Space Flight Mechanics

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Lecture No. # 17

Three Body Problem (Contd.)

Last time we have been discussing about the restricted three body problem and then, we have started to find out about, about the lagrange points or that what we call as the librational points. We are in the synodic reference frame with respect to two primary bodies, one primary body and other secondary body, which are quite heavy with respect to a tertiary body, which is very small, infinite or say, infinitesimally small. So, at the librational point, this tertiary body, it appears to be at rest in the synodic reference frame. So, we continue with the same discussion in this lecture.

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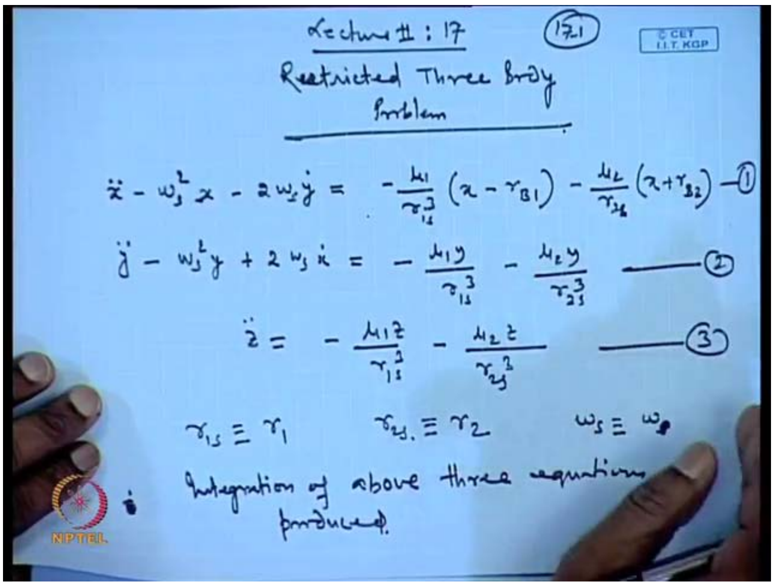

So, we started with the equations,

So, these are the three equations that we have started with so here for our convenience we wrote r 1 s is equal to r 1 and r 2 s equivalently as r 2 and omega s can be replaced as omega so if the integration of this three equations with respect to t it produce the jacobi's integral.

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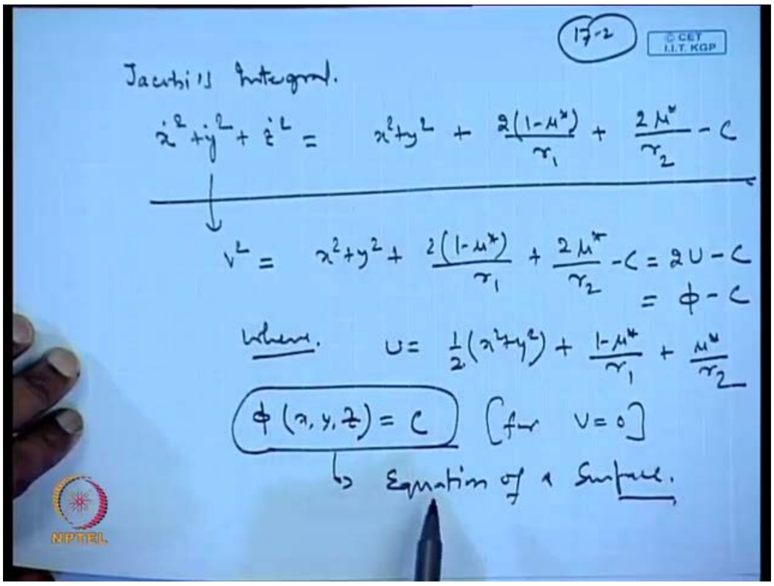

So, that we wrote as integration of above three equations produced Jacobi's integral, which we wrote as x dot square plus y dot square plus z dot square is equal to x square plus y square plus 2 1 minus mu star, and the left hand side here, this is nothing, but v square. So, the same thing we wrote as v square x square y plus 2 1 minus mu star by r 1 plus by r 2 minus c and this can be written as 2 u minus c is equal to phi minus c, where u is equal to 1 by 2 times x square plus y square plus 1 minus mu star by r 1 plus mu star by r 2.

So, here the phi, which is appearing, so phi we wrote as a function of x, y, z and for v equal to 0 phi, which is a function of x, y and z. so, this describes the equation of a surface, so we will keep discussing about these things expanded further. So, what we were discussing about? The librational points and the, or the equilibrium points, which are also called the Lagrange's points. So, librational points are the Lagrange's points. They arise once we set x double dot y double dot and z double dot equal to 0 and x dot y dot z dot equal to 0 and inter substitute into these equations, and resulting equations, they provide us the librational points. So, we need to solve them.

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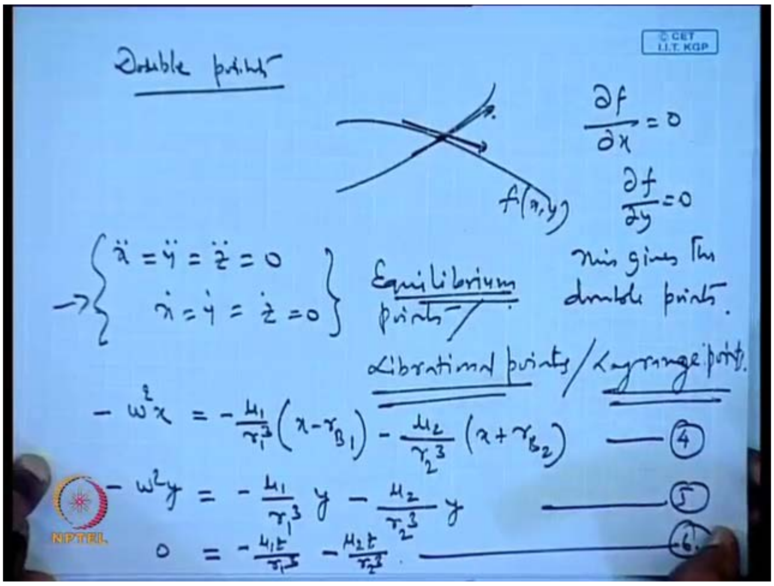

So, also I hesitate, that the librational points, that is nothing, but the double points. So, double points, they can be defined as if a curve is given, so this curve may intersect itself in a point. So, at that point, at that point we can have two tangents. So, if this curve is defined by f x, y, so dow fx, dow f by dow x equal to 0 and dow f by dow y equal to 0. This gives the double points, so if we do the (( )) operation on this equation, so we have the Jacobi's integral available.

So, we be differentiate the Jacobi's integral or the function phi is equal to, c is available to us, so we can work with this and we can prove this. So, phi equal to c, so here we can write it as this v square is equal to phi minus c. So, with this equation it is possible to work out using this partial differential and find out the librational points or either, you directly use the equation, which we have developed earlier and set into this. The x double dot y double dot x dot y dot and z double dot equal to 0. Of course, here z dot is not present in this equation but, this the equilibrium point also refers to the x dot y dot equal to, so we have the x double dot y double dot z double dot equal to 0 and x dot y dot z dot equal to 0. This gives us the equilibrium points or what we call as the librational points or the Lagrange points, Lagrange points.

