Space Flight Mechanics
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Lecture No. # 14
Three Body Problem (Contd.)

So, last time we have been working with the three body system. So, instead of considering the three body system, we considered a generalized form of this as of a n body system or say n number of particle; heavy particles are there, and they are moving mutually under; they are moving under their mutual gravitational attraction. So what will be the general properties of this motion. So, already we have seen that in the case of the two body system; we had the total energy of the system was constant, then total momentum of the system was constant were, and besides the system centre of mass it was moving with the constant velocity.

Because it was it is moving under their mutual attraction, it is a free from the external forces. But in the two body system of one should take care of particularly, this point that total energy that was derived; it is not the really the total energy. Once, we are describing the relative motion; so, in the relative motion the v square; that appears, it is the relative velocity, and therefore it does not indicate the absolute velocity or the absolute energy. So, the energy obtained in that manner, it is not really the total energy or the velocity, the kinetic energy plus potential energy, but it was indicating the total energy, in some in the relative sense; it is a, because we are using the relative distance as relative velocity. And therefore, we get the same kind of thing as we get in the case of the absolute term, but those terms replaced with the relative velocity and the relative distance.

And therefore, it does not really indicate the total energy of the system, the total absolute energy of the system. So, we should be very careful about this. So, in this context; we are start with the n body's problem, and because of the three body problem, it will just be a particular case of the n body problem. So, after this we will go into the restricted three body problem, where we try to solve the three body motion, assuming that two of the masses are heavy, and one is quite negligible as compared to these two masses just as we have done in the case of the two body system. If our central mass is very heavy as compared to the smaller mass, then it virtually gets reduced to the central force motion; what we call the1 body problem,

So, the same notation we will follow here the same approach, we will apply here, and work out for the three body problem. And, we will have a lot of insight into the problem; how the motion is taking place, and how the system will evolve; weather, the system will be stable or not, many things can be derived. So, within the limits of the number of lecture, we will try to cover as much as possible.

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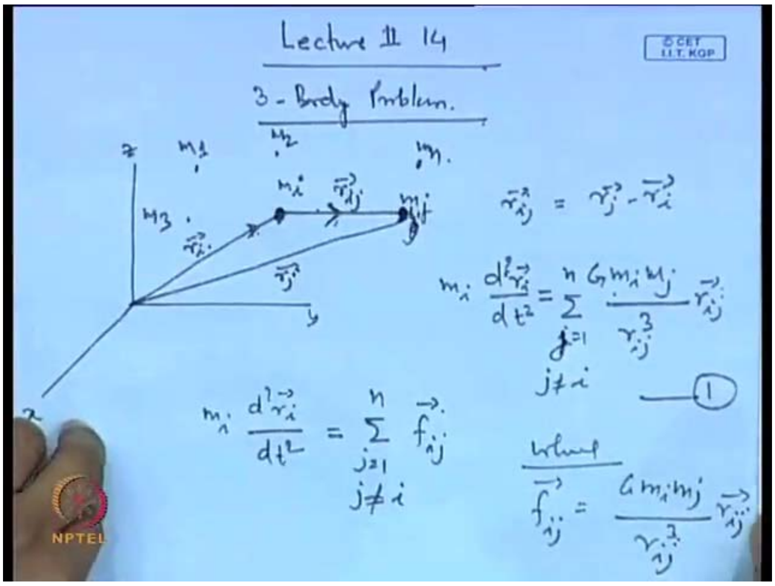

So, starting with our initial reference frame x. y, and z, so, we have a number of particles; heavy particles available 1, 2, say 3 then here i t h particle, and here j t h particle, then the n t h particle is here. So, this is m i, m n, m j, m 3. So the vector connect; there is a vector connecting this, and the vector for the i th mass and the j th mass they are referred as r i, and r j, and the vector connecting this we write as; r ij. So, r ij is nothing but r j minus r i. So, the motion of the i th particle; this can be written as in the inertial reference frame of course, d t square; this will be the force acting on the i th particle. So, the force acting on the i th particle due to the j th particle. So, this will be m i times m j multiplied by G and divided by r ij cube r ij, and direction of r ij is the same as the force is acting therefore, here the positive sign is coming. Now, what will be the effect, if we take all the particles together, so, we sum over all the particles; i is equal to1 to n or j is equal to1 to n and j not equal to i.

