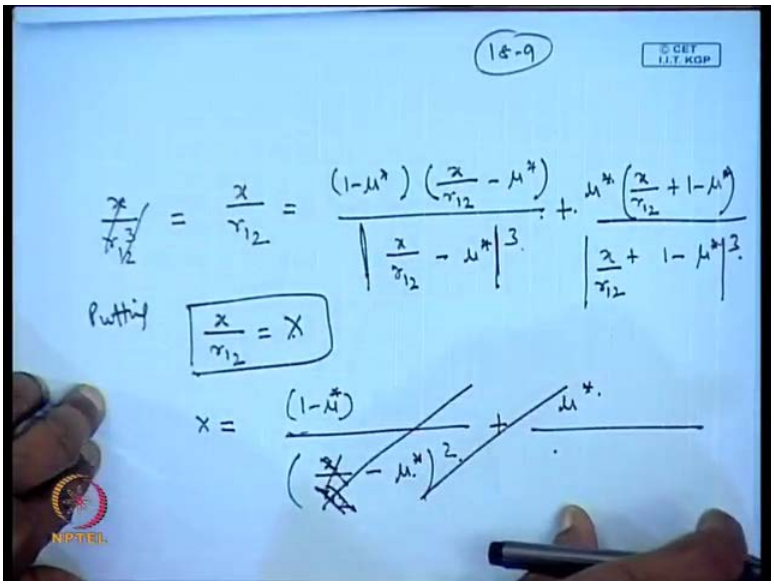

So, from here we have done so, in this equation we can cancel out this r 12, r 12 cube, r 12 cube. So, we can cancel it out on both the sides after canceling this what remains r 12, r 12 here this place. So, this can be brought on the left hand side. So, we directly write here x by r 12, this is equal to 1 minus mu star times x by r 12 minus mu star divided by x by r 12 minus mu star whole cube plus mu star times plus 1 minus mu star, this is here we are missing 1 by 2. So, introduce here x by r 12. So, this is x divided by r 12, 1 minus mu star whole cube.

Now, putting x by r 12 is equal to capital X. So, the left hand side will become x equal to 1 minus mu star, here 1 term we can cancel out. So, provided this quantity here, this is the quantity which is modulus, we are writing the modulus cube. So, if x y r 12 is greater than mu star we cancel out 1 of the term and then we can write this as x by r 12 minus mu star whole square plus and this we are going replace this by capital X. So, we are replacing this by capital X to the next page.

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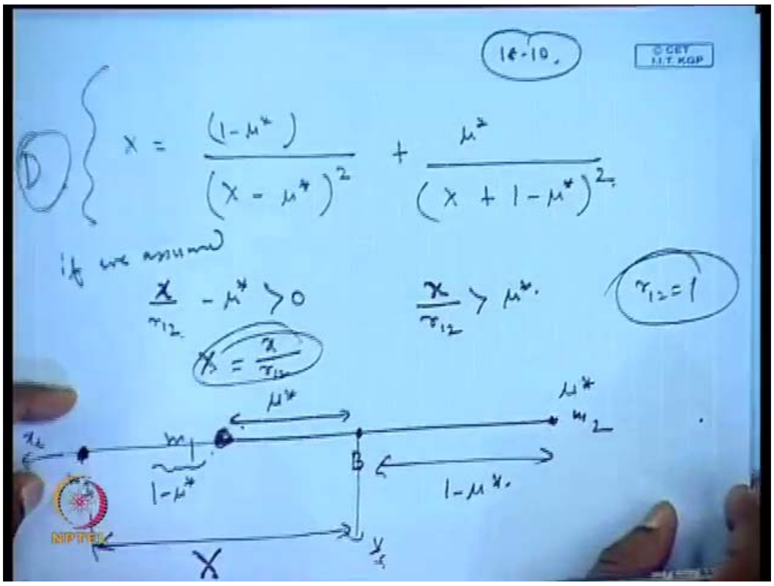

So, in this equation this becomes left hand is X, 1 minus mu star divided by X minus mu star square plus. So, remember that mu star is a positive quantity and 1 minus mu this is also a positive quantity and what we are assuming here that this quantity is greater than 0. So, see the implication of this so, if we assume that x by r 12 minus mu star is greater than 0, means x by r 12 is greater than mu star. Now, look into the figures the normalized conditions, that we have mentioned earlier, this is the mass m 1, this is the mass m 2 and somewhere B was the bary center.

