Space Flight Mechanics

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Lecture No. # 15

Three Body Problem (contd.)

In the last lecture, we have been discussing about the three body problem. So, one of the major problem of this three body motion, what we saw that there this problem is absolutely not solvable even the two body problem, we saw that in the absolute sense the two body problem cannot be solved, only in some relative things, we were able to get some meaning full solution to that, but in the three body problem, obviously we were far away from any tangible solution, but if we do some assumption, when we make certain assumptions, then under that assumption it is a possible that the solution to three body problem can be obtained.

Say, in the case of the earth and the moon system, we can have a satellite,where which can be treated as a third body, so satellite can be treated as a mass of as a third body of negligible mass. So, under that assumption, if we start working, so it is a definitely we are going to get certain solution which are very intuitive, andit gives us a feeling how the third body is moving with respect to these two primary bodies. So, let us start with the various basic assumptions, which we make for the motion of the three bodies in this case.

So from this we will call as a circularly restricted three body problem. So, the first assumption is the primary and secondary bodies move in circular orbit. Circular orbit say about their center of mass so the benefit of this assumption that the orbit is circular, you can directly see it orbit is not circular, so obviously here omega of the primary body it will be changing with primary bodies will be changing with time. So, in the circular orbit what we have that the angular motion, say the angular velocity of the satellite if this is the radius r, v is the velocity. So, in a circular orbit, this v remains a constant therefore, omega is also a constant but, in elliptical orbit we get extra complication in the sense that here omega becomes of function of r again. So, these beside this assumption, the second assumption we make the third body the third body is very small as compared to the two larger bodies.

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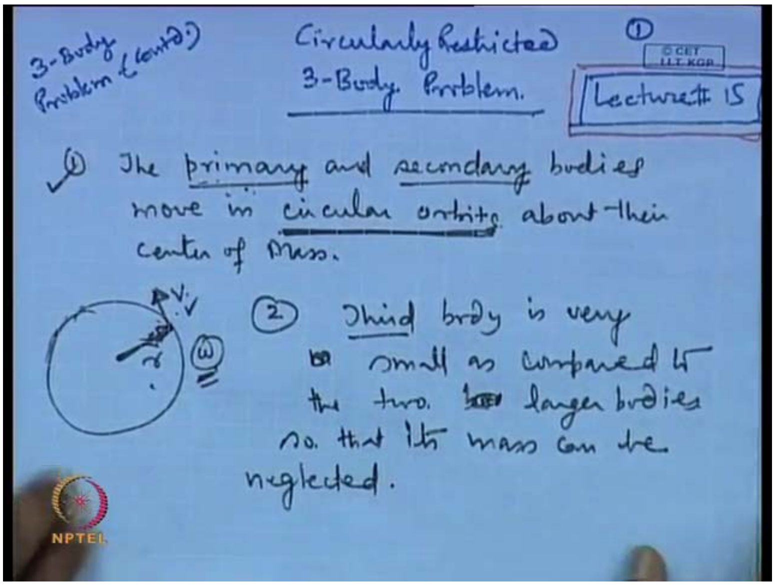

So here, the primary this is assumed to be quite large as compared to the secondary body and the third body, it will be assumed to be quite small as compared to this primary and the secondary body and so that it is mass can be neglected. So, mass can be neglected and the third assumption which is again which gives further restriction can be made or sometimes it is a made is that.

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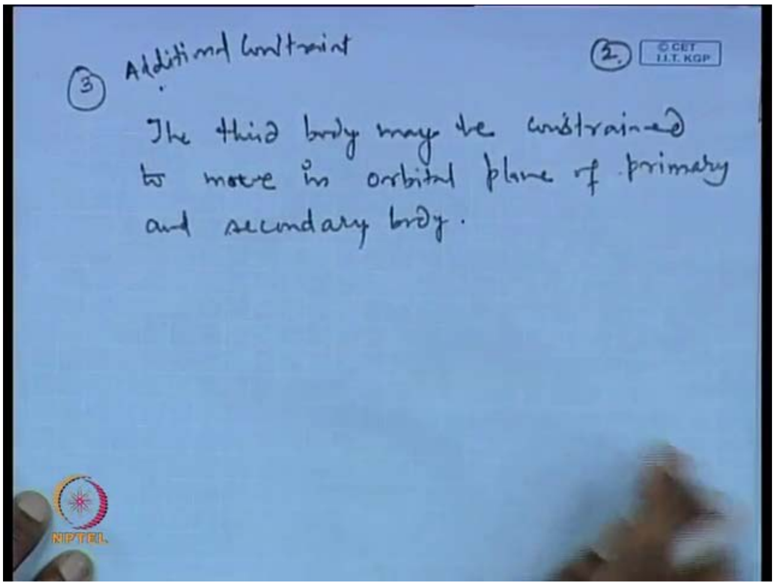

This is additional constraint, I will put this as a additional constraint only underthesetwo constraints we can work, butsometimes this may be available, so additional constraintso under this isthe third bodyconstrained to move in the orbital planarprimary and secondary body.