So, here the particle m j is here, so, if we take all the particles, and so all the particles will be attracting this i th particle towards itself; and this is the equation of motion that we are getting. Therefore, we can write m i times d square r i by d t square this is equal to f ij summation j is equal to 1 to n j is not equal to I, where f ij, this quantity is nothing but G times m i m j by r ij whole cube r ij. So, this is our equation number 2. Now, if we take the summation of on the both side; that is assuming that the, we are summing the this is the force acting on the all the particles so if we sum it up for all the particles.

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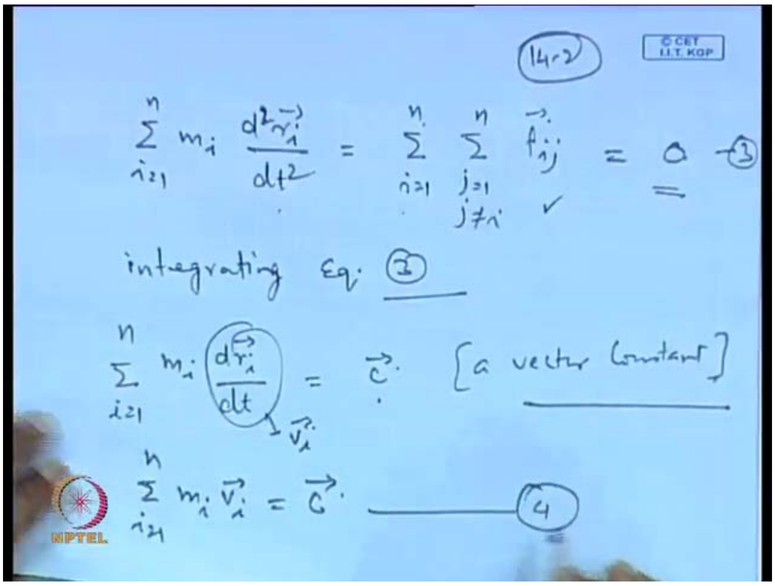

So, 14 2 is your; now, taking the summation i is equal to 1 to n d t square, this becomes i is equal to 1 to n and j is equal to 1 to n and j not equal to i f ij; and this quantity will turn out to be 0. Because the forces they adjust in pair like f ij, we indicate that f ij is the force acting on this, so, this is the f ij; which is directed from here to here. So, f ji will be of force present it will be directed from here to here. So, this pair will; this will exist in pair, so, once it adds it up, so the corresponding pair will cancel out. And therefore, we get the summation here 0, you can just expand for 2, taking 3 particle, and then check it; this will turn out to be 0. Therefore, we can integrate this; the integrating; this is our equation number 3, integrating equation 3, we get m i times d r i by d t this is equal to c; a constant; a vector constant. And, if you look into this; this m i times v i this d r i by d t is nothing but velocity, this is the velocity; this point is nothing but v i. So, this gives the total linear; this whole summation will give the total linear momentum of all the particles

So, if the summation of what it says that the linear of momentum of all the particles taken together; it is a constant, which is equal to C. So, we have basically m i times v i i is equal to1 to n this is equal to C. This is our equation number 4.

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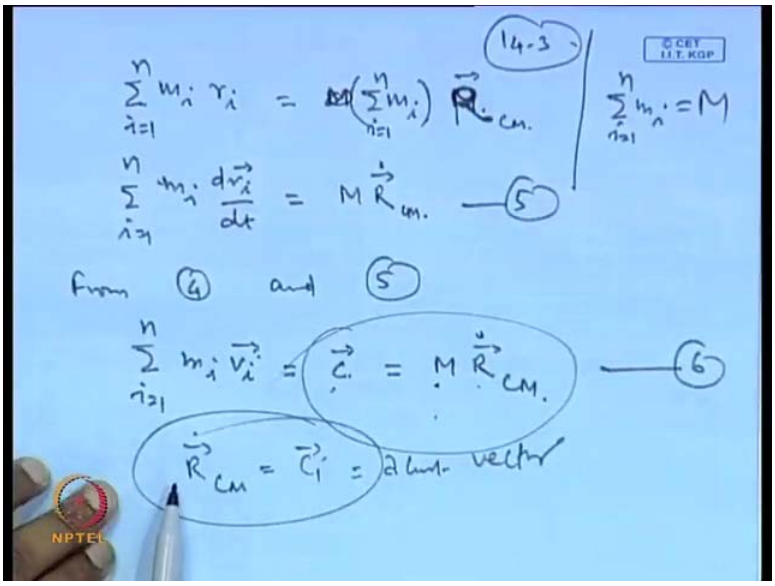