So, this distance we wrote as mu star and this mass we replaced in terms of 1 minus mu star, the right hand side mass we replaced in terms of mu star and this distance we wrote as 1 minus mu star. So, the implication of this is x by r 12 minus mu star this is greater than 0 means, this mass is the tertiary mass or the test, the smallest that tiny particle we are using or the small mass which is lying here, which we have written as m 3.

So, it is and x direction we are taking here as positive and y s downward as positive. So, mass is lying somewhere here in this place. So, and therefore, the then X by r 12 which is the distance this r 12. So, on the normalized scale r 12 becomes nothing, but equal to 1. So, after this normalization this is the distance that, we have written X by x is equal to, capital X is equal to x by r 12. So, this is the normalized distance. Now, this is the distance from here to here which is here appearing as capital X.

So, this distance minus mu star obviously this is a positive quantity from here. So, this is the 1 equation that we are getting here after this simplification. So, what we ultimately got, this was our main equation. So, this is our equation number, let us say this is equation number C and this is our equation number D. So, in this C, equation number C, if we are using the point on the left hand side so, this is the result that, we are getting if the point lies on the right hand side on the right extreme, we get some other result. This will get modified if it is lying in mid between so, we get some other result.

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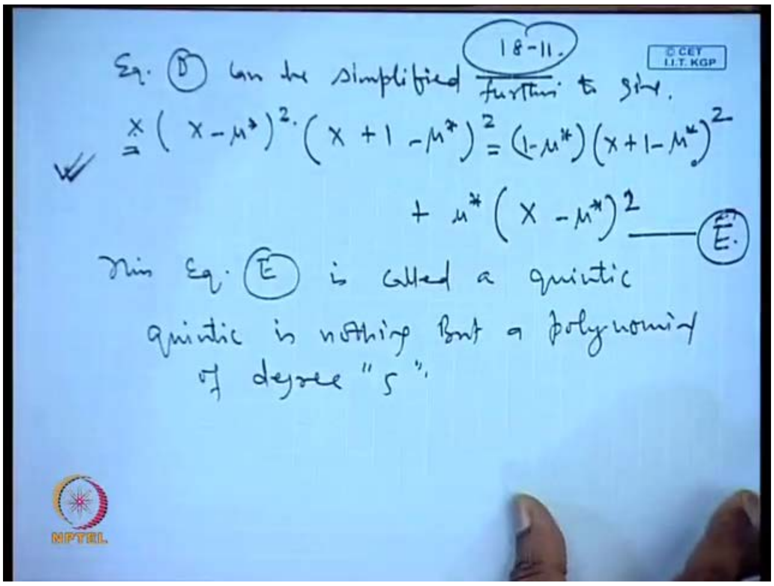

So, this equation we can simplify further and we can write this as X times, this equation number D. Equation D can be simplified further to give X times X minus mu star whole square times X plus 1 minus mu star whole square, this is equal to 1 minus mu star times X plus 1 minus mu star plus mu star times X minus mu star whole square here also whole square. And this, we termed as equation number E. So, this equation E is called a Quintic.

So, Quintic is nothing but a polynomial of degree 5. So, this equation is of degree 5 and this will give you 1 positive root and here obviously X, X in this case is positive. So, you will get only 1 positive solution out of this and that gives you the 1 librational point which we will write as. So, this point we will write as L 3 and on the right hand side. So, this is our point L 3, right hand side of this m 2, we will get another point, which we will write as L 2 and in mid between we get another point which we will write as L 1. So, we still have to work for L 1 and L 2.

Now, in this again going back to equation number C, if we replace in this equation x by minus x everywhere. So, we get the new condition that is the point, once it is a lying here over the right of the point m 2, this is the point m 2. So, if the point is lying right of this.

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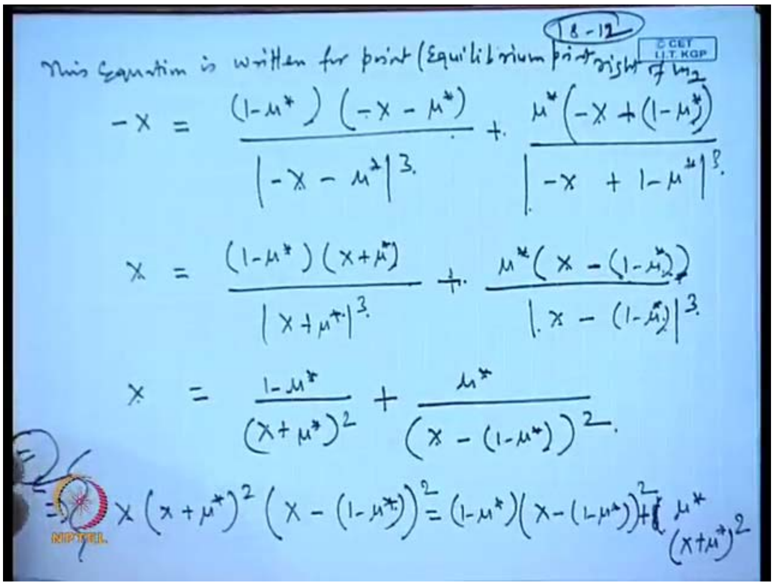