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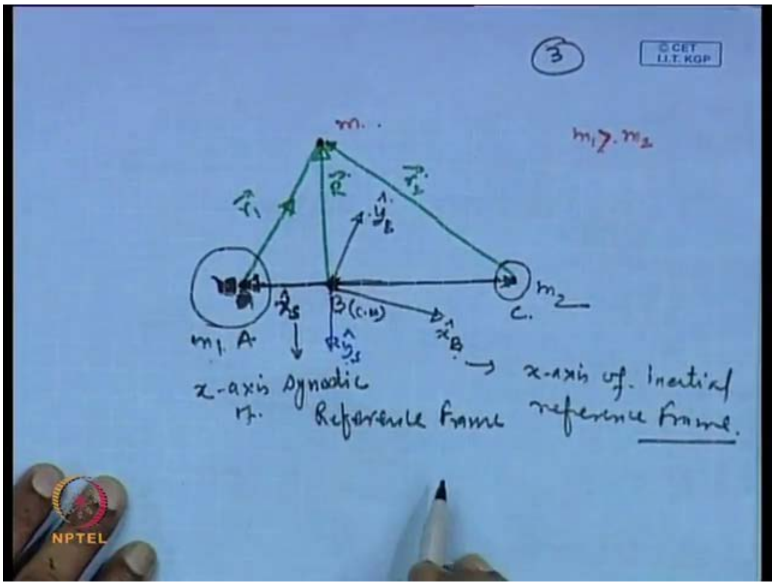

So, let us consider way of two masses; m1 and this is m2 the center of mass of these two masses, it lies at say B, we write as a bary center. So, both of them will be moving about the center of mass in a circular orbit, so this will describe a circular orbit of radius from here to here. This will continuously move similarly, m2 will also move but, they will always remain along the same line, it will never happen that this line the angle between, from here to here let us put here as a and this as c. So, that side a and this is c this is mass m1. So, both the masses they must lie along the same line, you can see yourself that, if they are not lying along the same line and bary center cannot be somewhere else they have to always over this same line and therefore, their period will also be the same.

Now, let us assume that we take a reference frame which is fixed to the bary center but, not rotating along with the satellites. So, this bary center the reference frame fixed at the bary center, this will fall as a inertial reference frame, because the bary center is, we know that, if the two particles are n number of particles they are free from the external forces,if they are free from the external forces then they are center of mass most with a constant velocity.Sotherefore, we can fix an inertial reference frame at the bary center.

So, in this case we take the bary center, let us sayas x B is the unit vector and this direction and then we havey B is the unit vector perpendicular to the x B and x B and y B, they are lying in the orbital plane of m1 and m2 therefore, z B it will be coming out of this plane. So, z B will be perpendicular to the plane of m1 and m2, the orbit of the m1 and m2.

Similarly, we here this is the center of mass similarly, we fix another reference frame, which will be rotating at the same rate as this line joining the mass m1 and m2 is rotating means, that will be the period of the reference frame, which is fixed at this point at the bary center, but it is a rotating along with the masses and it is a period will be same as the period of the period of these two masses. Therefore, we take the direction of this, let us say this is the x s comma the x s cap. So, this is the x s of the here the subscript s stands for the synodic this is called synodic reference frame. And this is our x body, x axis of the x axis of synodic reference frame and this is the x axis of inertial reference frame.

So, the y axis of the synodic reference frame it will be pointing downward,so here x s and this becomes y s caps, because this is the unit vector in the y direction for the synodic reference frame. So, the whole reference frame will be rotating along with the this to prime primary and the secondary bodies and then the z x which is the z x of the synodic reference frame, it is again coming out of the plane of the paper or the which ishere in this case, we are assuming to with a plane of this two masses it is a plane of the orbit of these two masses. Let us assume that, m is the mass of the satellite which is quite negligible as compared to the mass m1 and m2 and we are assuming here also m1 is greater than m2, r is the radius vector of the satellite in the inertial reference frame. Similarly, we can have this, we can write as r1 and this is as r2, so r1 and r2 are the radius vector of the satellite mass m, this is the third body from the mass m1 and from the mass m2.

