Space Flight Mechanics

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Lecture No. # 16

Three Body Problem (Contd.)

In the last lecture we have been discussing about the restricted three body problem, so we continue with the same problem.

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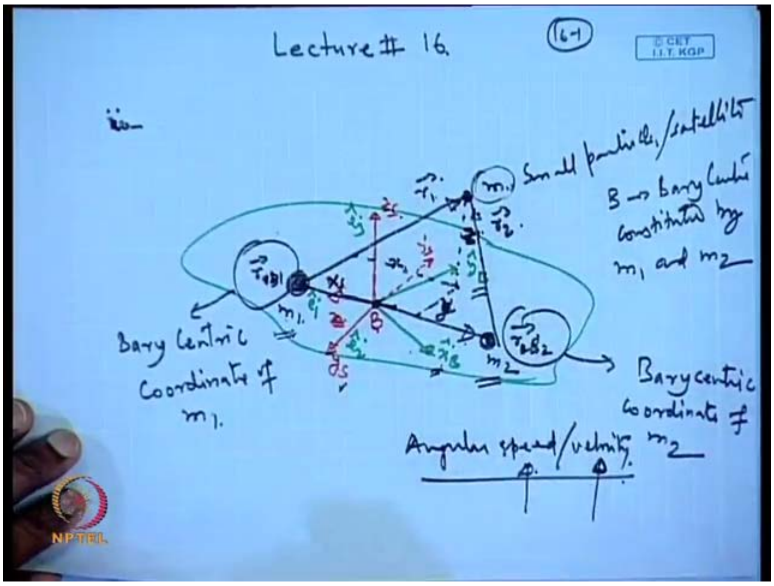

This is the figure for the motion of the satellite, which is moving with respect to the two bodies. So, here the major body, which is the primary body is m 1, written as m 1, and the secondary body is written as m 2. And as compared to this primary and the secondary body, the satellite or the small particle, whatever it is, so they, this is very small, so, so that it does not perturb the motion of m 1 and m 2. So, under this assumption we had started and then, this wrote, that the m 1 and m 2, they are moving in a circular orbit.

So, we had the two assumptions, that the m 1 and m 2, they are moving under the, in the circular orbit and the third body is very small as compared to the primary body and the secondary body. And the third one, which was the restricted, restricted constraint, so that we wrote as, that the orbital plane of the mass m can be restricted to move in the orbital plane of m 1 and m 2.

So, now in this figure we have, B is the bary center, this is bary center constituted by m 1 and m 2. So, in the bary center we have the reference frame centered about the bary center, which is referred to as x B and the y B. So, these are the (( )), these are the directions of the x x and the y x of the initial coordinate system or because it is a center about the bary center and it is non-rotating. Moreover, its direction is fixed, so let us say this is x B and y B and z B will be perpendicular to x B and y B, which is not found here in this case.

Now, we have the synodic reference frame simultaneously we define and that we wrote as the coordinate for the synodic reference frame. It was written as the x s coordinate, which is directed along the m 1 and then the next one was the y s, which is perpendicular to the x s and the z s, which will be perpendicular to plane of x s and y s. So, this constitutes the synodic reference frame. So, what is the property of this synodic reference frame? We have already discussed, that the synodic reference frame x s direction will always be fixed along m 1, so as the m 1 and m 2 are rotating simultaneously. x s is also rotating, so synodic reference frame is nothing, but a rotating reference frame and it rotates at the same angular speed as the combine.

This combine planets m 1 and m 2 are the combined heavy particles m 1 and m 2 are having the angular velocity, angular velocity or here we say the angular speed, so angular speed or angular velocity. So, once we say, that it is a velocity, then that direction is fixed and the case of this direction is combined here. In the case of angular speed direction is not combined, so the magnitude-wise both are same and also the direction-wise because they move in the same direction. So, if suppose m 1 and m 2, they are rotating anticlockwise, so x s, y s, it is also rotating anticlockwise, so direction also remains same. Moreover, we have the coordinates of the satellite or the small particle, this m is a small particle small particle plus satellite, this is the coordinate r 1 and the coordinate of the, coordinate from m 2, this is r 2.

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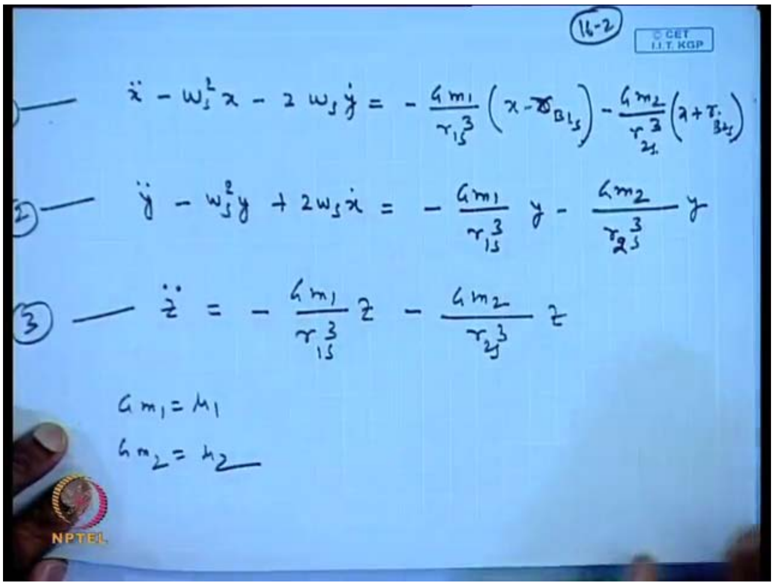

So, we develop the relative motion or we develop the relative motion of the small particle with respect, in the synodic reference frame, that is, in the moving reference frame. And we wrote these equations, we developed these equations earlier.

So, thereafter, what we did? We multiplied the equation number 1 by x dot 2 x dot; the equation number 2 by 2 y dot and equation number 3 by 2 z dot, and then added them.