So, the same thing can be written as so, x by r 12 we have written as capital X. So, this becomes minus X equal to 1 minus mu star minus X minus mu star by the minus X minus mu star whole cube plus mu star times minus X plus 1 minus mu star whole cube. So, this equation is written for point, the equilibrium point right of m 2. Now, we can take out the minus sign here from this place and this can be written as 1 minus mu star times X plus mu and here also this becomes X plus mu star whole cube and then here, in this place this becomes 1 plus mu star and minus sign, if we take it out from this place.

So, what we are doing, we are taking out minus sign from everywhere and canceling with this sign. So, this gets us X minus 1 minus mu and this also can be written as X minus 1 minus mu. So, this is the quantity which is. Now, we can see from this place once we have written in this fashion. So, what we are assuming again going back to this figure. So, we are assuming, we are already taking the sign of here replacing by this minus X, we are assuming here that this distance is now positive. So, this will give us another equation. So, this gives us another equation which we can write as 1 minus mu star divided by X plus mu star whole square plus this is our equation, next equation. So, this equation, we can write as 11, this is 12, this was equation number E. So, this we term this as equation number F.

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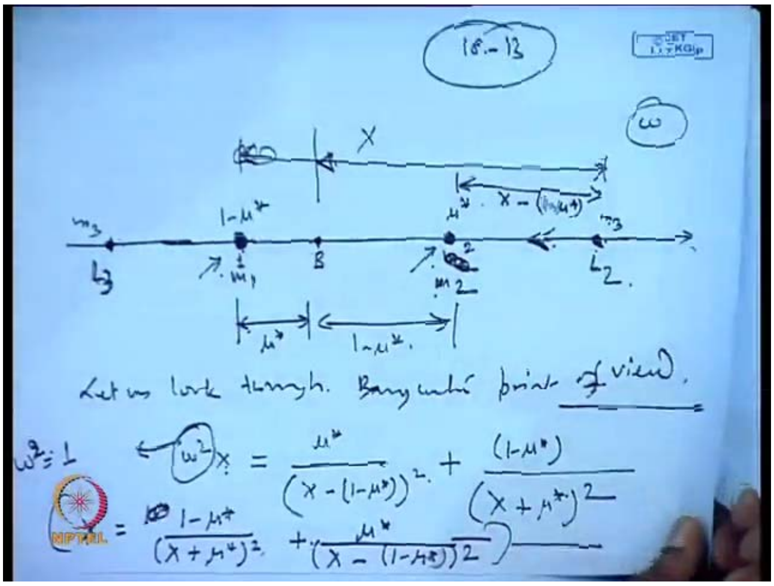

Now, from this 2 equations E and F. This is our equation number F and this is equation number E of the same thing again, I am going to explain in a rather very simple way. See exactly what we have done. This is our point m 1 and this is the point m 2. So, what exactly is happening? This is the bary center here. So, about the bary center these two masses are moving with the angular velocity omega. Now, they will always remain in the same line, because they have the same angular velocity.

So, they will move about this bary center. Now, if we look into the equation E. So, equation E we derived for the point which was lying here which we wrote as L 3 on the left hand side. Another point, we worked out which was lying here on the right hand side which we wrote as L 2. Now, exactly what is happening here look this distance we have termed as mu star and this distance we have termed as 1 minus mu star and this mass as 1 minus mu star and this mass as mu star.

Now, what happens in reality once it is a moving so, this point is also moving, because we are looking for the equilibrium points. Equilibrium points means in the synodic reference frame this point will appear to be stationary with respect to this two, means, it will neither move in this place neither move in this place neither, it will move in this place. So, if it is remaining here or either this point it is a remaining here, in the here in this place only. So, you can see that what are the forces are acting on this point in the bary centric reference frame. So, if you look from that point of view also, it will give you very good insight. So, let us look it through the, let us look through bary center point of view.