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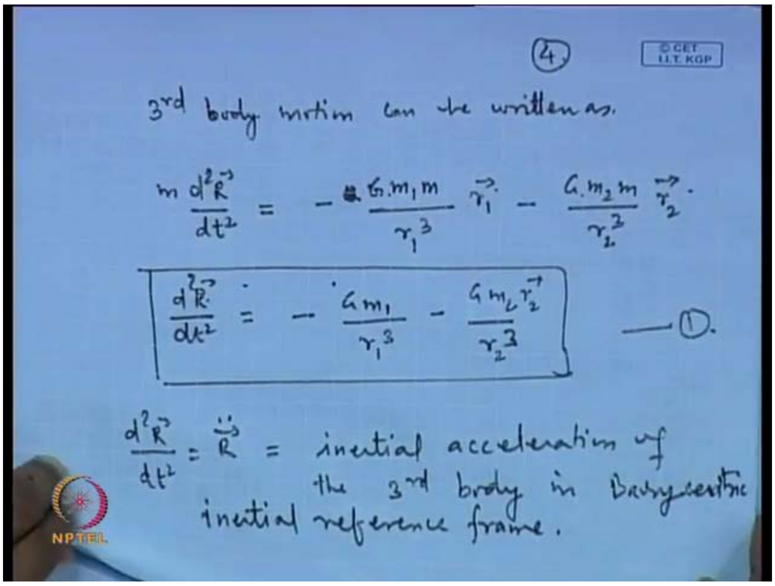

So, the equation of motion of the mass the body 3; so, the third body motion can be written as minus k times or G times m1 m divided by r 1 whole cube. So, our assumption that this mass third body mass is quite a small as compared to these two primary masses. So, that simplifies the situation that this two motion of this two masses can be certain quite easily as we have done earlier; for the two body case and the where the mass of the third body, because it is a very small, so it is a not at all affecting the motion of this two primary and the secondary bodies. Otherwise, if this mass becomes quite large, then the equally will affect the motion ofthe primary and the secondary mass and therefore, it would not be it would not be possible to solve this motion at all.

So, under this assumption we get d square R d t square, now here d square R d t square you can write this as R double dot this is nothing but, the inertial acceleration of the third body in bary centric inertial reference frame.

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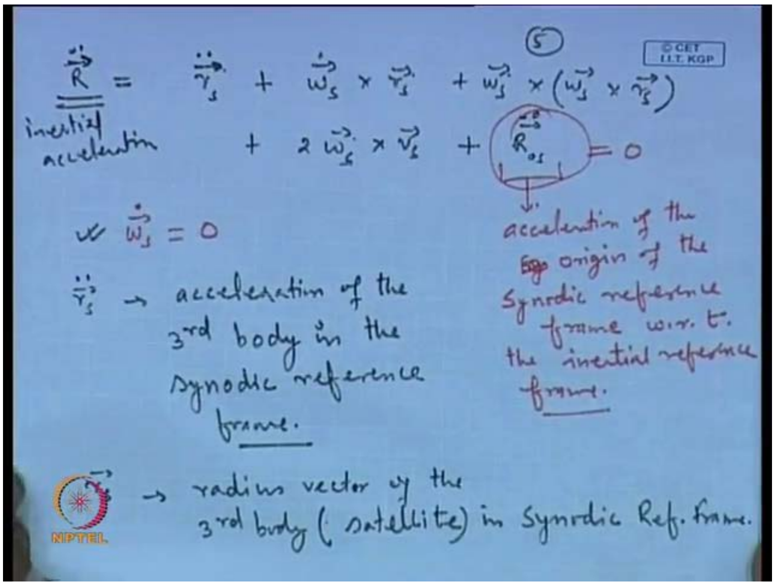

So,R double dot which is the inertial acceleration of the third body. So, this can be written in terms of the motion of the say, if here in this case this is our third body. So, this acceleration, we have got in terms of in the inertial reference frame but, we want to describe the motion of the third body with respect to the moving reference frame. So, what does it signify it signifies that our two masses are moving in this plane and with respect to this two masses means, the line joining this two, then how the motion of the third body can be described, one is this becomes again the relative motion this is not the motion in absolute sense.

So, this is the relative motion that, we are trying to ascertain so it is a very important that, if we have thethird bodymoving with respect to thistwo primaries bodies. So, what will be its location at any other time or in a any other time in the future with respect to these two primary and the secondary body,so it will give us insight say, I want to locate certain satellite with respect to two bodies, then how the satellite motion will get affected with respect to these two bodies. So, that must be known before hand, so that can be done if we assume that if we try to resolve the motion the inertial motion with, if we try toget from the inertial motion the motion with respect to the moving reference frame which is the synodic reference frame here.

So, we tried to convert the inertial motion in synodic reference frame, so for doing that we know a very simple kinematic relationship which we have done earlier in our earlier lecture. So, R double dot which is nothing, but the inertial acceleration this can be written assay r double dot s plus, so what exactly r double dot s is, I will come to know thatso here this part this is the acceleration of the origin of the synodic reference frame with respect to the inertial reference frame.So because the synodic reference frame is fixed at the bary center, so it is not having any transnational motion and therefore, this quantity becomes equal to 0.