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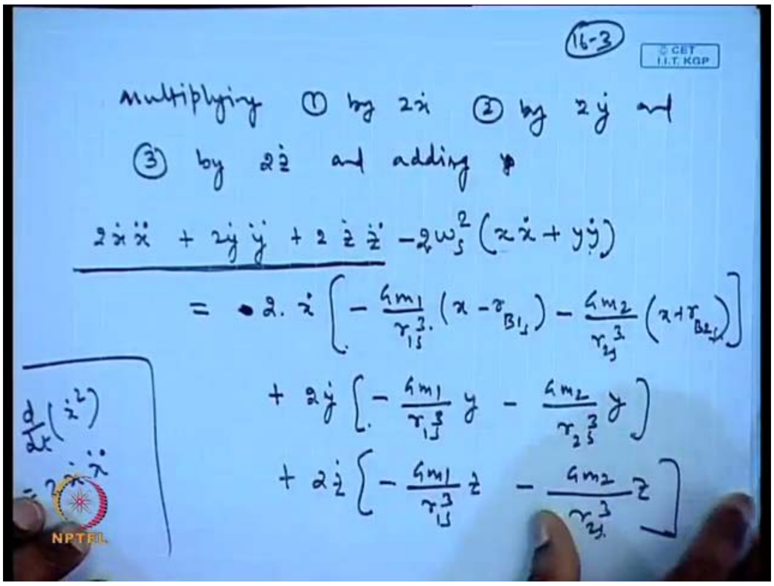

So, once we do this operation, so multiplying 1 by 2 x dot, 2 by 2 y dot and 3 by 2 z dot and adding, this gives us 2 x dot times x double dot plus 2 y dot times y double dot z dot times z double dot minus omega s square x times x dot plus y times y dot. This 2 factor is there, is equal to 2 x dot times the quantity minus G m by r 1 s whole cube x minus r B 1 s whole cube r B 2 s 2 y dot times minus G m by r 1 s cube r 2 s whole cube y plus 2 z times.

Now, we can simplify this equation, so this you can see from here, this is nothing, but you can write that derivative of the x dot a square and like that. Suppose, here we can show it, d by dt x dot square, this will be nothing, but 2 times x dot times x double dot. Similarly, other terms in this can be written and similarly, here this can be written in terms of x square and y square.

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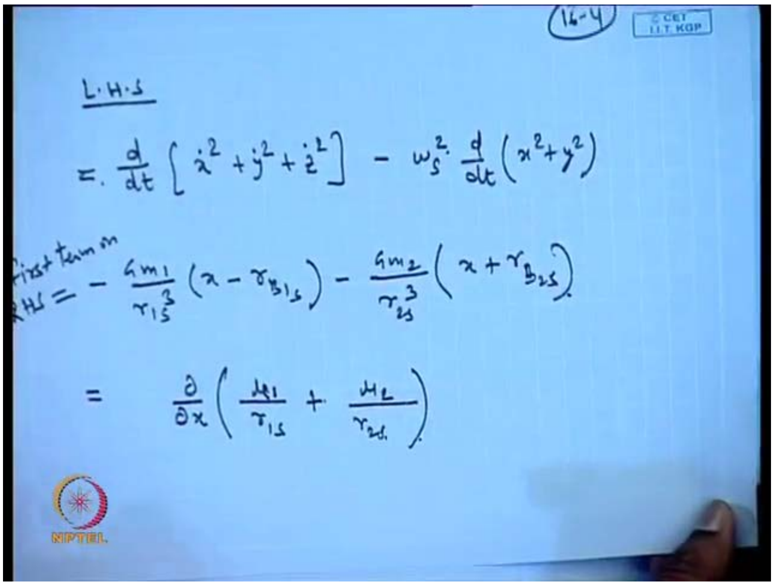

So, next, simplifying it, so the LHS, we take the LHS, LHS becomes d by dt x dot square y dot square and z dot square minus omega s square times d by dt x square plus y square. Now, taking the RHS, so for RHS, working out the RHS of this equation we need to work out further and we will see, that this gets into a very simple form. So, let us first take minus Gm 1 r 1s whole cube x minus r B1s minus Gm 2 by r 2s whole cube x plus r B 2s.

So, this is our first term on RHS, so this can be written in a very simple format as, dow by dow x times mu by r 1s, this is mu 1, r 2s. So, we can check this and verify it. Similarly, other terms can be written, but let us first work out this particular term.

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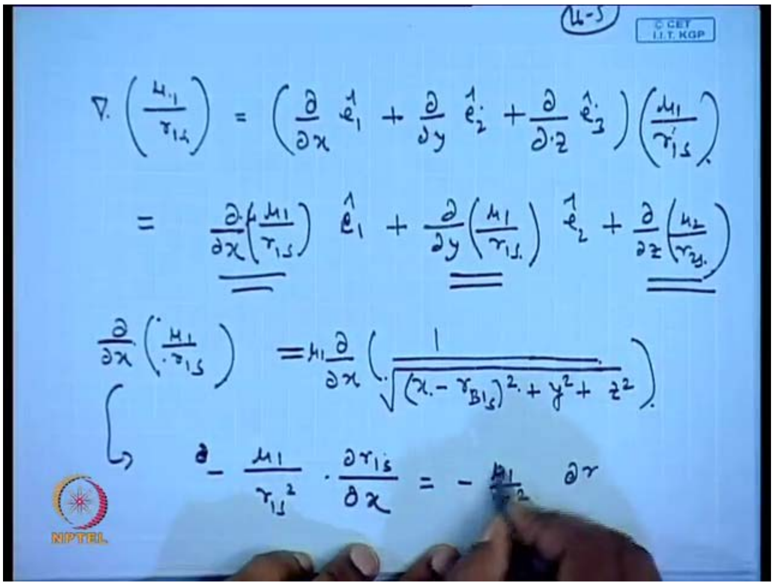

Taking up the next term, sorry, taking this again, so we have, let us say, mu 1 by r 1 s, this quantity is given, if we operate on this by the del operator, so we can write this as, dow by dow x e 1 dow by dow y e 2 times dow by dow z e 3. And this is operator on y, this is to be the quantity to be operated upon, this is mu by r 1 s.