So, rather than working with this differentiation, this partial differential, we start working with this and set into this equation, this constant. So, the solution to this will be, so last time we have worked out all those things and we saw, that the resulting equations, they are minus omega square x r 1 cube x minus r B 1 minus mu 2 by r 2 cube x plus r B 2, and the this is our equation number 4, this is our equation number 5. And then 0 equal to minus mu 1 z by r 1 cube minus mu 2 z by r 2 cube, this is equation number 6.

So, now this, 4, 3 equations, equation number 4, 5 and 6, they can be solved to find out the values of x, y and z.

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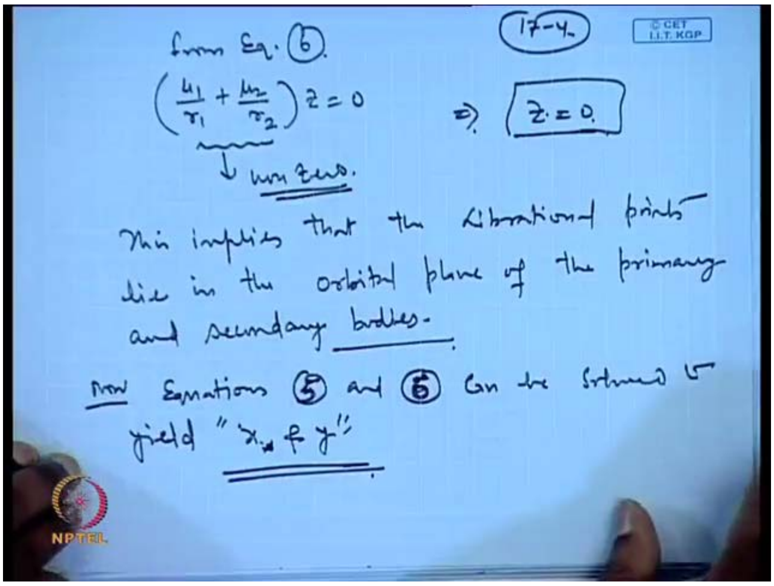

So, from equation number 6, from this equation, this is very easy to solve, we can see, that this equation can be written as mu 1 by r 1 plus mu 2 by r 2 times z equal to 0 from equation 6. Now, the quantity, which is present here, this is non-zero and therefore this implies, that z equal to 0. So, this is the solution, that we get for z equal to 0. So, this simply implies, that the point, this implies, that the librational points lie in the orbital plane of the primary and secondary bodies. Now, we can proceed to solve the equations 4 and 5.

Now, equations, 4 and 5, 5 and 6, 5 and 6 can be solved to yield, x, x and y.

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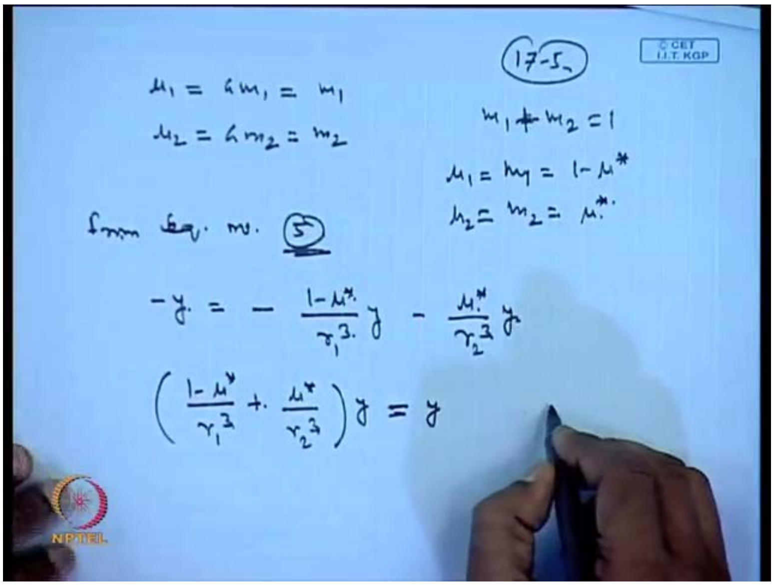

So, solving the equation 5 and 6 we can do in the normalized form and also, we can do in the not non-normalized form, so both are possible. So, on the normalized scale what we have seen, that we set mu 1 is equal to Gm 1 and G we set as 1, so therefore, this got reduced to m 1. And similarly, we have mu 2 is equal to, Gm 2 is equal to m 2. Moreover, we wrote m 1 as m 1 plus m 2, this is equal to 1 and then we defined m 1 as 1 minus mu star and m 2 is equal to, mu, mu star. So, we have this equation number, sorry, this is equation number 4 and 5, this is not 5 and 6, so we have equation number 4 and equation number 5, which is related to the x and y coordinates.

So, from equation number 5 we have minus y equal to mu 1 is nothing, but m 1 is equal to 1 minus mu star and mu 2, similarly is this quantity here. So, we can simply write this as 1 minus mu star. You can have a look of this equation again, so we are replacing in this quantity, mu 1 is nothing, but 1 minus mu star. Similarly, mu 2 is mu star. So, this will be replaced by 1 minus mu star divided by r 1 cube y minus mu star r 2 cube and y. We can rearrange them, so 1 minus mu star by r 1 cube plus by r 2 cube.

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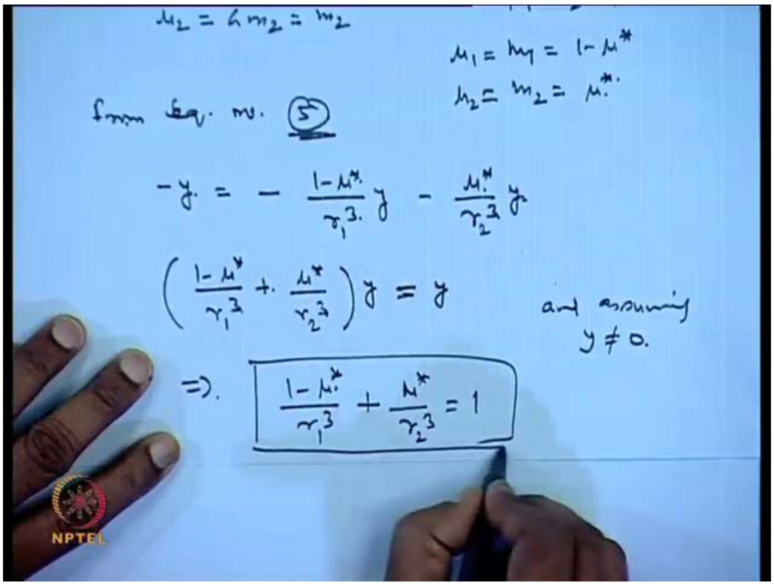

And assuming y not equal to 0 in this equation, so this leads to this, implies 1 minus mu star by r 1 cube plus mu star by r 2 cube, this is equal to 1.