Besides, omega s dot this quantity equal to 0, why this is so because we are assuming that the two primary and the two secondary bodies, these two bodies they are moving in a circular orbit and therefore, the omega s is the rate at which these two are moving about their center of mass and therefore, if we fix a reference frame at this point at the bary center and let it move at the same rate means the omega s in the reference frame will also move at the same rate. So therefore, the benefit of the circular orbit is that the here we are not getting any change in the angular rate ofthe synodic reference frame and therefore, omega s dot can beset to 0.

So, this two greatly simplifies our situation, now r s double dot this is the acceleration of the third body in the synodic reference frame in a small r s, this is the radius vector of the third body which here in this case is a satellite in synodic reference frame. So, we have omega s dot equal to 0 and R o s double dot this quantity also equal to 0.

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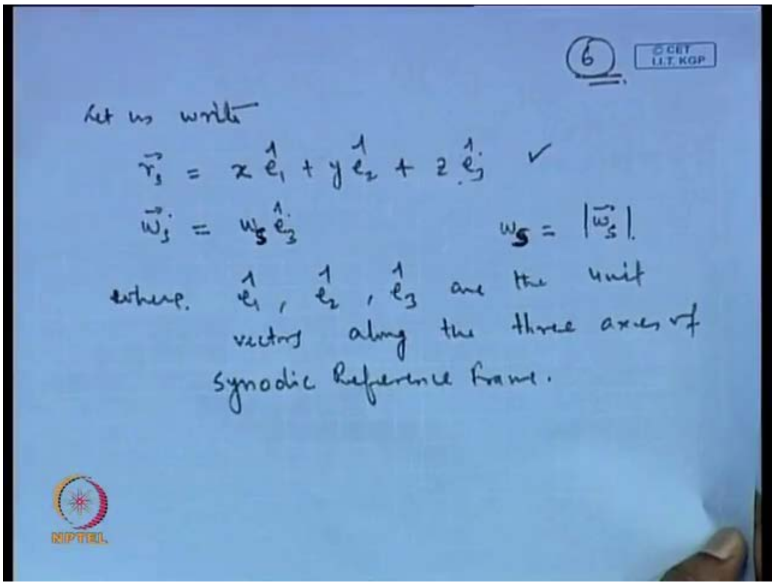

Now let us write, so here e1, e2 and e3 are the unit vector in the along the three axis of the synodic reference frame. So here, we have actually used the notations in this place this is the unit vector along the x direction and similarly, this is the unit vector along they direction, so this thing actually becomes e2 and this becomes nothing but, e1 cap and e3cap will be coming out of this. So, this is just for the age of notation writing this is much easier so we can follow this, so r s is equal to x times this x times e1 cap plus y times e2 cap and z times e2 cap this is a simple relationship. Similarly, omega s the angular velocity of the synodic reference frame it can be written as omega s times e3 cap. So, where omega3 is the magnitude of the omega s vector and e3 is the unit vector in the along the z axis of the synodic reference frame. Omega3 is nothing but, or omega s we can write as this as omega s along the three axis of synodic reference frame, now we can evaluate the r double dot.

Now, we can evaluate this is the inertial acceleration, so by inserting all this quantities here so for this omega is dot this becomes equal to 0. So, one by one we will be able to a certain all of them and therefore, we can express the inertial acceleration in terms of the components of the quantities which are given here.

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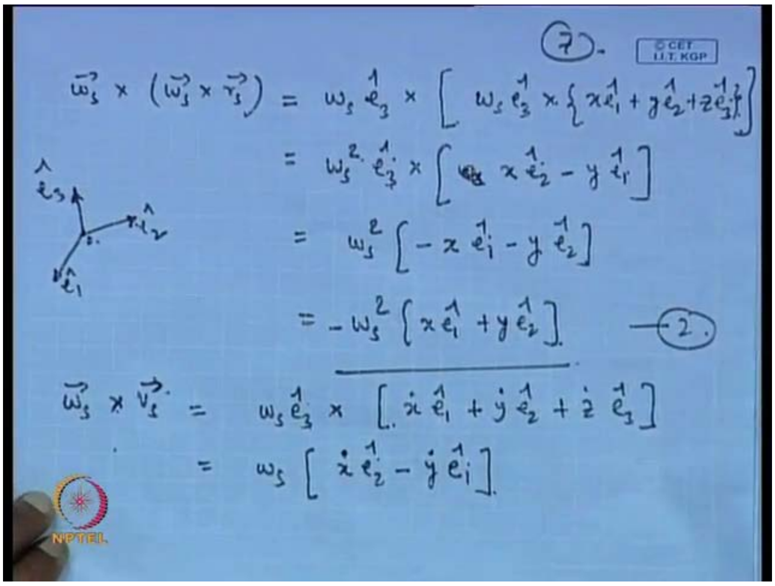