And already, we also know that, the center of mass of the n number of particles; this will be given by summation m i i is equal to1 to n times R, where R is the centre of mass; this is the R vector for centre of mass. Therefore, if we differentiate this quantity here, so, this will appear as d r i by d t summation i is equal to 1 to n, and this quantity is constant therefore, we can write here total m i i is equal to 1 to n is equal to capital M. Therefore, this becomes M R dot CM. So, from 4 and 5, we see that m i times v i summation i is equal to 1 to n this is equal to constant C, so this is nothing but m times R dot C M. So, this implies that, because here M is constant, and therefore, in this part if M is constant means R dot; and the left hand side is constant vector; and therefore, R dot C M; this becomes a constant vector. Therefore, R dot C M; this is a constant vector let us say C 1; a constant vector.

So, this simply implies that the centre of mass of the n body system, it moves with a constant velocity. So, this same thing is true for the three body system. The centre of mass of the three body it will move with a constant velocity. Once, the system is free from the external forces, so, it will not have any acceleration. So centre of mass will not have any acceleration. So, either three body or two body this is valid or either n body.

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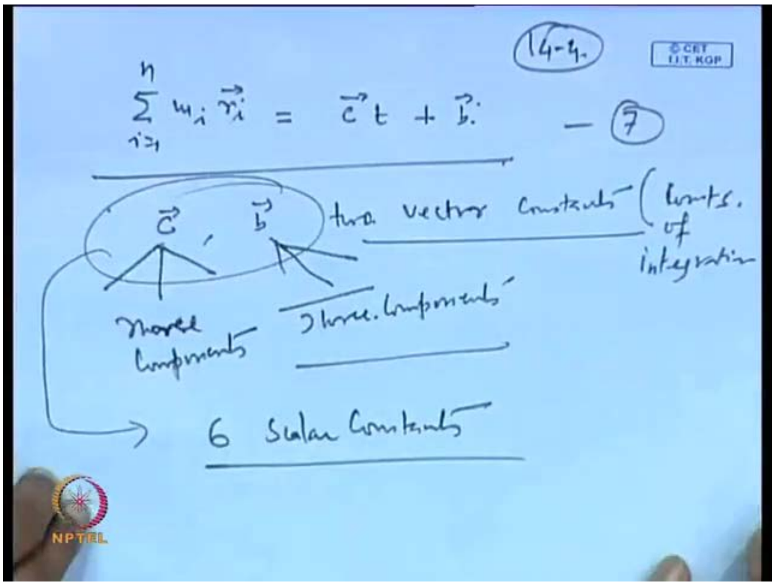

And, further integrating it we can write m i times C t plus b. So, here we have two constants of integration C and b; these are the two vector constants; constants of integration. So, each of the vector will have three components, this vector is also having three components, so, this gives us all together a total of 6 scalar constants. So, thus we get 6 scalar constants here.

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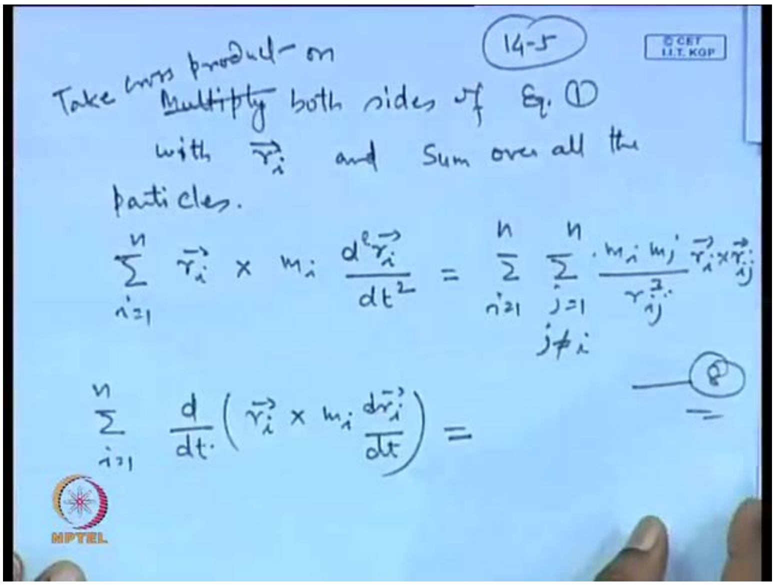