So, if we are taking the point number this the small mass m 3 here in this place or either small mass m 3 here in this place. So, you can look that the force acting due to this bigger mass m 2 on this point mass m 3, this is on this side and also the force due to this will be on this side. Now, the whole thing is also moving like this. In reality what is happening the mass m 1 and m 2 they are moving about their bary center. So, if the mass m 3 has to keep in the same line so, this also must move like this. So, what exactly we need to do here that if omega is the angular velocity of this mass m 1 and m 2.

So, omega square and times the normalized distance that, we are showing this is the distance x, omega square times x which is the something you can call this as the centrifugal force or if you can, it is a better to describe in terms of a centripetal force, because the centrifugal force is not a good option. It is an, because we are writing it from the bary center point of view and therefore, it is a good to say here that this is a centripetal force to move this mass in the around this bary center. So, the mass m 3 this will also be moving around the bary center. So, to move this necessary centripetal acceleration this must be provided by the mass m 2 and the mass m 1.

So, in this case both are acting in the same direction, the force due to the mass m 2 also acting in this direction and force due to mass m 1 is also acting in this direction. So, both of them combine. Now, we can write this so, in terms of the normalized masses. So, this is mu square mu star and the mass of m 3, if we multiply on the left hand side and on the right hand side so that cancels out. So, you can yourself little bit explore it so, this becomes mu star times, the distance between this and this mass. So, this is the distance here, this distance we can write as the distance from here to here. This X minus 1 minus mu star. So, this becomes X minus 1 minus mu star whole square and another one due to this will be 1 minus mu star, which is the mass of this particle. So, the distance is total from here to here is this sorry distance, we have taken from here, this is the distance taken from this place, from the bary center this is the distance X.

So, the distance from here to here will be X plus this distance is mu star. So, this becomes mu star and omega square obviously on the normalized scale omega square equal to 1, we have already done worked out all these things. So, once omega square is equal to 1, this quantity. So, simply what we get X is equal to mu star, 1 minus mu star divided by X plus mu star whole square and plus mu star divided by X minus 1 minus mu star whole square. Now, compare it with equation number F that, just now we have written, look into this equation and the equation number F that we have written here in this place. Is it the same? This X is equal to 1 minus mu star divided by X plus mu star square and then mu star by X minus 1 minus mu star square.

So, here through this route it is a very easy to write. Similarly, for the point number this is of for the smallest mass these are the two primary masses and this is the tertiary mass, which is infinitely as small as compared to this two. So, as not to is disturb the motion of this two so, under this assumption we are working so, if the point is lying here the same thing can be written as omega square X is equal to X equal to.

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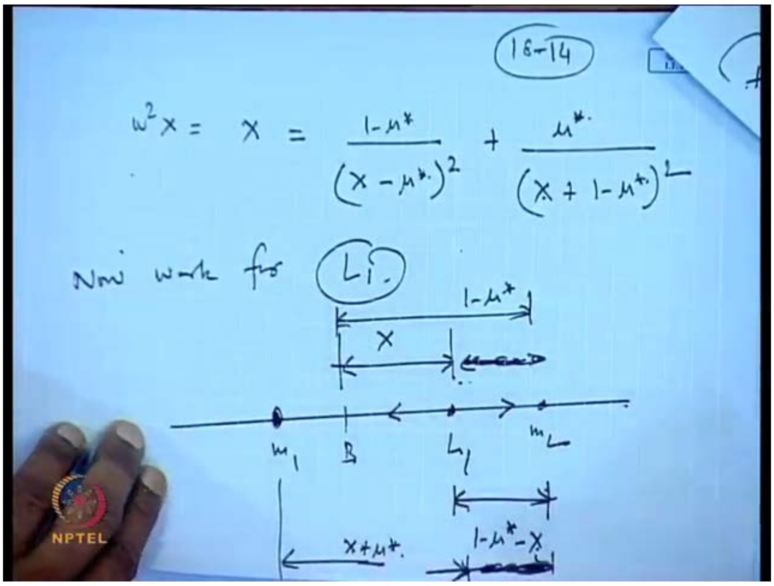

Now, look into this mass is pulling it in this direction and this mass is also pulling in this direction, while and this two together provide the necessary centripetal acceleration. So, if we balance all this forces so, what we get here 1 minus mu star divided by and the distance now, from here to here this distance will be so, this distance if we write from here to here as X. So, this distance becomes X minus mu star, because the distance we are measuring from the bary center so, if we subtract this mu star from this X. So, what we get is X minus mu star.