So, taking first omega s cross, one by one we work it out so we have e1, e2 and e3.So, e3cross e3, this is nothing, but e2. So, this becomes omega s, omega s we have already taken out so this is x times e2 then we have e3 cross e2, so e3 cross e2 this is minus e1and this is y times e1 and e3 cross e3 this becomes0. Now, again working out this, so e3 cross e2, e3 cross e2, this is minus e1 so x times e1cap and e3 cross e1 is nothing but, e2 so this is minus y times e2 capthis is our equation number2. Also, we can estimate here itself omega crossv s so v s is the velocity of the satellite for the velocity of the third body with respect to the synodic reference frame this become omega s e3 cap cross e3 cross e1 this is e2 so x dot e2 cap e3 cross e2 is minus y dot e1 cap and e3 cross e3 is 0.

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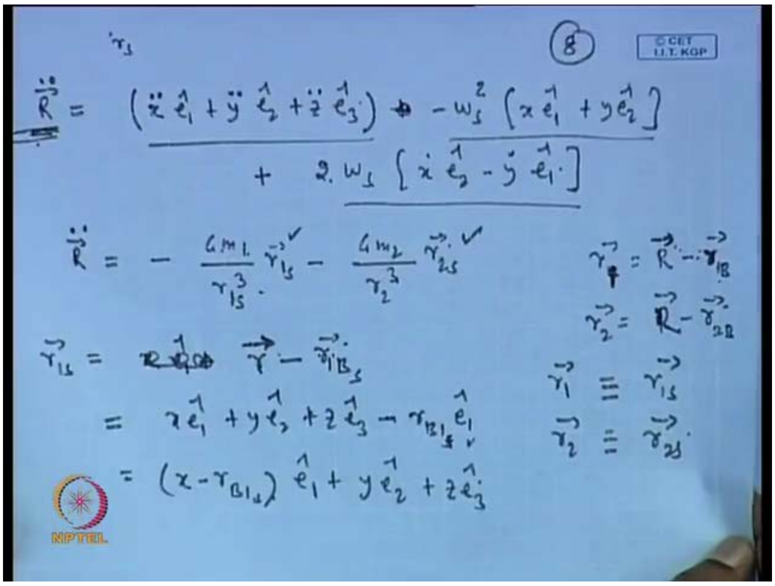

Therefore, R double dot, this can be written as, so r s double dot. Now, the quantity we have shown here r s double dot this is a acceleration of the third body in the synodic reference frame. So this will be nothing but, x double dot e1 plus y double dot e 2 plus z double dot e 3 cap and rest put other things so omega cross r s this quantity, we have just now determined this quantity is minus omega s square x e 1 cap plus y e 2 cap and then the third quantity which is remaining is two times omega cross v s, so this becomes two times omega s times.So, this is the inertial acceleration which are expressed in terms of the acceleration along the body, this synodic reference frame components this are the acceleration with respect to the synodic reference frame of the third body and these are raising because of the rotation of the synodic reference frame itself. So, if we can recognize these acceleration this acceleration is nothing but, the coriollis term.

While this term you can visualize as the, what you call as the centripetal acceleration, so it is a very easy to work it out and look into this so whenever some particles for somebody is moving with respect to a rotating reference frame to it fills some the what is called the coriolis force, so the this is expressed here this is the coriollis term this is the centripetal term coriolis. So, also R double dot this we can write as we have earlier written as minus G times m1 by r1 whole cube. Now, if we look into the original figure this figure s or 1 is a in this equation r1 is basically this is with respect to the inertial reference frame means these are taken in bary centric reference frame. But if we look into the this quantity and our reference frame which is a synodic reference frame this is rotating at omega, but what happens either we take the inertial reference frame or the synodic reference frame these are the invariant quantities, they do not differonly the components along their body axis will change but, as you hold this quantity remains invariant.

So therefore, we can express this r the r 1 and this r 1 and r 2 in the synodic reference frame itself and therefore, we can compute the corresponding sum, so what exactly we are doing so we know the acceleration here in the synodic reference frame and the left hand side is the inertial reference frame acceleration in the inertial reference frame. So, inertial reference frame components can be broken along the synodic reference frame and that makes us easier to put thisall these expression in a compact for. So, following this we can put a subscript here to indicate this these are in the synodic reference frame because these are the invariant quantitiesso we have r 1 s which is nothing but, the r vector minus the mass of the or we write it as r 1 B which is the radius vector of the this quantity we can write as r 1 B this is the vector from here to here.

So to the mass1 this is the vector r 1 B so r 1 becomes r minus r 1 B similarly, we here we can write this as r 2 B so r 2 that can be written as so we can write here as r 1 equal to r minus r 1 B and r 2 is equal to r minus r 2 B. Now, more over a in the same notation will also follow in the quantity r 1 and r 2 they are invariant so these are equivalent to r 1 s and r 2 to r 2 s. So, we can replace in terms of this here in this quantities, so once we replace it so this expression will become further simplified. So,r 1 s r 1 s then we can write as x e 1 means, then we are expressing in terms of the synodic reference frame so r minus where r is the radius vector of the third body with respect to the synodic reference frame. So, this becomes r minus r 1 b in the synodic reference frame.