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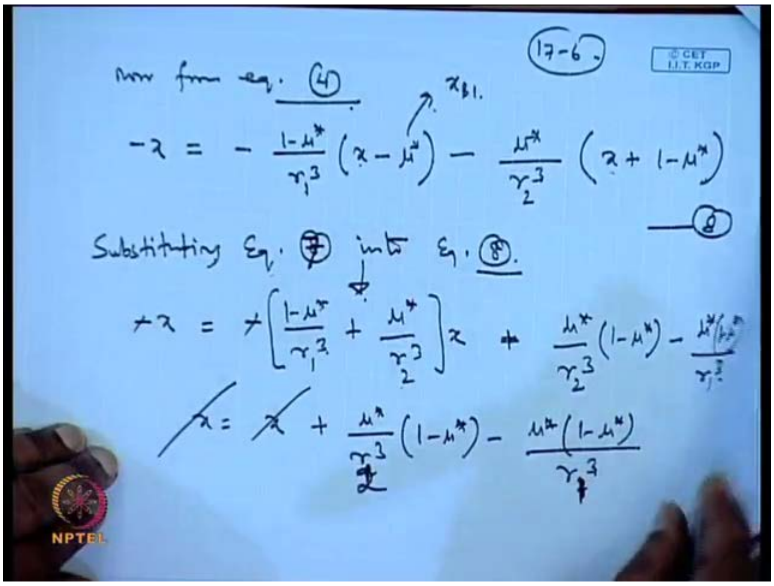

Now, from equation 4, minus x is equal to minus 1 minus mu star by r 1 cube x minus mu star. Here, this quantity is nothing, but x B 1, this we have explained in earlier lecture, mu star by r 2 cube times x plus 1 minus mu star.

So, the solution, that we got here, this we can write as equation number 7. Let us say this is equation number 8. So, substituting, equation 8 into, equation 7 into equation 8, so first we will sort out the common terms, so this we can write as minus sign, we can cancel on both sides, so let us remove the minus sign at this stage itself. So, 1 minus mu star r 1 cube x and from here we sort it out, that is, mu star by r 2 cube times x and then from this place we will have plus mu star r 2 whole cube times 1 minus mu star minus mu star times 1 minus mu star divided by r 1 whole cube. So, if we substitute from equation from equation 7 this quantity into this, so this become x is equal to x plus mu star r 2 whole cube 1 minus mu star.

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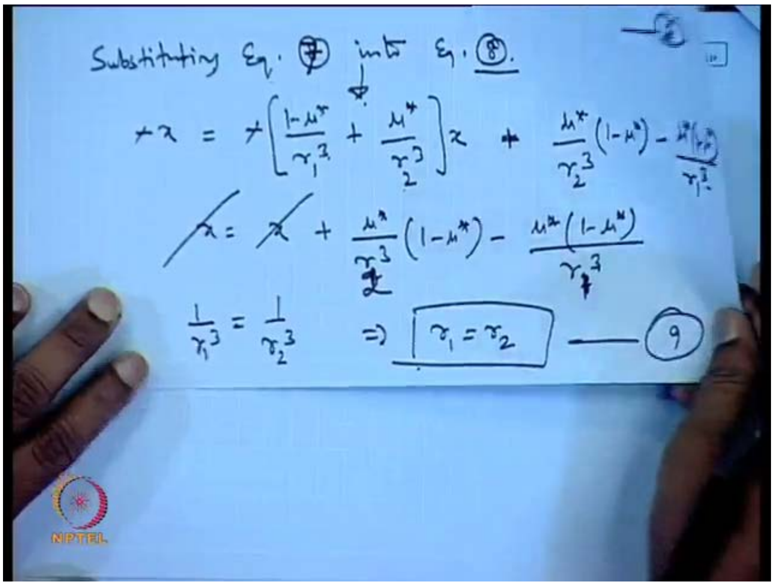

This is r 1 r 2 whole cube and this is r 1 whole cube, so these 2 cancel out, leaving us with r 1 cube is equal to r 2 cube. So, this simply implies, r 1 is equal to r 2, so this is our result number 9.

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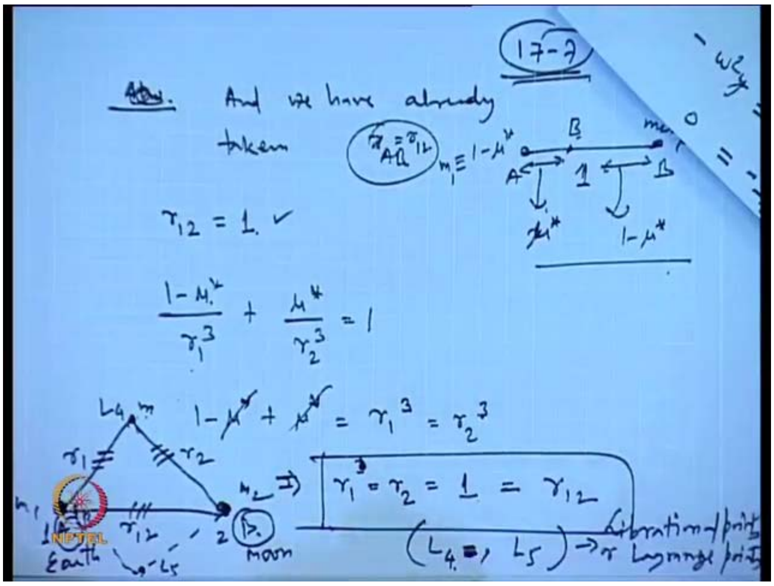

And we have already taken the distance x AB, that is, the distance between these two particles A and B as 1. And if this is the point B, so from here we measured this distance as, this mass was 1 minus mu star and this distance we measured as, wrote as mu star and this distance we wrote as 1 minus mu star, and this mass as mu star. Therefore, this is m 2, so this we are replacing with mu star and m 1 we are replacing with 1 minus mu star.

So, already our distance r 12 or x AB we can write as r 12. So, r 12 equal to 1, this we have already written. So, if we have that some equation number 7, this is our equation number 7, now in this equation if we substitute r 1 is equal to r 2, so this will yield us 1 minus mu star times mu star is equal to r 1 cube is equal to r 2 cube. So, this cancels out and this implies r 1 is equal to r 2 is equal to 1. And therefore, this quantity is nothing, but equal to r 12.

So, it shows, that the equilibrium points, they lie in a equilateral triangle. So, here the points we have, the point number 1, which is indicated by A; the point number 2, which is indicated by B. So, this is m 1, m 2 and the mass m. So, this distance was our r 1 and this distance was r 2 and this is r 12. So, this one is equal to this one and this one, and on a normalized scale they are equal to 1. Therefore, they, these three points, they lie in the equilateral triangle. So, from here it is obvious, that one point will lie here, another point will appear here downwards, somewhere here in this place.

So, if I, we have the earth here in this place and this is our moon, so for the earth moon system one point will appear here, another point will appear here in this point and they will appear always at the same distance because here, because it is equilibrium point. So, this mass is going to appear from these two places to be always present in the same point. So, as the moon is moving and the earth is also moving, so the they have their bary centre somewhere lying. Suppose, this is the heavy, heaviest mass, the earth is here. So, suppose bary centre is lying here in this place somewhere, this is our point B, so earth will be moving along, around this point and the moon will also be moving around this point. So, this is our earth and this is our moon. So, you got this point, they are moving like this and while the m is remaining at a constant distance from this point and this point, so it will always appear in the same point.