So, this whole square and similarly, for the second mass which is m 2, this we write as mu star divided by and the here the distance will add X plus 1 minus mu star whole square. Now, you compare this with equation number E, equation number D. So, now, look into this equation, isn't it exactly, it is the both are same. So, in very simple way using this figure we have been able to work out this problem. So, instead of doing so much manipulation, if we use work the normalized scale so, it is a very easy to look into what is exactly it is happening. So, lastly we are left with the point number 1. Now, work for L 1.

So, we have the mass here m 1 here is the mass m 2. Now, the point is lying here in this place. So, to work out for this again B is the distance here, this is L 1, this is the distance X and this is the distance from here to here, this is the distance 1 minus mu star. So, this distance will be 1 minus mu star minus X and the distance from here to here, this will be this 1 minus mu star, we are considering the point this 1. So, the distance from L 1 we are taking from L 1, not from here. So, this distance becomes X plus mu star and this distance become X plus mu star.

So, considering the mass m 3 here in this place now you can see that this mass is pulling it in this direction while this mass is pulling it in this direction. So, this is a different case than the previous two, because here the 2 gravitational forces are opposing each other and then since, the bary center is here. So, this mass will also move along with this about this bary center so, the whole thing is moving with the same angular velocity. So, it will go like this oppose. So, if it goes like this then the angular velocity being the same as the angular velocity of m 1 and m 2.

So, to provide the necessary centripetal acceleration in this place so, what we need to do we subtract the, from the, this gravitational acceleration we subtract this gravitational acceleration and that will provide the necessary centripetal acceleration to move in the orbit around this bary center B.

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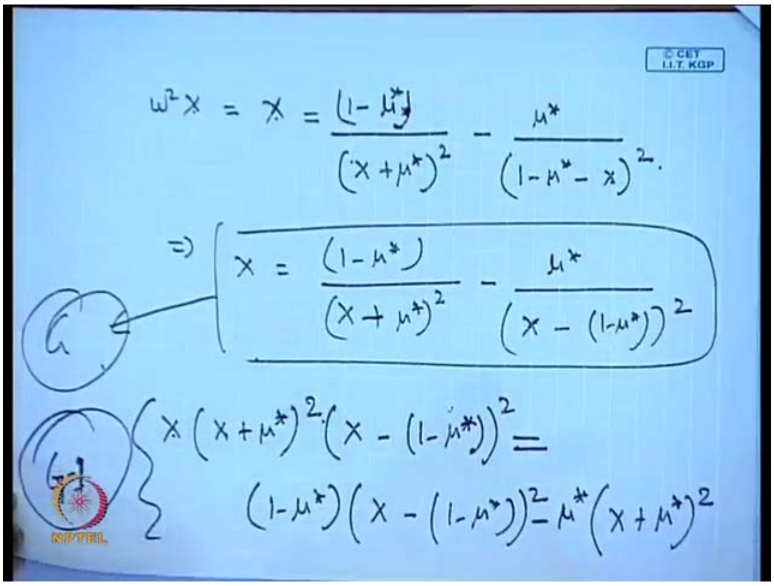

We have from here omega square X is equal to X, because omega now in this case equal to 1 and therefore, this quantity then becomes mu star, this is the mass which we have written as 1 minus mu star and this is the mass we have written as mu star. So, 1 minus mu star is so, divided by the distance between this and this point, the square of the distance. So, this distance is X plus mu star whole square and then from this what is to be subtracted, the force due to the mass m 2.

So, if we subtract the force due to the mass m 2. So, that becomes mu star divided by 1 minus mu star minus X, this is whole square. So, this implies X is equal to 1 minus mu star by X plus mu star whole square minus mu star divided by X, we can take out the minus sign and the square obviously this will not matter. So, this matter, so, this becomes 1 minus mu star whole square. So, this is our equation number D, F, we write this as the equation number G. So, this can be further written as X times, X plus mu star square times X minus 1 minus mu star, whole square this is equal to 1 minus mu star times.

So, this is another Quintic. So, let us write this as equation number G 1, equation number G 1 and this as our, we have written as the equation number, if we can note as this and this becomes equation number F 1 and other equation that we got, this was equation this we have written as E. So, rather we can write this as E 1. So 18, 16.