Now, from if we take the equation 1, multiply both sides; so multiply both sides of equation 1 with r I, multiply means take the cross product; take cross product on both side of equation 1 with r i, and sum over all the particles. So, if you do this; so, we will have r i cross, it is the same treatment as we have done for the two body, only thing little bit mathematical development it differs i is equal to 1 to n, and on the right hand side we will have i is equal to 1 to n j is equal to1 to n j not equal to i. So, this is our equation number 8. As we can check, we have done earlier for the left hand side; this can be written as d by d t r i cross m i d r i by d t; this we have done several times, so, you can check yourself. Right hand side, we have to do little bit work here.

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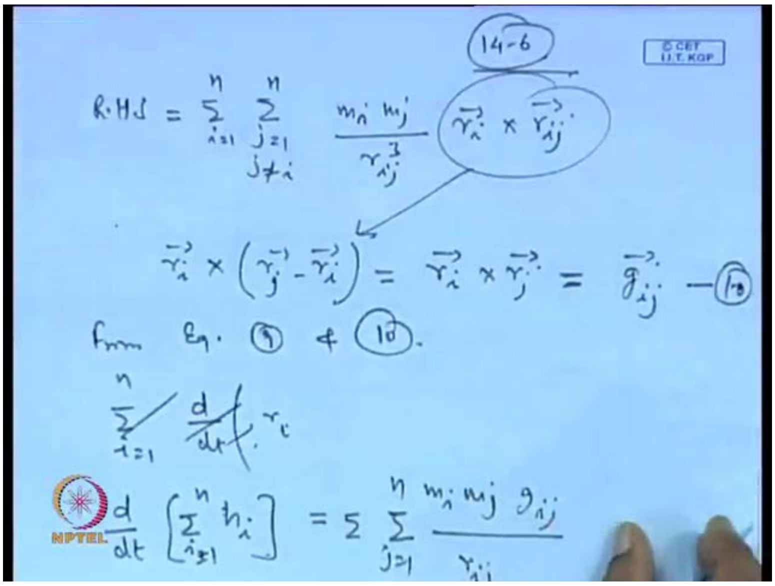

So, taking the R.H.S, now here this quantity; this is nothing but r i cross r j minus r i. So this will get reduced to r i cross r j. So, this let us write it as g j i. So, this is our equation number; this is equation 9 you can complete it here; double summation i is equal to 1 to n j is equal to1 to n j not equal to i m i m j r i cross r i j divided by r i j whole cube. So, this is our equation number 9, and this is say equation number 10. So, from equation 9, and 10 we can write, and if you look into the equation number 9; so, the quantity which is present here this you can write as h i this is nothing but the angular momentum of the i th particle. So, if we sum the all the angular momentum of the particle. So, the take the differential operator outside, and this is the summation operator inside. So, the same thing here we can write as d by d t taking the differential operator inside this becomes summation i is equal to1 to n. And from here we can write it as m i m j times g i j j is equal to 1 to n i is equal to 1 to n and j not equal to i.

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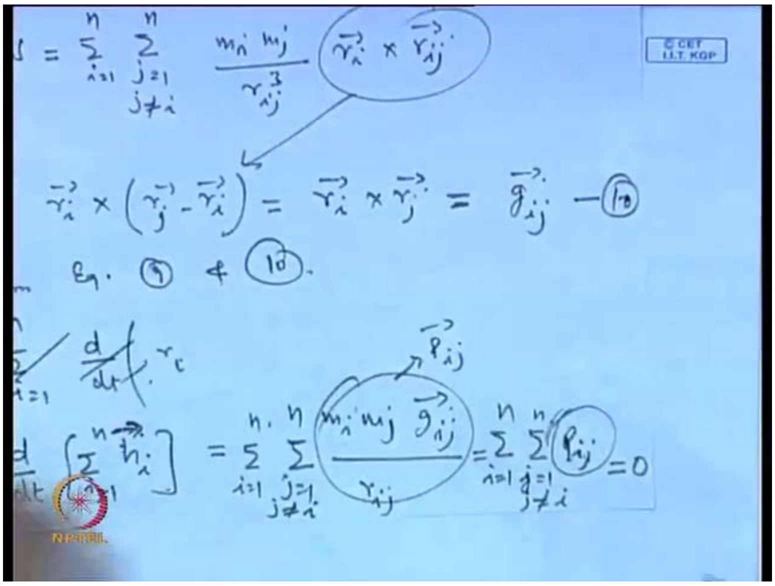