So, this we can then express as x times e 1 cap plus y timese 2 cap z times e 3 cap minus r b1 s and because this is in the synodic reference frame this is directed along the x axis.

So here, we will have this as e 1 cap and this isyou can see here this is a directed along the synodicx body x therefore, whatever the magnitude of this vector is this and this is multiplied by the unit vector. This again we can bring together and write them as e 1 plus y e 2 plus z e 3.

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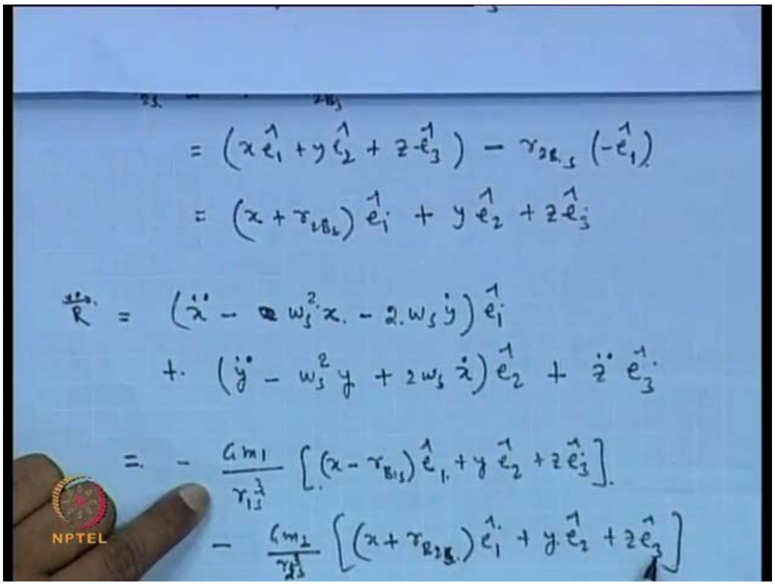

Similarly, we can write for the r 2 s also,so r 2 s this can be written as r minus r 2 b s which is x e 1 plus y e 2 plus z e 3 and minus r b2 s and this is directed along the r 2 b,r 2 b is here just directed opposite to the even direction it is here and therefore, we can write this as and e 1 is positive in this direction so minus e 1 cap so this becomes x plus r 2 b s e 1 cap plus y e 2 plus z e 3. So, once we have written in this fashion then we can assemble all the things together look into this equation R double dot here the e 1 term can be brought together e 2 terms can be brought together and e 3 will be left alone and we can simplify this equation.

So R double dot, now this can be written as x double dot minus x times or omega s square x minus 2 omega s y dot this times e 1 cap plus y double dot minus omega s square y plus 2 omega s omega s times x dot this is e 2 cap and lastly the z 3. The z component it can be brought. So, once we have expressed this so this will be equal to nothing but, G times m1 r 1 s whole cube and then r 1 s vector we have to bring in here so that is x minus r b1 s z e 3 cap minus G m2 divided by r r 2 s cube x plus r b2 s x plus r b2 s times e 1 plus y e 2 cap plus z e 3 cap. Now, you can see that on the left hand side all the terms are grouped together with e 1e 2 cap and e 3 cap similarly, on the right hand side we can bring the terms together with the e 1 cap e 2 cap and e 3 cap and compare them so we get three second order differential equation. So this three second order differential equation they describe the relative motion of the third body with respect to these two primary and the secondary bodies.

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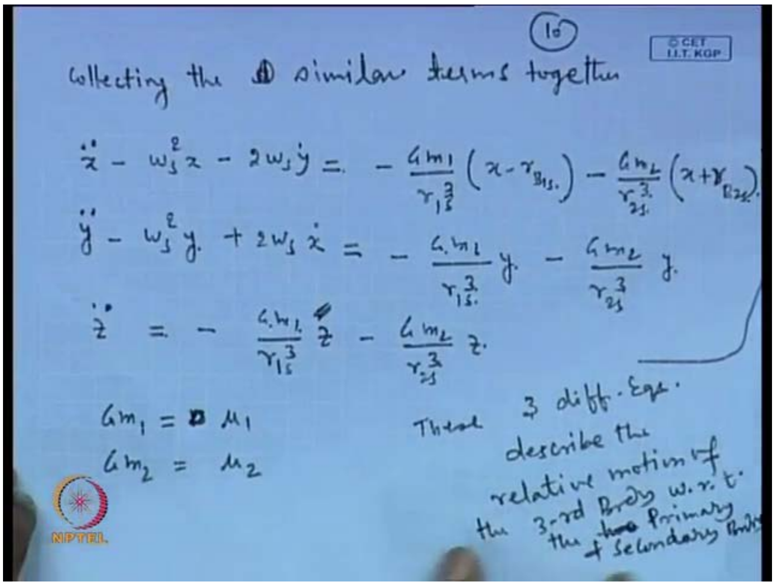