And later on, we will discuss more about this graphically, how the whole thing looks like. So, beside these two points, so this we can name as the L 4 and the L 5. So, L 5 will appear down, downwards somewhere in this place. So, this, this solution will, later on we will see, we will find out using the, by solving these equations 4, 5 and 6. We will find out the values x and y from this place and obviously, of z also, and these are going to give us the points L 4 and L 5. So, these are our librational points; librational points are Lagrange points.

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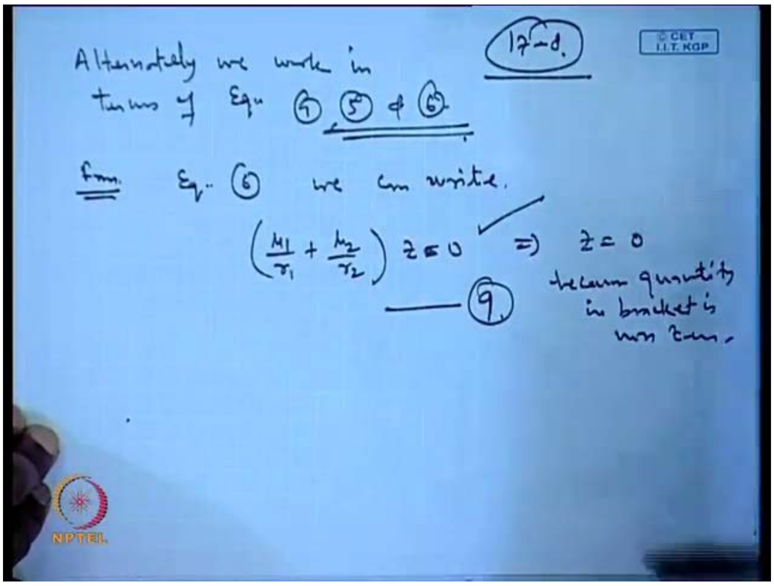

So, alternately, we work in terms of equation 4, 5 and 6. So, this is going to produce a more general result and from there we will further develop and look into graphically what does it mean to have the librational points and how according to the variation in v. So, either I can have the 0 velocity, which is the bonding surface beyond ,which the particle cannot go or if we have v, which is positive, which is always positive. So, if we have certain positive value of the v, so how the particles will be moving? The 3rd particle will be moving with respect to this primary and the secondary bodies m 1 and m 2.

So, from, from equation 6 we can write, this is our equation number 6, so from here it is apparent z equal to 0. So, from equation 6 we have mu 1 plus r 1 plus mu 2 by r 2 and z equal to 0. So, simply this implies z equal to 0 because quantity in bracket is non-zero. So, here we term this as equation number 9; so, this is our equation number 9.

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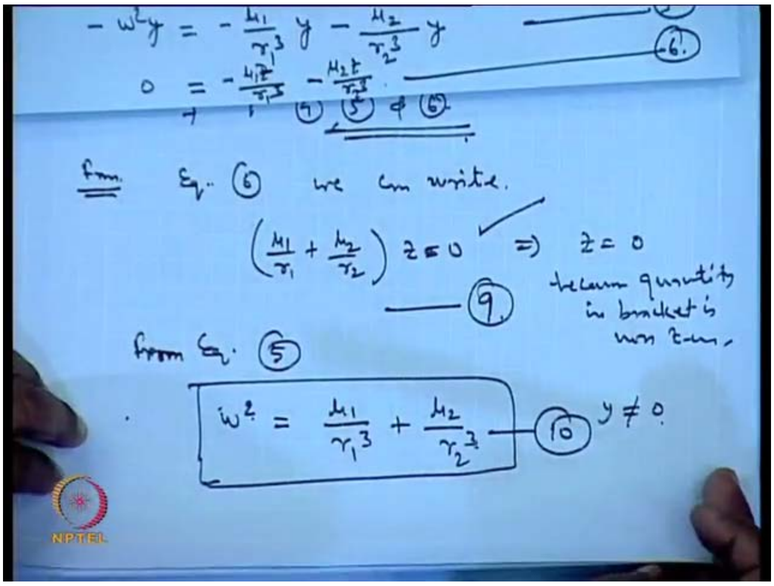

Now, for, from equation number 5 we get omega square, this is equal to mu 1 by r 1 cube plus mu 2 by r 2 cube, assuming why this is not equal to 0. So, this is our equation number 10.

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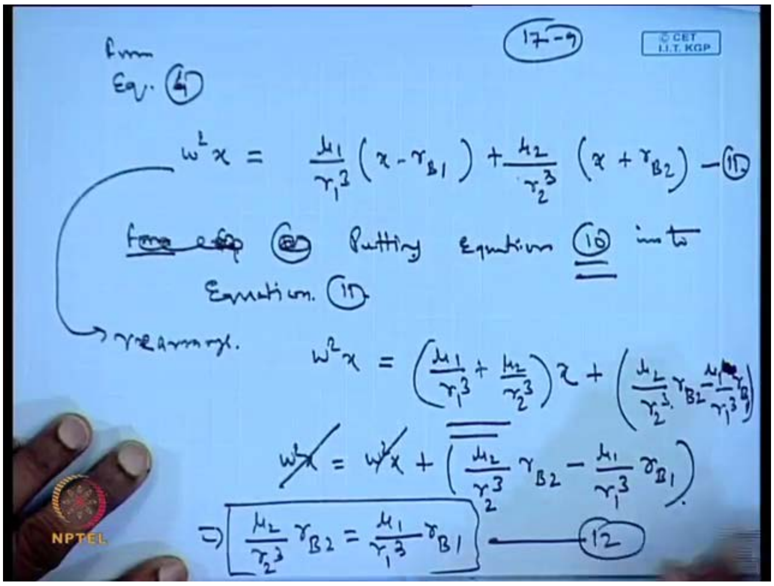

And equation number 4 gives us, so from equation 4 omega square x, this will be equal to mu 1 by r 1 cube mu 2 by r 2 cube x plus r B 2. This is equation number 11.

Now, from equation 10, so if we put the equation 10, equation 10 into equation 11, so what we can do? So, before this we need to little rearrange this, so rearrange. So, already we have done the rearrangement earlier, this we can write as mu 1 by r 1 cube plus mu 2 by r 2 cube x plus mu 2 by r 2 cube r B 2 minus mu 1 by r 1 cube r B 1. So, this quantity we can substitute from equation number 10. So, this quantity is nothing, but omega square x. So, omega square x and omega square x this cancels out, as earlier we have done in the normalized from, but here in this format we are going to get little more inside. So, this is mu 2 minus mu 1, so they cancel out and this implies mu 2 by r 2 cube times r B 2, and this we term as our equation number 12.