So, once we operate on this we can see, that we can write this as dow by dow x e 1 cap plus dow by dow y mu 1 by r 1 s. Now, we can take each of the term in this separately and work it out.

So, taking the 1st term, so this will be nothing, but we can take out mu 1 outside, r 1 we know, that r 1 is nothing, but x minus r B 1 s s square plus y square plus z square. So, this is underroot, so from here itself we could have written in a way, dow by dow x is equal to mu 1 by r 1 s square minus dow r 1 s by dow x. So, we need to just find out the derivative of r 1 with respect to s. So, this quantity, this will lead to minus mu 1 by r 1 s square and dow r 1 by dow x, then we have to work it out.

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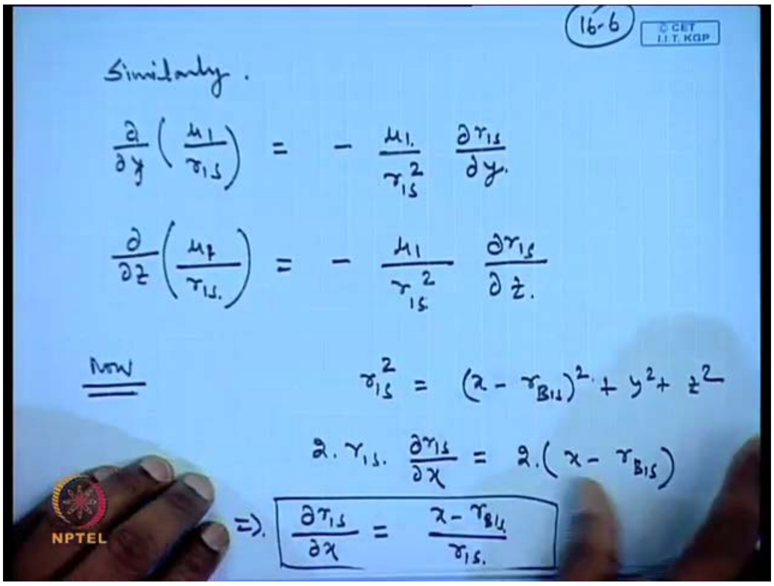

Similarly, we have, we will work it out, we will work out this equation later on, let us work out the all other terms. So, dow by dow y mu 1 by r 1 s, this will also be equal to mu 1 times r 1 s whole square times dhow r 1 s by dhow y. And generally, dhow by dhow z mu 1 by dhow z. Now, r 1 s, this is nothing, but x minus r B 1 s square. This s square is equal to this s square plus y square plus z square. So, differentiating with respect to x, so 2 times r 1 s times dow r 1 s by dow x and this implies dow r 1 s by dow x is equal to x minus r B 1 x by r 1 s.

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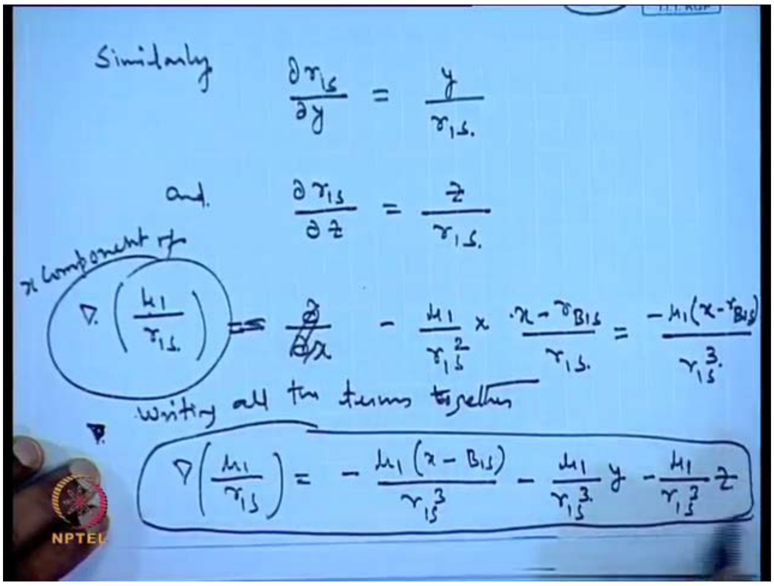

So, similarly, dow r 1 s by dow y, this becomes equal to y by r 1 s and dow r 1 s by dow z, this is equal to z by r 1 s. Now, we can put this expression back into the equation where we required, so we need to put into this equation. So, therefore, del mu 1 by r 1 s, this becomes dow by dow x minus mu 1 by r 1 s square times x minus r B 1 s by r 1 s. This is equal to minus mu 1 times x minus r B 1 s divided by r 1 s whole cube; whole cube. This is just one component of the, x component of this, the x component of del x is this quantity. So, adding all the terms, therefore, writing all the terms together, together that gives us del mu 1 by r 1 s is equal to minus mu 1 by x minus B 1 s by r 1 s cube.

And similarly, other terms here we can supplement and they will become equal to mu 1 by r 1 s cube times y minus mu 1 by r 1 s cube times z. And moreover, what we have missed here, we can look into this equation here, we have the vectors, the unit vectors e 1, e 2 and e 3 is missing out here. So, e 3 is present, so we supplemented with the unit vector e 1, e 2 and e 3.