So, if you look into this particular part, so, again the same logic applies here; here this is h i is vector sign is there, so, the same logic applies here; this will exist in pair here; this is we are considering for the i th particle. So, the i th particle; this is the for the i th particle, this is the whole term we are getting. So, we consider the j th particle; for the j th particle also the same term will exist, but this will just differ in sign. Therefore, let us say, this whole term we can write this as something p i j. So, this becomes the; this will be summation over p i j and j is equal to1 to n and i equal to1 to n j not equal to i. and, this existing; because this existing in pair therefore, this quantity is equal to 0.

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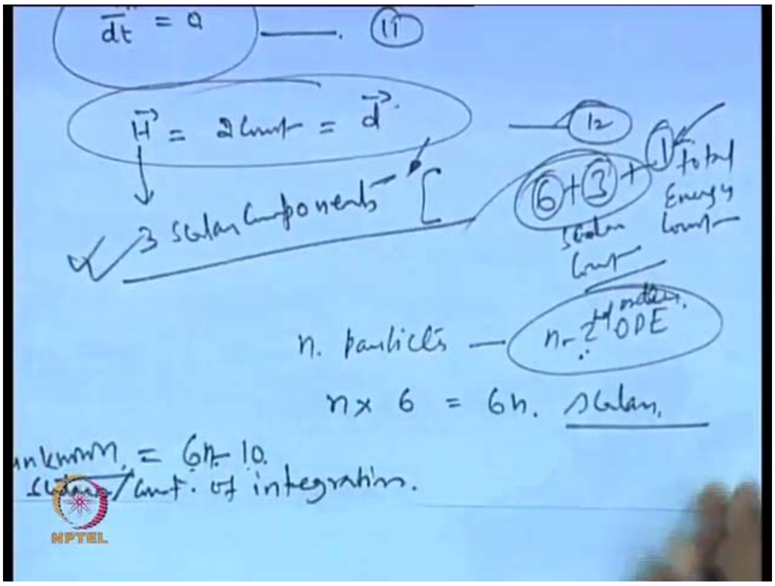

So, ultimately what we get d by d t and summation this h i i equal to 1 to n this we can write as H. So, what we get d H by d t this quantity equal to 0. This is our equation number 11. This is nothing but what it are states that the rate of change of the angular momenta of all the particles is 0. This is the capital H; is the total angular momenta. This is of angular momentum is sum of the angular momentum of all the particles. So, it gives us the angular momentum of the system. So, here from here we can say that H dot; H is a constant. Let us say, this is vector d means the angular momenta of the whole system; it will point in a particular direction in the space. So, this will give us 3 components; 3 scalar components; this gives us 3 scalar components.

And, this vector will be pointing in a particular direction in the initials space. So, this is constant of integration of the motion or the equation of the motion we have been trying to find out its general properties. So, this gives us 3 constants. So already we got 6 constants for the linear; telling that the centre of mass of the system it moves with a constant velocity. So, there we got a total of 6 numbers of scalar constants, and here we are getting again 3 scalar constants. So, till together; we have tilled now 6 plus 3 scalar constants. And one scalar constant we will get for the total energy of the system it remains constant as usual we have got our earlier so what it implies that all together we are just going to get a total of 10 number of scalar constant so we would not be able to identify more than this number of constant while for considering that we have n number of particles.

So, for n number of particles; we can write n number of second order ordinary differential equation; second order ordinary differential equation. So, how many scalar constants will then enter here. So, we have n particles for each particle we get 6 scalar constants, because it is a second order differential equation. So, 6 into n; this is 6 n is scalar constants are involved, so, out of that we will be able to identify only 6 minus 10; 6 n minus 10. So, these are only 10; we are able to identify, the remaining are unknown; unknown are 6 n minus 10 and 6 plus 3, and 1 will come from; plus 1 comes from; this 1 from the total energy; total energy constant, which will derive soon.