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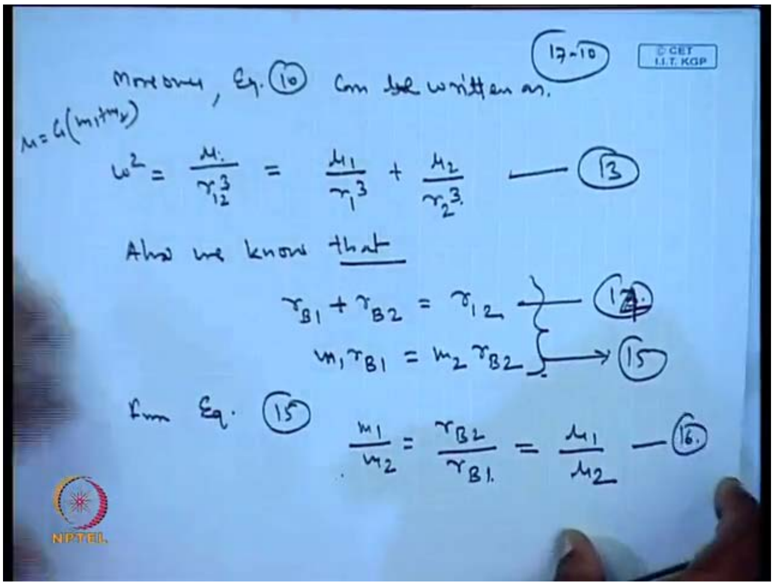

Moreover, equation 10 can be written as mu by r 12 whole cube because omega square is equal to nothing, but mu by r 12 whole cube, where r 12 is the distance between the points A and B and mu is nothing, but here, mu is nothing, but G times m 1 plus m 2. So, this is equal to mu 1 by r 1 cube plus mu 2 by r 2 cube.

Also, we know, that r B 1 plus r B 2, this is equal to r 12, the total distance between the point 1 and 2. And moreover, m 1 times r B 1 equal to m 2 times r B 2. So, these two results can be utilized along with the equation, that we have just now written, equation number 12 to data useful results. So, from this we can write as 12, this is 13th, this we can term as 14 and this 15. So, from equation 15 we have m 1 by m 2 is equal to r B 2 by r B 1 and m 1 by m 2. If we multiply up and down by G, so this will be nothing, but equal to mu 1 by mu 2. So, this is our equation number 16.

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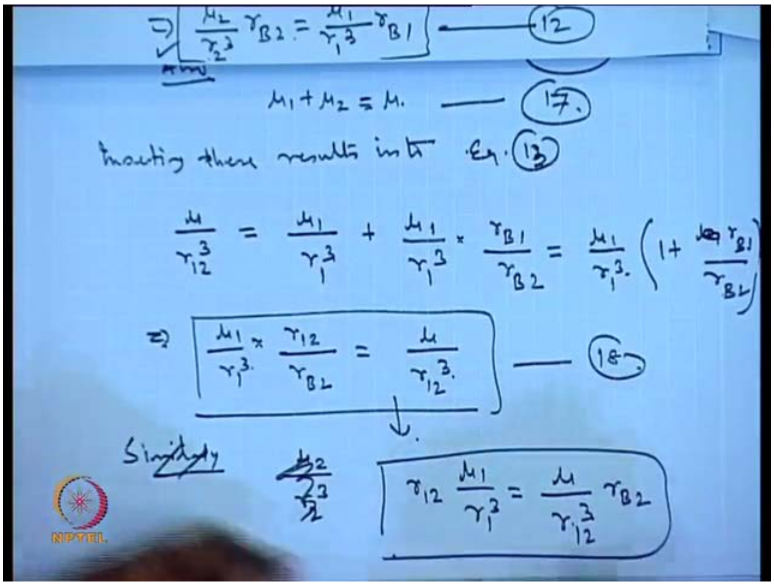

Also, we will have mu 1 plus mu 2 is equal to mu, this we write as equation number 17. Now, it is inserting into the equation number, equation number 13. So, inserting these results into equation 13.

So, we will have mu by r 12 whole cube, this is equal to mu 1 by r 1 cube plus mu 1 by, now mu 2 by r 2 cube we need to replace, so mu 2 by r 2 cube we can replace from equation number 12; mu 2 by r 2 cube, it can be replaced from equation 12. So, mu 2 by r 2 cube, this becomes mu 1 by r 1 cube times r B 1 by r B 2. We can take it out mu 1 by r 1 cube, this gives us r B 1 r B 2. So, this implies mu 1 by r 1 cube and this we can sum it up. So, r B 1 plus r B 2, this is nothing, but equal to r 12. So, we can write this as r 12 divided by r B 2 is equal to mu by r 12 whole cube.

Now, similarly, mu 2 by r 2 cube, so we will do little more expansion, we will write it little more properly. So, we will write this as mu 1 by r 1 cube is equal to mu by, r, r 12 is equal to r 12 whole cube times r B 2.

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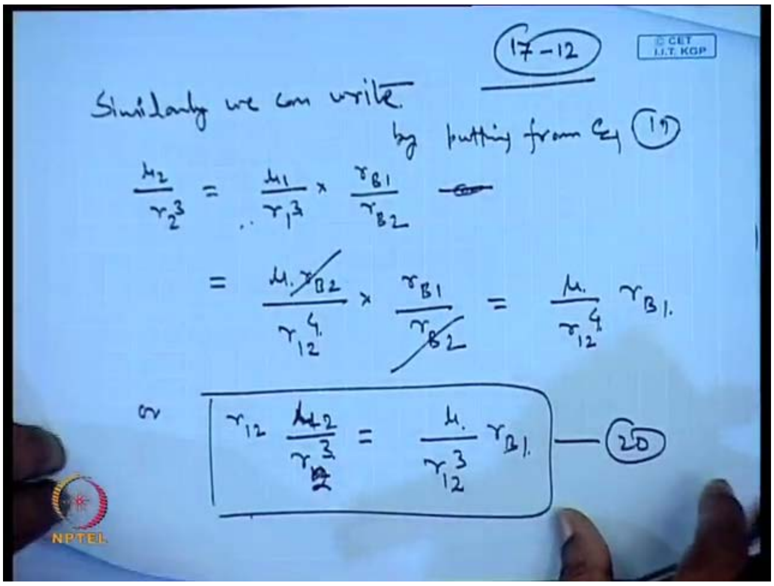

Similarly, we can write, this is the result, that we got earlier mu 2 by r 2 cube, this is a result, that we had in the equation number 12. So, we, we utilize this result, so this quantity will be equal to, now mu times r B2. So, we have here this equation we can utilize, let us name this also as equation number 19.

So, from here we will have, so if we can take out from mu 1 by r 1 cube from here and insert into this equation here in this place, so this is our, and inserting into this equation. Similarly, we can write this by putting from equation 19. So, mu 1 and mu 1 by r 1 cube we are replacing here, so this comes in terms of this quantity is here, mu times r B 2 divide the r 2 to the power of 4 times r B 1 by r B 2, this cancels out and what we get is… or r 12 times mu 2 by r 12 r 2 whole cube. This is equal to mu by r 12 whole cube r B1

Now, all these equations can be utilized, so this is our equation number 20.