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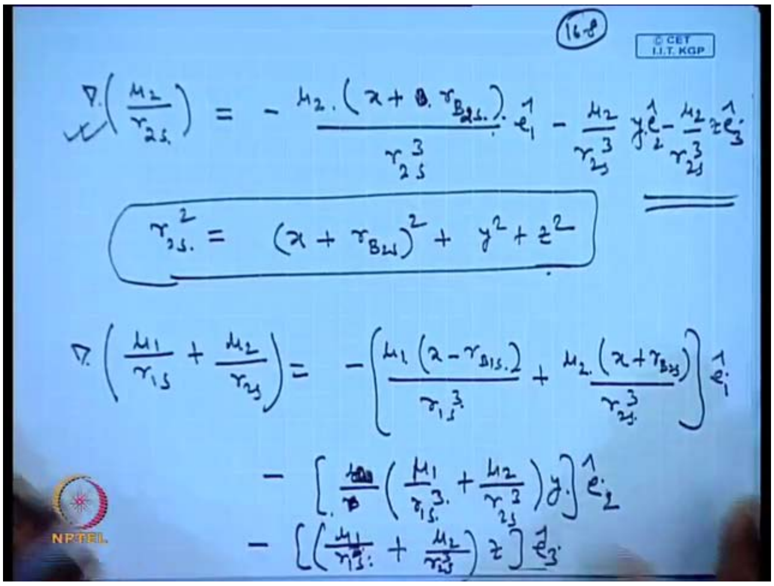

Thus, after deriving this, similarly we can write del by mu 2 by r 2 s, this is equal to minus mu 2 times x plus having the quantity here, this is r B 1 s. So, this is r B 1 s, this will be r B 2 s, how it is coming I will you show next. So, this is r 2 s whole cube e 1 minus mu 2 by r 2 s whole cube y minus mu 2 by r 2 s whole cube z. This is e 2 and this is e 3, so we have r 2 s. This we have written as x plus r B 2 s whole square plus y square plus z square.

So, with this quantity if you do this operation workout, as we have done earlier, for del mu 1 by r 1 s, so you get this quantity exactly, what is written here. Therefore, del mu 1 by r 1 s plus mu 2 by r 2 s, this can be written as, we can sum them up. So, after summing them up they will become minus mu 1 x minus r B 1 s by r 1 s whole cube plus mu 2 times x plus r B 2 s divided by r 2 s whole cube time c 1 minus mu 2 by mu 2 times 1 by, this is (( )) present and y is present here also. So, this is, we write here as mu 1 by r 1 s whole cube plus mu 2 by r 2 s whole cube by this is e 2 cap minus. Similarly, we will have mu 1 by r 1 s whole cube plus mu 2 by r 2 s whole cube and z e 3 cap.

Now, going back to the equation where we started. Now, let us look into this equation. So, you can see, that the quantity, which is present here inside, they are nothing, but the terms, that we have developed just right now.

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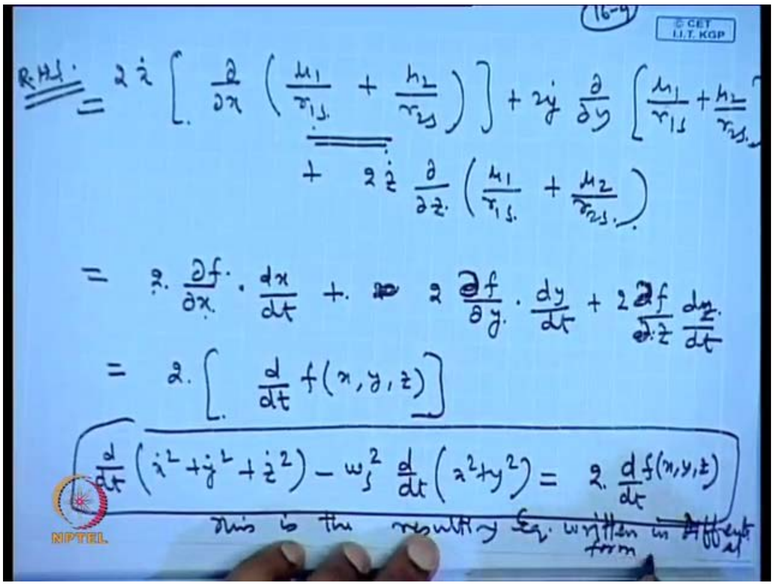

Therefore, we can write it further as, so if we are given the quantity, say we need to find out the quantity x dot times 2 x dot times, 2 x dot time

This was our right hand side, which we need to evaluate.

So, this is very simple to work out and let us see how to manage it. So, can we write it as 2 times dow by dow x and the quantity in the bracket, let us write this, as this quantity is f mu 1 by r 1 s and mu 2 by plus mu 2 by r 2 s. We write this as f, so this quantity becomes 2 times dow f by dow x times dx by dt plus 2 df by dy times dy by dt plus 2 df by 2 dow f, this is dhow f, 2 dow f by dow z times dz by dt. So, we can take 2 outside and this becomes nothing, but d by dt of function f x, y and z.

So, we started with these three equations, then multiplied them by 2x dot 2y dot and 2z dot, respectively. And we wrote this in this format, so the left hand side of this equation was written as this quantity here and then, we started working out the right hand term. So, right hand side we showed, that it can be represented in terms of del and this is a very useful representation. But as far as integration is concerned, this can be integrated in this format. So, once we get into this format, now we have the left hand side, which is written as d by dt x dot square y dot square plus z dot square minus omega s square d by dt times s square plus y square. And on the right hand side, just now we got the equation as 2 times d by dt f x, y, z. Now, we can integrate this equation, so this is the resulting equation written in differential.