So, this 1; this 6; and this 3; this is total making it 10, so, 6 n minus 10. So, these are the unknown scalar; scalars or constants of integration; constants of integration. Therefore, it is not possible to solve the equation of motion in absolute sense for either n number of particles or 3 and even for 2. We have seen that in absolute sense, we are not solving it, just we are looking it into the relative motion or the once we are shifting the center of mass to the centre of mass of the whole system to the primary body; primary body; centre of mass assuming that the particle, which is revolving it it is very small. Therefore, the centre of mass will coincide with the centre of mass of the primary body. So, in that case, we got the solution, and solution was very easy and in fact to derive.

So but, as we go up for 2; for 2 body, absolute sense we are not doing for 3 body in even for the relative sense, it is not possible to work it out. But for few restricted cases, where we impose certain more constants on the system; and therefore we will be able to solve for those cases but, not in general cases.

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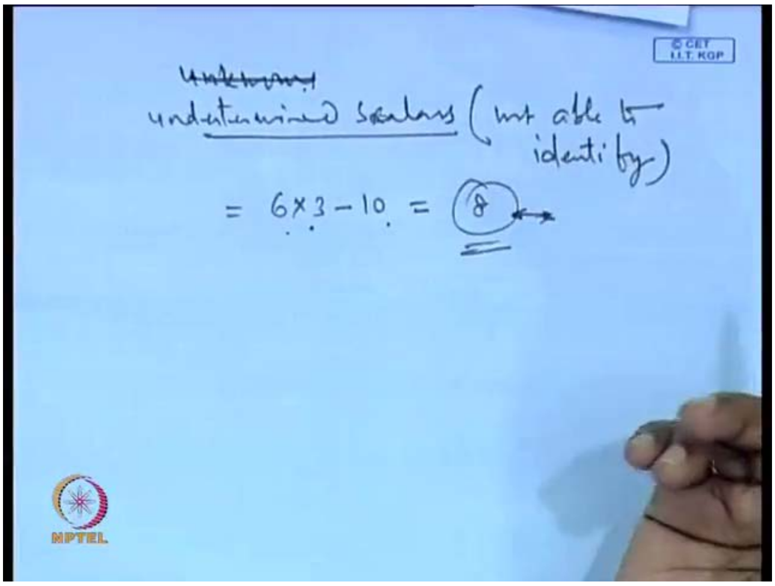

Because for 3 body system, we can see that, we get the total unknowns; the total unknowns are saying, the undetermined, undetermined scalars are which we are not able to identify, so, not able to identify. This is for the 3 particle system; this is 6 into 3 minus 10, so this equal to 8. So, in the case of 2 body system we had this 6 into 2 minus 10, so, we were getting only 2, but here the undetermined scalars are the scalars, which we are not able to identify the constant of integration, which we are not able to identify; it mounts up to 8. So, as we increase the order of the system means; as the number of particles we keep adding to the system that the more it becomes complex, and it remains unsolvable.

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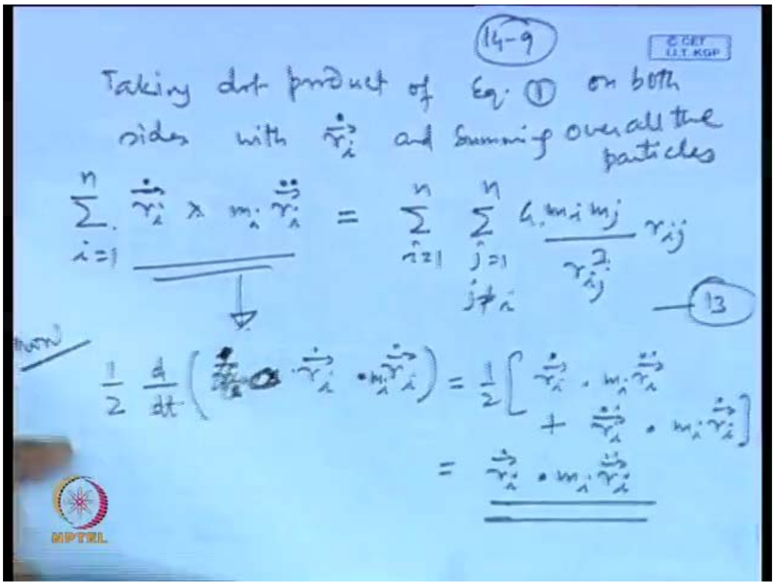