So, collecting the similar terms together we can write x double dot minus2 omega s y dot this will be equal to minus G m1r 1 s whole cube B1 s minus G m2 by r 2 s cubex plus r b2 s. Similarly, the other 2 equations can be written, so the other 2 equations then for the y motion y double dot minus omega s square x y plus2 omega s x dot, this will be equal to minus G m1r 1 s whole cube y minus G m2r 2 s whole cube. And lastly, the z double dot this can be written as minus G m1 byr 1 s whole cube z minus G m2r 2 s whole cube times z, for simplification we can assume g m1 is equal to let us say mu 1 and G m2 is equal to mu 2. Now, these are the three differential equations which describe the these three differential equations, describe the relative motion of third body with respect to the two primary and secondary bodies with respect to the primary and secondary bodies. Now, to solve this equations so let us term this equation as a this as b and this as c.

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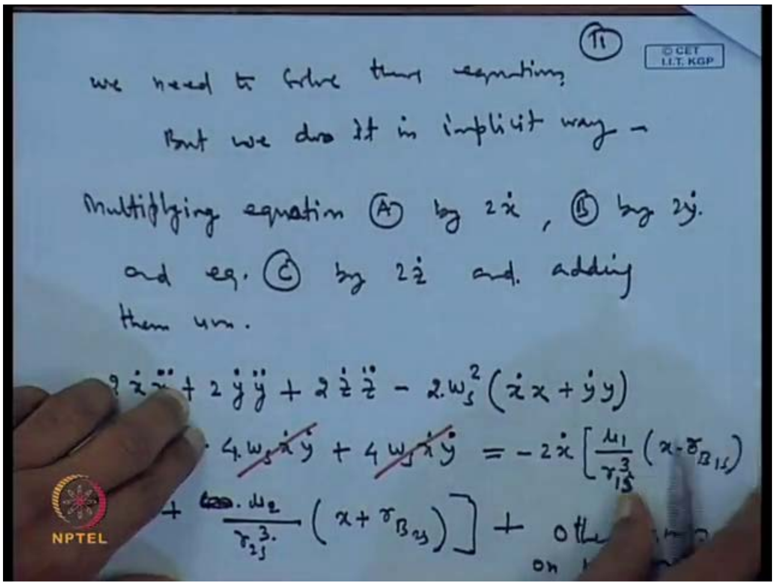

To solve this equations, butwe do it in a implicit way, but we do it in implicit way and this we achieved by multiplying equations A,B and C. These are the equations A,B and C, and this we multiply by2 x dot2 y dot and2 z dot respectively, and add them up. So, multiplying equations A by2 x dot B by2 y dot and equation C by2 z dot and adding them up. So, you can see that the terms that we get this will be 2 x dot times x double dot which is coming from this equation. Similarly, from the equation we will get the term 2 y dot times2 y double dot and from the equation c, we get2 z dot times2 z double dot. Now, next we take this two terms; omega x, omega s square x and omega s square y and here,we do not gave any other term, so only one term is present in the equation number c.

So, next we get two times omega s square andx dot times x plus y dot times y and thereafter, we have this two terms and this two terms are of opposite sign and this, we are multiplying by x dot and this by y dot, so they will cancel out each other but, still I write here in this place, so we will have two into two, this is four and this is four times omega s times x dot y dot and plus four times omega s times x dot y dot. So, these two terms will cancel out. On the right hand side, we will have minus 2x dot times rest other terms from here, we have to multiply by x dot2 x dot2 y dot and2 z dot and copy it here in this place. So, G m1 we are writing as mu1, so2 x dot mu1 by r 1 s cube times x minus r b1 s next this term we have to pick up so this equation is bit long.

But we have to write it here, so this is G m, so G m2 we replace as mu m2. So, this is mu m2 divided by r 2 s whole cube andx plus r b2 s x plus r b2 s this term the term which is present here this is appearing andwe close the bracket here. So, we will have to take to the next page plus other terms other terms on next page. So, left hand side we cancel out these two terms.