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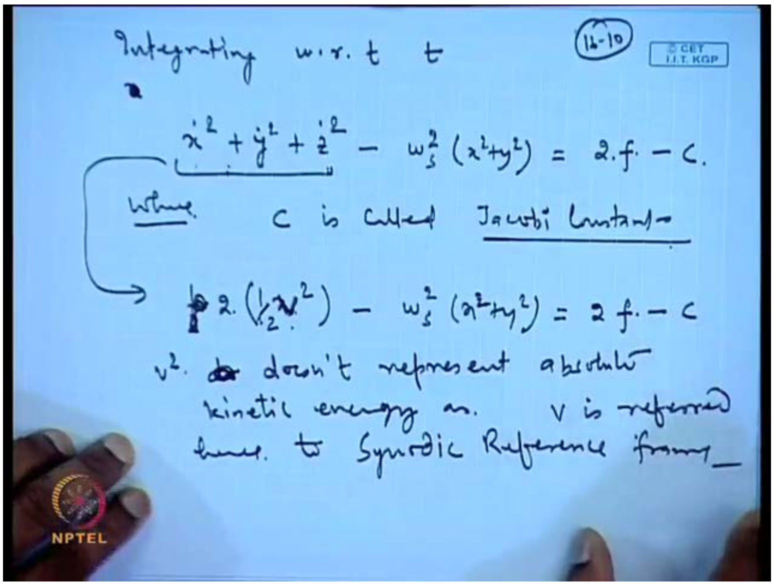

Now, this equation can be integrated, so after integration, if we integrated with respect to c t, so integrating with respect to t, so this gives us s square x dot square plus y dot square plus z dot square is equal to minus omega s square is equal to 2 f. And say, quantity c here, c is called, this is a Jacobi constant; so, this is Jacobi constant. And now, once we have written in this format, so we can see, that, that the quantity, which appears here, if we divided by 1 by 2, so this can be written as 1 by 2 times v square minus omega s square x square plus y square. We will take 1 by 2 inside and write it in this format, 2 f minus c. So, v square term, here it appears in, though this is not multiplied by mass, but this is, some, something, which represent the kinetic energy.

Now, what happens here, here in this case because x, y and z, they are referred to the synodic reference frame. So, this v is not really the v square, it does not really represent the absolute kinetic energy. So, here, v square does not represent absolute kinetic energy as v is referred here to synodic reference frame.

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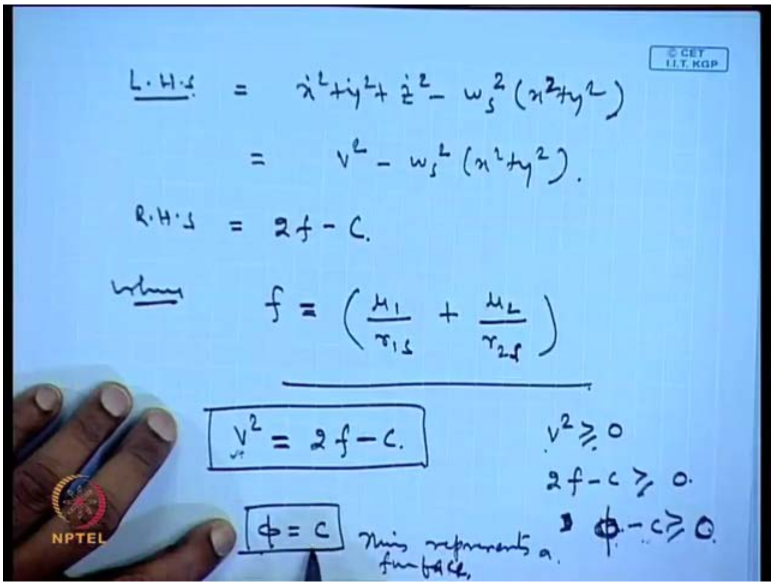

So, what we have got here? The left hand side, this appears in this format x dot square plus y dot square plus z dot square minus omega s square x square plus y square. This is equal to v square minus omega s square plus y square.

Now, this problem can be, we have to further look into what this function f is. So, the function f be defined as mu 1 by r 1 s plus mu 2 by r 2 s and the RHS we have written as 2 f minus c, where f is mu 1 by r 1 s plus mu 2 by r 2 s. Now, to get any meaning out of this, the equation that we have developed 2 f minus c, from here we can see, that v square is a quantity, which will be always positive, for at most it can be 0. So, v square, this will be always greater than equal to 0 and therefore, we can write 2 f minus c, this will be always greater than equal to 0 and we can write 2 f as another function, say phi. So, phi minus c, this will be greater than equal to 0. Therefore, if we write phi equal to c, this represents surface.

So, this is the equation of a surface and when phi equal to c, so at that time v becomes equal to 0 and this says, that on the surface the velocity of the particle, the 3rd particle, which is moving, that will be equal to 0. So, that implies the small satellite or the small particle that we took, which is moving with respect to the primary and the secondary body, so it cannot escape this boundary, which is defined by phi is equal to c, which is a surface basically.

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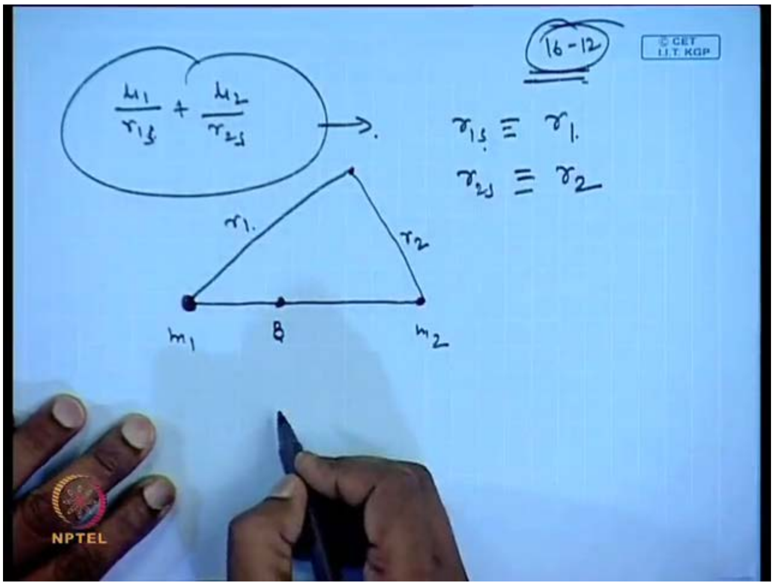

So, taking now mu 1 by r 1 s plus mu 2 by r 2 s, so in this we can do the simplification, that r 1 s, we drop the subscript s and this we write simply as r 1 and r 2 s. We simply write as r 2, this makes the representation very easy, so we have here two ways of treating the problem of, we can see that later on, that in such a system are the two masses, are two primary mass, one primary mass and other secondary mass is there, which are very heavy with respect to the tertiary body, which is a particle or a small satellite. So, it is infinitesimally small such that it does not affect the motion of the, the primary and the secondary bodies. That is why, is here the emphasis is given on the infinitesimal word, that it is not going to disturb the motion of these two bodies.