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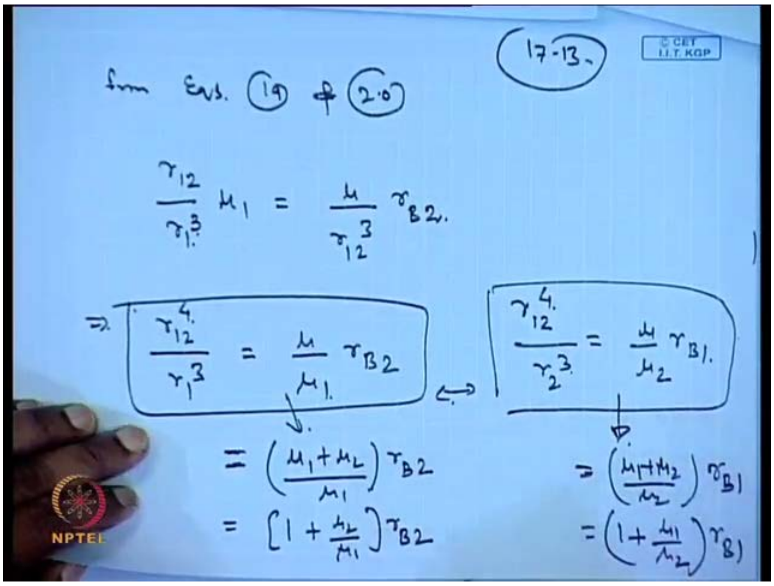

To find out the relevant relationship between these quantities, so we can look into equation number 19 and 20. So, from equations 19 and 20, r 12 whole cube, r 12 by r 1, this r 12 by r 1 whole cube times mu 1, this is equal to mu r B2, this is one equation. Another we get r 12 to the power 4 from equation number 20 to the power r 2 cube time is equal to mu by mu 2 times r B1. So, these are the two equations after rewriting we are getting. So, from here the useful results can be derived. So, if we expand them further, so this can be written as mu 1 plus mu 2 divided by mu 1 times r B2. So, this is nothing, but 1 plus mu 2 by mu 1 times r B2 and here this will yield us mu 1 plus mu 2 divided by mu 2 times r B1, 1 plus mu 1 by mu 2 times r B1.

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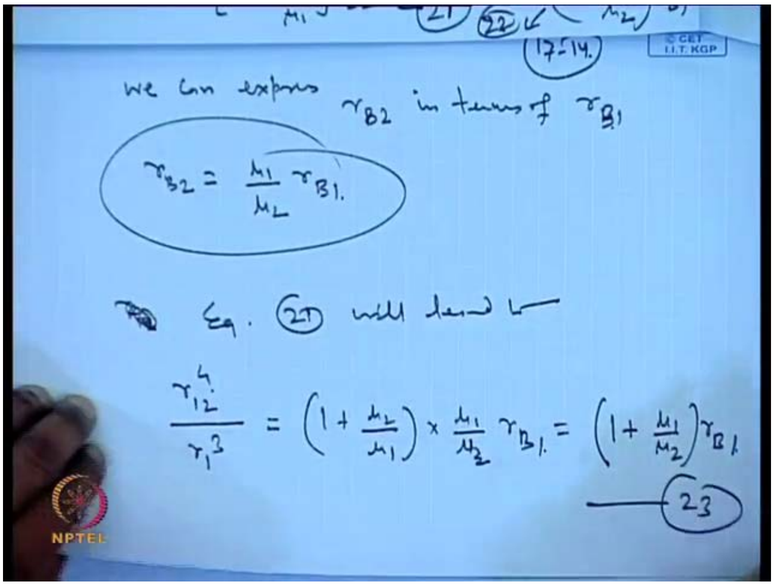

Now, both the equations, they are available here. So, these equations, let us say this is our 21 and this is 22. So, r B2 can be expressed in terms of r B1, so we can express r B2 in terms of r B 1, which we have written earlier. So, r B2 is nothing, but mu 1 by mu 2 times r B1. So, use this result to resolve the things further. So, this becomes the equation, 20, 21 will become equation 20 will lead to r 12 to the power 4 r 1 cube. This becomes 1 plus mu 2 by mu 1 times mu 1 by mu 2 times r B1. So, here we can cancel out mu 1 if we multiply in sides, so this becomes 1 plus mu 1 by mu 2 times r B1. So this we name as equation number 23.

Now, we can compare equation number 23 and this equation number 22. So, we have in equation number 22, r 12 to the power 4 by r 2 whole cube equal to, on the right hand side, 1 plus mu 1 by mu 2 times r B1. And this equation on the right hand side also, we have 1 plus mu 1 by mu 2 times r B1, while the left side differs.

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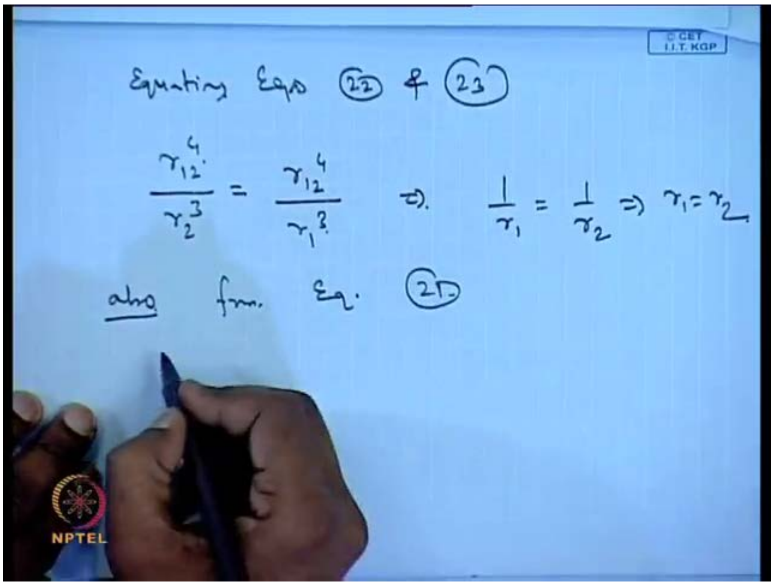

So, by equating these two equations 22 and 23, equating equations 22 and 23, this will give us r 12 to the power 4 by r 2 whole cube equal to r 12 to the power 4 by r 1 whole cube and this implies 1 by r 1 is equal to 1 by r 2 and this implies r 1 is equal to r 2.

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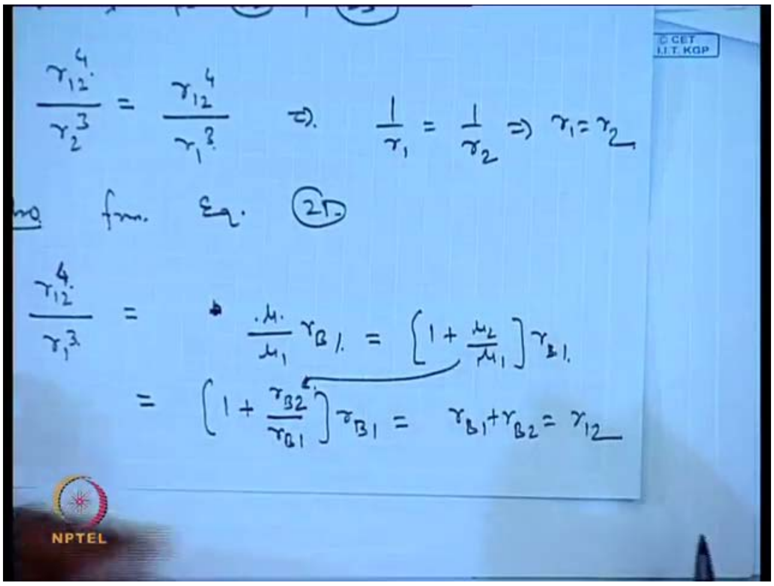

Also, from equation 21, r 1 to the power 4 divided by r 1 cube, this is equal to mu by mu 1 times r B1 is equal to, this we have written as 1 plus and mu 2 by mu 1, this quantity we can replace as r B2 by r B1 multiplied by r B1. So, this leads up to r B1 plus r B2 is equal to r 12.