So, now if we check the scalar product; dot product of; so, taking dot product of equation 1 product of equation 1 on both sides with r i dot. So, doing this we will get r i dot cross and summing; summing over all the particles i is equal to1 to n r i j whole cube for we have missed out this G the universal gravity constant, and from page number 5 onwards; so, we introduce this G here. So, from page number 5 onwards we are just introducing it, so, here also the G is missing so putting the G here. Now, this equation is our equation number thirteen, left hand side is that very easy to see as earlier we are also done it this will be 1 by 2 times d by d t r i times r i dot times we write it as r i dot times r i dot, if we take the derivative of this you can directly check this quantity will result. Let us verify here this so this will be1 by 2, now r i dot times m i r i double dot plus r i double dot dot m i times r i dot. So, this is nothing but r i dot times m i r i double dot so this quantity we can replace with this. So 14-10, here we can write 1 by 2 m i, we can bring it here.

And here, we have the vector sign now this is our equation number 14, we need to work out the right hand side equation number 14, we are missing here also the dot product we can write her r i dot; dot product of this whole quantity, so, we put here G and put it in bracket. So, this is the dot product we have to take. So, here this dot product we have to still work out, and this is a dot product we are taking with this whole quantity. So, basically in this we have to solve for this dot product. Now, if we look into this particular equation, so, what we will find that we have the term r i dot r i j. Similarly, we will have another term which will look like r j dot r j i so they will exist in pair. So, we can take advantage of this particular thing.

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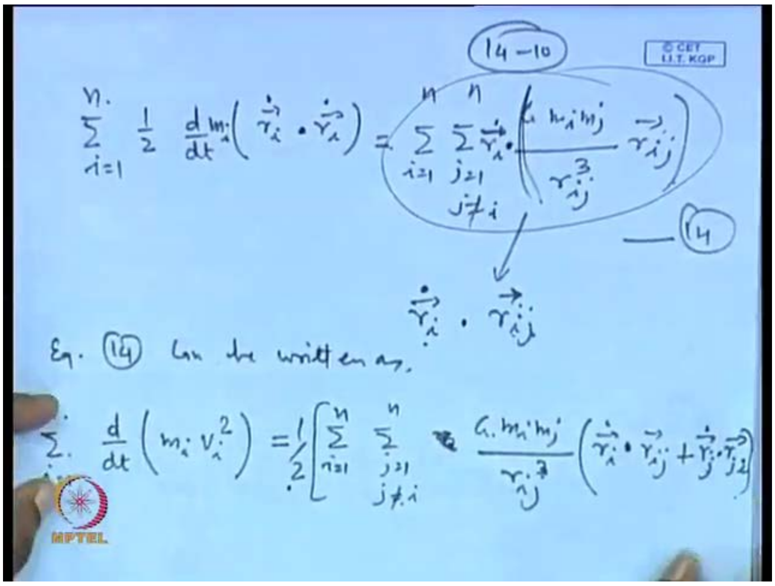

And, the equation number 14 then can be written as; equation 14 can be written as summation i is equal to1 to n, and this d by d t, and here this is nothing but the dot product of v i v i, so we write it as v i square, and on the right hand side, we will have i is equal to1 to; we will put it in this way 1 by 2 n j is equal to 1 to n; G m i m j divided by r i j whole cube we can write it in this way r i dot r ij plus r j dot r ji here dot dot present. So, in the derivative, so 1 by 2 we have put because. These terms are equivalent this kind of pairs will appear in this on the right hand side so this we take the pair of them and the divide by 1 by 2. So, once we do this,