So, once we start working with this, then we can see, that the satellite, there are some equilibrium points where the, this particle, the small particle are the satellite it will remain stationary with respect to this primary and the secondary body, that is, it will, if some observer is sitting on these two planets or say, one sun is there and another earth is there and with respect to this another planet is, another planet is there. So, saying this case, moon is there, which is orbiting the earth. So, we will have few points, which is called the librational points or the equilibrium points. So, if this librational points or the equilibrium points, if the particle is kept in those points, then for the observer, which is sitting either on the earth or either on the sun it will remain stationary forever. So, that implies it does not move at all with respect to these two points, say it will maintain a constant distance from the primary body and the secondary body.

Now, development can be done in two ways. So, if we can normalize the whole system and then we work our, either we can go without normalizing for developing the equilibrium points of the system. So, first us, first of all let us look into the normalized system, the how to represent it. Now, let us take the two points, which are body m 1 and m 2. So, m 1 is heavy as compared to m 2, this is the bary center B and we are writing the distance of the small body as r 1 and this as r 2 and this distance we wrote as r and the distance of the point, this one we have written as, this we have written as r B 1 and this we wrote as r B 2.

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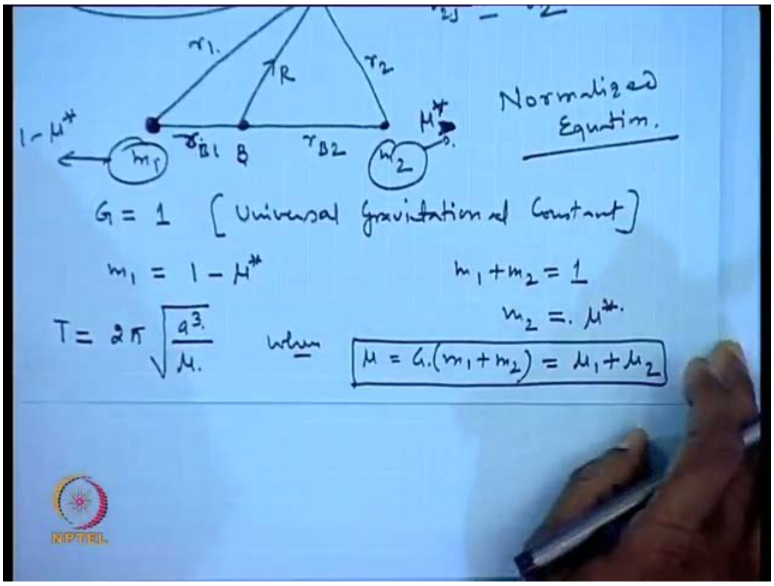

So, we will use a system, let us use G is equal to 1, this is universal, so we are working of normalized equation. We choose G is equal to 1 and also, we choose m 1, we write as 1 minus mu star. So, here basically, we will be writing it has m 1 plus m 2 equal to 1 and if we choose m 1 as 1 minus mu star, so simply m 2 becomes mu star. So, this we are going to replace as 1 minus mu star, a particular of mass 1 minus mu star and m 2 as a particular of particle of mass mu 2 mu star. Now, time period of the system is, will be given, write 2 pi a cube by mu under root where mu is equal to G times m 1 plus m 2. So, this we can write mu is equal to mu 1 plus mu 2.

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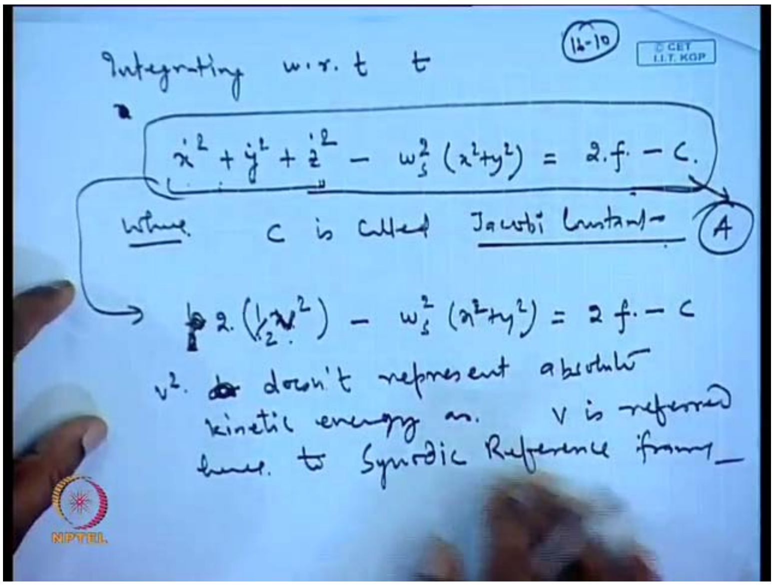

So, if we choose G is equal to 1, as I stated earlier, so we will have 2 pie by T is equal to mu by a cube underroot. Now, this quantity is nothing but, the angular velocity; this is our angular velocity. Now, we choose also the distance between m 1 and m 2 to be unity, that is, the distance between the points A and B equal to 1. So, r, we will write as r 1 2 or either r AB. This quantity is nothing, but equal to 1 and this is nothing but, the quantity which is appearing here. So, this is our A, which appears in the, as we have done for the two body system, we saw, that the mu is nothing, but G times m 1 plus m 2. And this is the distance between the two particles, say it is that distance between the two particles.