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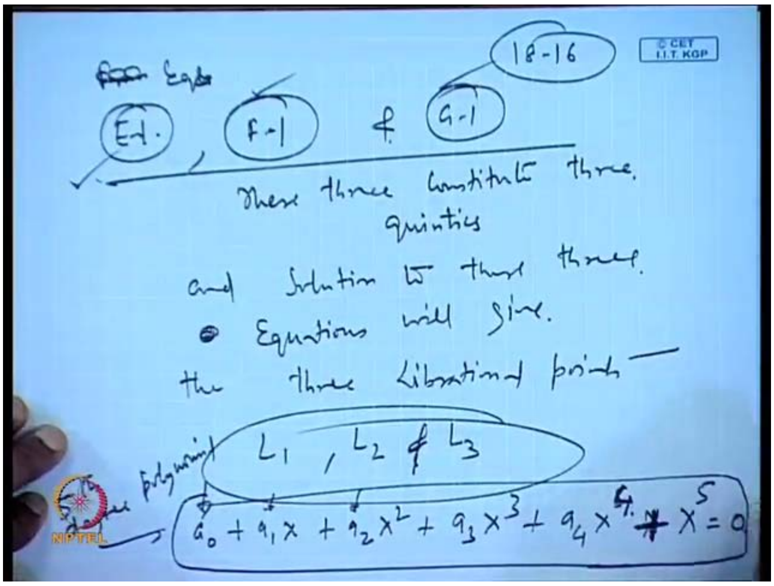

So, we have E 1 from E 1, Equations E 1, F 1 and G 1. These 3 constitute, 3 quintics and solution to these 3 equations we will give the 3 librational points L 1, L 2 and L 3. Now, remember the way we have shown here the L 1 lying on this side and L 2 lying on this side and L 3 lying on this side, L 2 lying on this side and L 1 here, this is quite arbitrary. Many of the books may refer as the L 1 as this, L 2 as this and L 3 as this. Somebody may refer this as the L 3 point. So, do not get confused with this. So, this convention is totally arbitrary.

So, if in the next lecture we continue with this. So, will directly write the solution of, we will little bit simplify this E 1, F 1 and the G 1 equation. So, of you will see that it will appear in the form of, this equations will appear, in this format a 0 plus a 1 X plus a 2 X square plus a 3 X cube plus a 4, X 5, a 4 plus X 5 equal to 0. So, this will basically constitute a fifth order, 5th degree polynomial, this is a 5th degree polynomial and this coefficients next time we will write for all the 3 equations E 1, F 1 and G 1. So, there after we will proceed to work out with the Jacobi's integral, that we have got and in that Jacobi integral they will see how the things are taking place. So, the next time what we are going to do if the Jacobi integral, which will the Jacobi integral solution, it will look something like this, what is shown in this in this figure.

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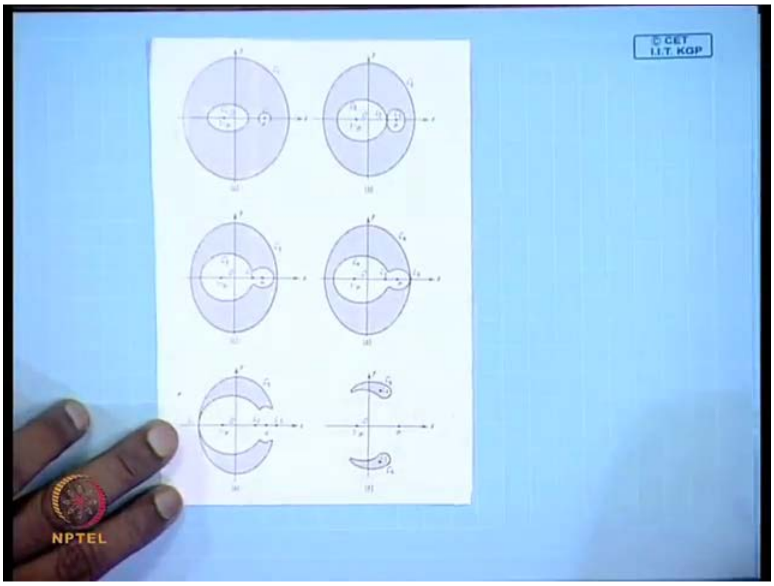

So, from there next time we are going to show that is the shaded region is non accessible region and the non shaded region that is the only accessible region. So, how this is taking place so, we will conclude this restricted three body problem in the next lecture. Thank you very much.