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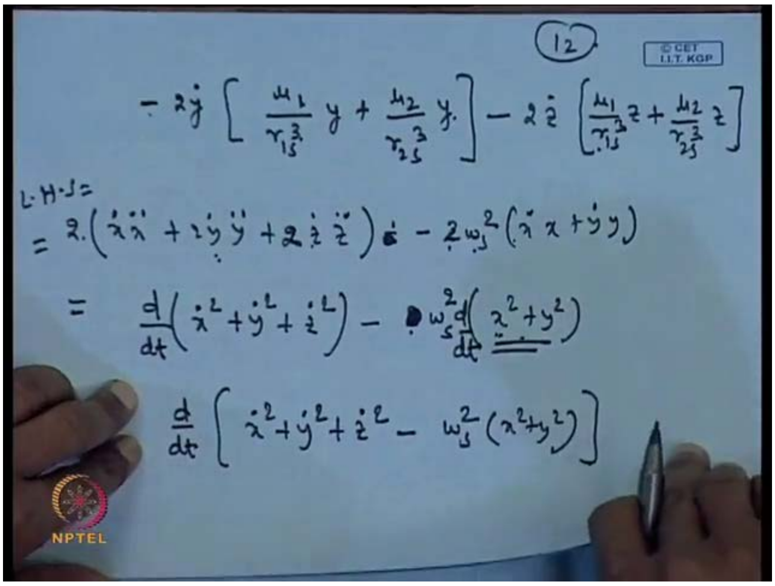

Here, so the other remaining terms are, then these terms is to be multiplied by 2 y dot. So, minus2 y dot times mu1 G m1, we are writing as mu 1 by r 1 s cube plus y this term becomes mu 2by r 2 s whole cube and this is also y minus sign has gone outside in. Here, in this place and then the last term is remaining here which is again minus sign, we can take it out side and this is 2 z dot multiplied by this is mu 1 and this is mu 2. So, accordingly we write here mu 1 by r 1 s whole cube and then multiplied by z plus mu 2 by r 2 s whole cube multiplied by z.so after completing this, then the whole equation canbe written as two times the L.H.S can be written as, two times x dot times x double dot plus two times y dot times y double dot plus two times z dot times z double dot equal to plus the other terms we have minus 2 times omega s square times x dot x plus y dot y this the we are writing the L.H.S, L.H.S equal to this.

So, this becomes the first term becomes d by d t x dot square plus y dot square plus z dot squareif you differentiate you get this quantity andthese term similarly, we can write here 2 omega s square times x square plus y square and this is d by d t sod by d t here we write so this is your L.H.S. So, this gets reduced to d by d t x dot square y dot square plus z dot square minus.The factor two which is appearing here, this factor will not be there, because once we differentiate this quantity here. So, two will be appearing so two is that two gets accounted here, so another way if we are writing it in this format this two gets absorbed here this side similarly, this two has got absorbed here in this side. So, this we get omega s square and plus x square plus y square.

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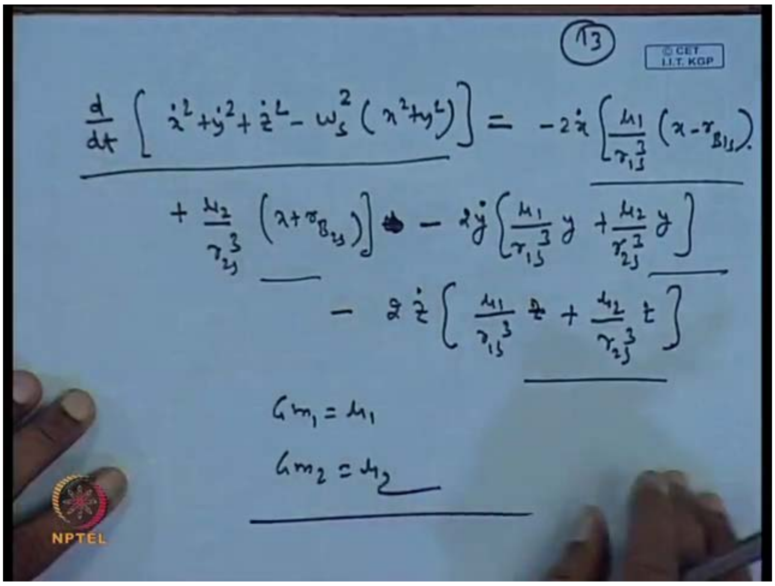

So finally, our equation gets reduced to the format d by d t, x dot square plus y dot square plus z dot square minus omega s square times x square plus y square this is equal to the all other terms. Now, we have to mention in this place, so this is equal to minus2 x dotx minus r b1 s plus mu 2 time r 2 s whole cube x plus r b2 s bracket close. Plus this two terms here, so this is this is starting with minus sign, so we put here minus sign2 y dot minus2 z dot times mu 1 by r 1 s whole cube times z plus mu 2 by r 2 s whole cube times z. So, and where we have written G m1 is equal to mu 1 and G m2 is equal to mu 2 so we have got equation in this format, but still it is not complete we need to integrate it.So, we will do that integration in the next lecture, time is getting over. So thank you very much we continue in the next lecture with the remaining portion.