So, from here we will have omega equal to G times m 1 plus m 2 divided by a cube underroot. So, G already we have chosen as 1 and m 1 plus m 2 we are defining as 1, and a cube, from here this turns out to be 1. So, therefore, this quantity becomes equal to 1. So, omega is equal to 1. Now, this is highly simplified, so if the equation that we developed earlier, this particular equation, that we have written here, let us tag this as the equation number A. so, here omega s or this will become equal to 1 under the assumed condition.

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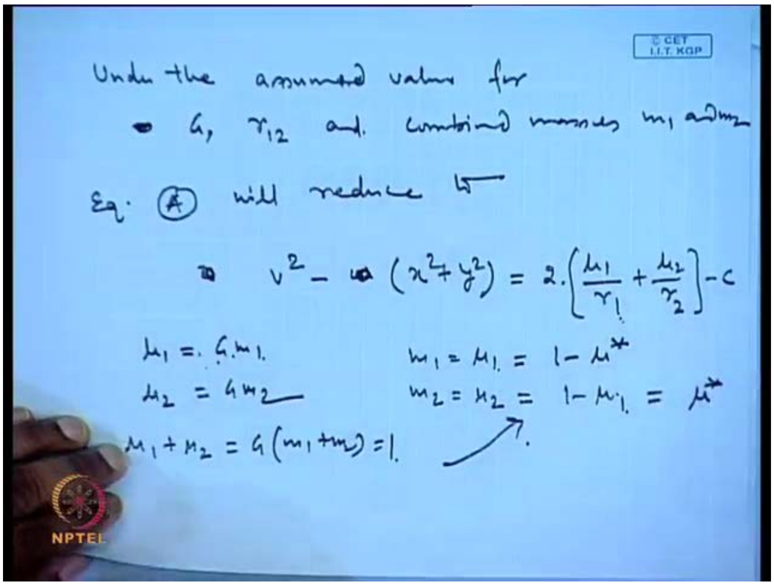

So, under the assumed value for G, then the distance r 1 2 and combined masses m 1 and m 2, equation A will reduce to v square minus omega v of, naturally writing as 1 for this is x square plus y square is equal to 2. And if we have written as mu 1 by r 1 plus mu 2 by r 2, here also we dropped the subscript s, so this minus c.

So, now we know, that mu 1 is G times m 1 and mu 2 is G times m 2. So, is it possible to, if describe this in a special format for, say G we take as 1, so m 1 becomes mu 1 and m 2 becomes mu 2 and naturally, then this mu 2 you can write as 1 minus mu 1 because mu 1 plus mu 2 is equal to G times m 1 plus m 2. This we have written as 1, so from here these quantities can be written.

So, therefore, our equation gets reduced to v square is equal to x square plus y square plus 2 times and m 1 we have written as, sorry, this m 1 we wrote as v times m 1 s mu 1. And then, this quantity we wrote as 1 minus mu star and so this quantity, then becomes m 2 is equal to mu star.

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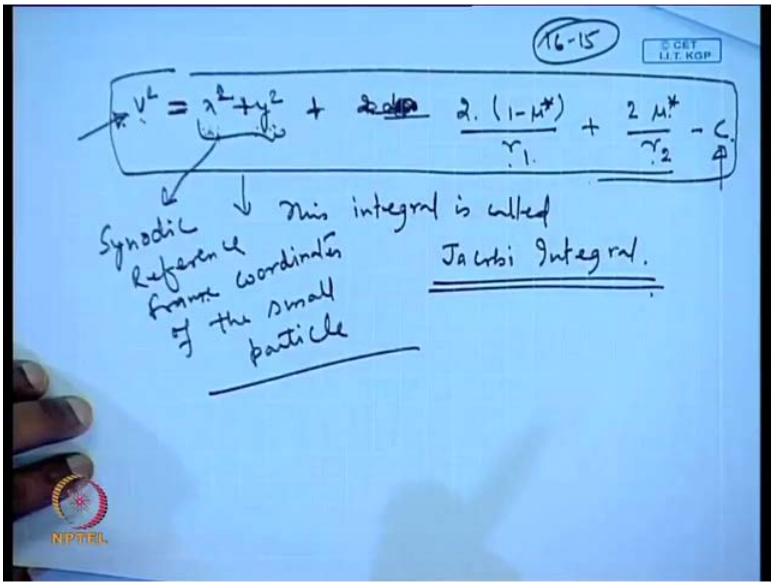

So, finally, we write here as this quantity 2 times 1 minus mu star by r 1 plus 2 times mu star by r 2 minus c. And this integral is called Jacobi integral. So, we can see that from a very complicated format, from a very complicated format we have got a very simple format, which is very useful. So, here v square, this is the velocity of the particle with respect to the synodic reference frame and x, by these are the coordinates of the particle with respect to the point that, we have chosen as the bary center. x and the, x and y, these are the coordinates of the particle with respect to the bary center, also because synodic reference frame is located in the same place. Therefore, x and y coordinate, now be converted into the synodic reference frame and therefore, this now referred to the synodic reference frame instead of bary centric reference frame. So, instead of bary centric, now we have the synodic reference frame coordinates of and here r 1 and r 2, these are the distances of the particle from the mass m 1 and m 2.

So, this equation, it will give us lot of insight in developing the Lagrange points showing how the third particle or the small particle it will behave with respect to these two primary bodies as we take different values for the c. So, but we before we do this we need to work out the equilibrium points.