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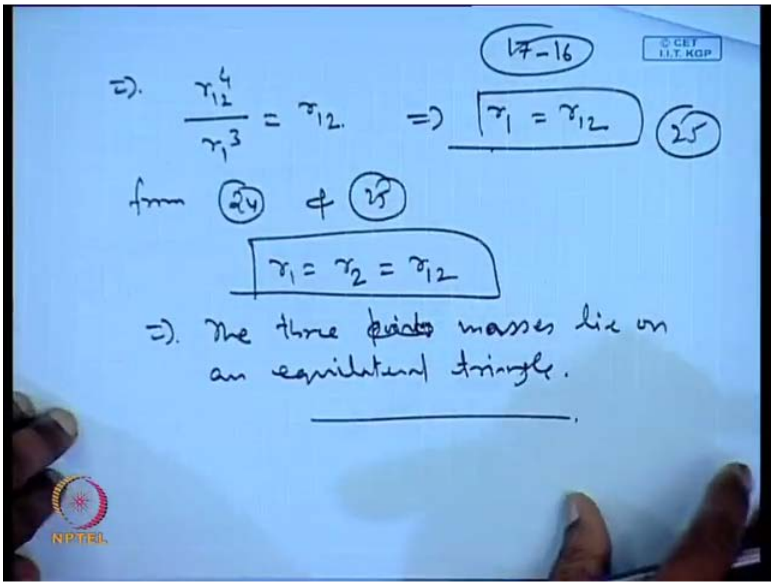

So, this implies, r 1 to the power 4 by r 1 whole cube is equal to, r r, r 12, and this implies r 1 is equal to r 12. Therefore, from this result, this is result number, we can write this as 24, and this we can write as 25. So, from 24 and 25 we get r 1 is equal to r 2 is equal to r 12. This is what the result we established on the normalized skill. So, there it was very easy while here it took some time to work it out. So, this implies the three points, three points are the three masses lie on an equilateral triangle.

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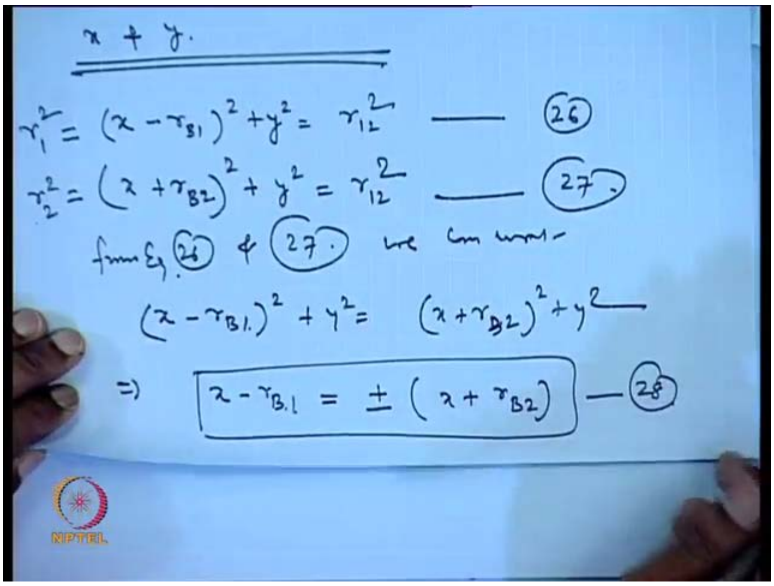

Now, we have once worked out all things, so next we can look into the, what will be the values of x and y. So, we need to work it out working for values of x and y. So, we know, that x minus r B1 square plus y square, this is the quantity r 12 square. And similarly, we have x plus r B 2, this is our equation number 26 and this is 27. So, utilizing the previous results and here we have already, we see, that the right hand side in both the cases. Therefore, they are referring to the same quantities because this distance we wrote as r 1, r 1 square and this distance we wrote as r 2 square. So, this quantity was, because we have already proved, that r 1 equal to r 2 is equal to r 12, so from 26 and, equation 26 and 27 we can write x minus r B1 x plus r B2 square plus y square. So, this implies x minus r B1 is equal to plus minus x plus r B2.

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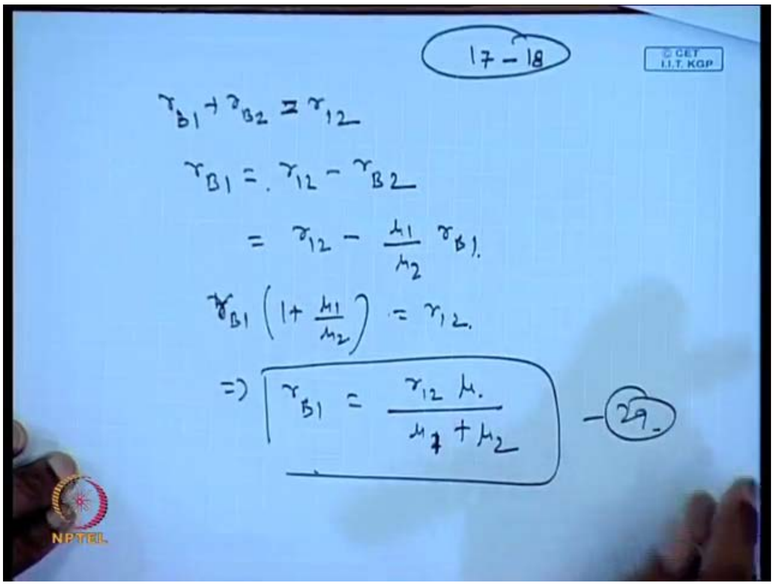

Next, we utilize some of the results, like we can prove it, r B1 plus r B2, this we have already written as equal to r 12. So, from here it can be written as r B2, so r B1 is equal to r 12 minus r B2 and r B2 again can be written as mu 1 by mu 2 times r B1. And therefore, r B1, earlier also we have proved this result, r 1 times r B1 times 1 plus mu 1 by mu 2, this becomes r 12 and this implies r B1 is equal to r 12 mu divided by mu 1 plus mu 2.