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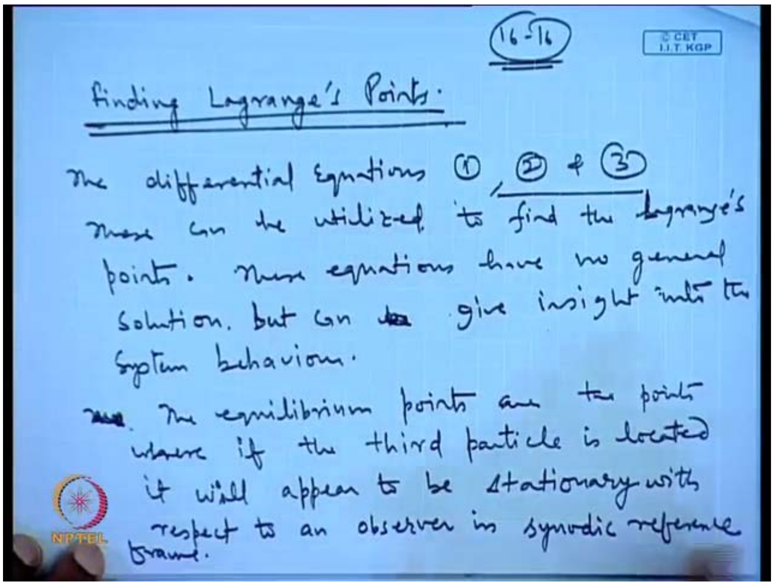

So, finding Lagrange's points, so the differential equation, the differential equations 1, 2 and 3, this can be utilized to find the Lagrange points, which are nothing, but the points of equilibrium. So, these equations have no general solution, but one useful format we have got into this, which will write as the equation number B. So, this, this equations have no general solution, but can be, but can give insight into the system behavior and one of the representation of this system behavior is given by this equation number B.

So, the equilibrium points, we can try to find out the equilibrium points. So, suppose in the synodic reference frame if we are looking for certain equilibrium points, which here in this case they will be also termed as the double points and what does mean by double point we will look later on, but right now we are just worried about what is, what are this equilibrium points.

So, if these equilibrium points we can develop by assuming, that say in the synodic reference frame, once we are taking so naturally in the synodic reference frame, if this particle is in equilibrium, so it must stay in the same point with respect to the primary and the secondary masses. So, that implies, that the acceleration and the velocity components of the particle will simply vanish and we utilize this fact to simplify the differential equations 1, 2 and 3.

So, the equilibrium points are the points where, if the 3rd particle is located, it will appear to be stationary with respect to an observer in synodic reference frame.

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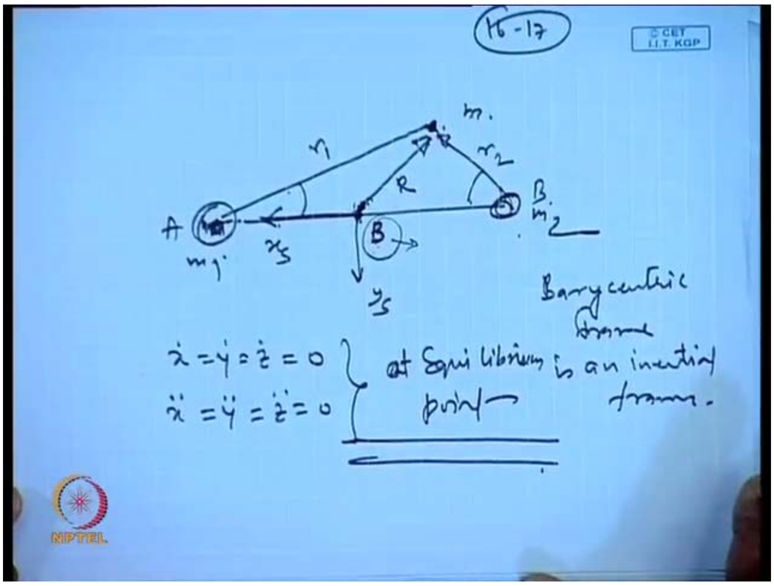

So, we will develop these equations, the equilibrium points mathematically in the next class, but today we can have at least, graphical picture of what exactly is happening. So, let us say, we have the mass m 1 and here is the mass m 2 and this is the synodic reference frame whose x s direction is here and y s direction is here. And a particle, which is lying in, suppose it is lying in this plane itself, so we have this mass m 1 and lying in this plane itself, and so this particle, let us suppose it is located here, which is the mass m. So, this implies, that the distance between the mass m 1 and this particle, which is r 1 and this distance, which is r 2 it remains constant and this angle will also remain constant because it does not appear to move from, if this is equilibrium point, so in this synodic reference frame it will appear to be the in the same point all the time and therefore, this configuration will remain intact for ever.

Now, instead of this in the synodic reference frame if we take the bary centric reference frame, then what is happening? So, in the bary centric reference frame you can see, that the person who is sitting at the bary centric reference frame, which is located, which is the center of mass of the particle m 1 and m 2. So, with respect to this, this synodic reference frame is moving. So, here, suppose this is the, r is the distance from the bary center of this mass m. So, about in the bary center, the person who is sitting at the bary center, that is, in the inertial reference frame, so in bary center is the bary centric frame. Bary centric frame is an inertial frame and therefore, the person who is sitting here and the whole thing, this, this line if joining m 1 and m 2, which is AB, this is moving at the angular speed of omega and therefore, this whole figure is rotating.

So, this particle will appear to move in a circular orbit to the person who is sitting in the bary centric reference frame at the point B. So, it will undergo along with m 1 and m 2, it will undergo a rotational motion and the bary centric reference frame, both the masses m 1 and m 2, they are also moving. So, simultaneously this is also moving along with them.

So, if this is just a graphical picture, so here next class we will see, that they are the points we have to find out. Basically, the points where this particular statement will be true, that x dot, y dot, z dot, this is equal to 0 and x double dot, y double dot, z double dot, this is also equal to 0. So at, at equilibrium, so rest of the things, rest of the equations workout, we continue in the next class.

Thank you very much.