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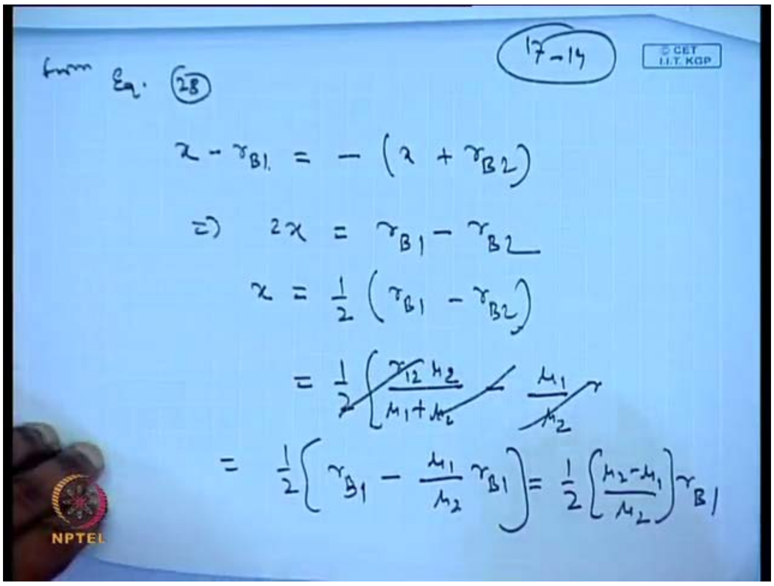

Now, from equation 28, from equation 28, taking negative sign we can write this as, x is equal to 1 by 2 r B1 minus r B2. So, r B1, r B2 we have right now expressed, so you can, we can use those expression to resolve it further. So, inserting for r B1 and r B2, this is r 12 mu divided by mu 1 plus here. We have in this equation mu 1 plus mu 2 r B1, this is mu 2 here, this is mu 2 minus r B2, we have written as mu 1 by mu 2 times r B1. So, we can simplify it first and then we can write it again 1 by 2 r B1 minus r B2. We can write as mu 1 by mu 2 times r B1. So, this becomes 1 by 2 mu 2 minus mu 1 by mu 2 times r B1.

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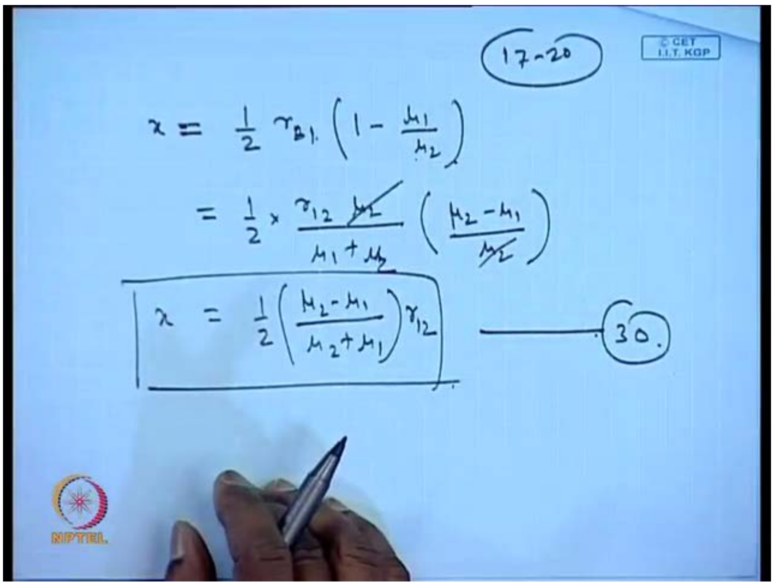

So, what we get x as 1 by 2 times r B 1 times 1 minus mu 1 by mu 2, and r B1 we have already written in terms of r 12. So, this becomes r 12 times mu 2 by mu 1 plus mu 2 and then multiplied by mu 2 minus mu 1 by mu 2, so they cancel out. Ultimately, we get mu 2 minus mu 1 by mu 2 plus mu 1. So, this is the value of x that we get and multiplied by of course, r 12. So, this is our equation number, this is 30.

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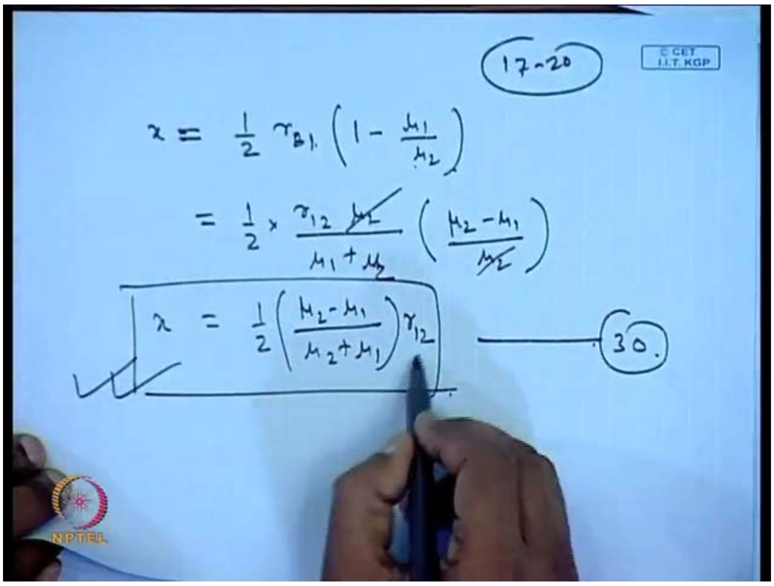

Now, we can work out for y, so we can work along the same line. x we can substitute from here, so this will be r 12 by 2 times mu 2 minus mu 1 by mu 2 plus mu 1. Now, r B1 can be written in terms of r 12. So, after that simplification you can write this as 1 by 2 mu 2 minus mu 1 mu 2 plus mu 1 r B1 by r 12 mu 1 by mu 2. And further simplification of this can be done, so we will continue next time. So, will see, that y can be written as plus minus underroot 3 by 2 times r 12.

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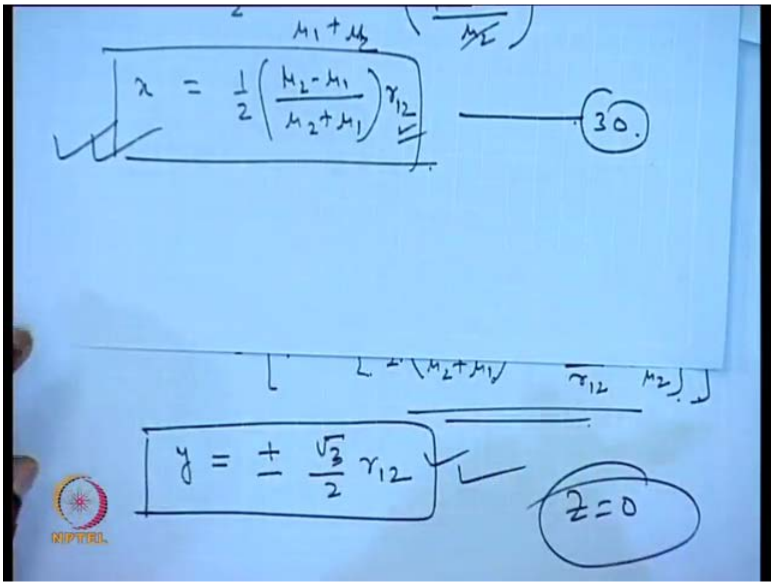

So, we can see, that the x and y, both have been written in terms of r 12, which is the distance between the points A and B, so these two useful results. And z we have already written, that, in that z equal to 0. So, using this we will be able to find out where the point is exactly located.

So, we continue with these things next time and beside this these are the two points, beside this we have three points, other points, which are collinear, that is, the line is the same, on the same line. So, those three points how to work them out, that we will see next time.

Thank you very much.