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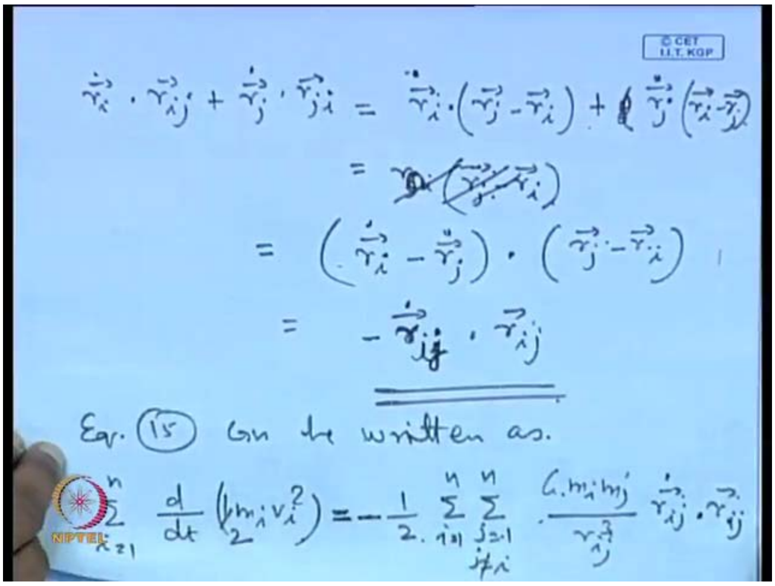

So, we need to just work out this quantity. So, let us look into this r ij is nothing but r j minus r i and plus r j dot times r i minus r j. So, this we can write as r j minus r i and r i dot minus r j dot dot product with r j minus r i here we can write it r j minus r i; minus sign will come outside, and then we can take the common r j minus r i. So, this comes out as a common, and we get this term. So, this term is nothing but we take the minus sign again outside, so, this becomes r j dot minus r i. So, we can put it as r j dot minus r i dot with minus sign taking it outside, so, this quantity can be written like this r ij and this quantity then becomes r i j.

So, the this is equation number 15; equation 15 can be written as 1 to n d by d t m i v i square and 1 by 2; we have missed out, so 1 by 2 is also present here, so 1 by 2 we can push it inside the bracket, and this will be equal to 1 by 2 times, summation j is equal to 1 to n i is equal to1 to n j not equal to i, and here G m i times m j divided by r ij whole cube, and multiplied by r ij dot; dot product r ij. So, this is what we are getting with a minus sign. So, minus sign we can put here in outside, this is our equation number 16. So, in this equation number 16, now we have to little bit simplify this, and then to of course this can be written in the form of differential format, and therefore we will be able to integrate this equation and finally, integrating this we will get the total energy of the system.

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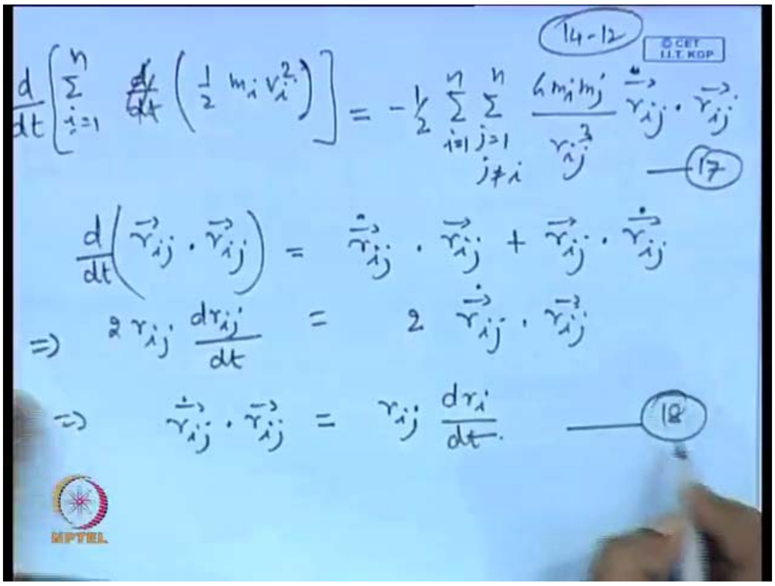

So, now we have this i equal to 1 to n d by d t 1 by 2 m i v i square, so, what we will do we will take the d by d t term outside and so, lets us take this d by d t term outside here summation operator inside, so, we get this term here and on the right hand side we have the quantity minus 1 by 2 summation r i j dot taking dot product r i j, now look into this quantity r i j dot r i j if I differentiate this quantity with respect to t, so what I get, so this is 2 r i j dot r i j, while on the left hand side we have r i j square. So, if we differentiate this we get 2 times r i j d r i j by d t, so this implies r i j dot r i j this becomes equal to r i j times d r i j by d t. So, this is our equation number 17 let us say this is 18.

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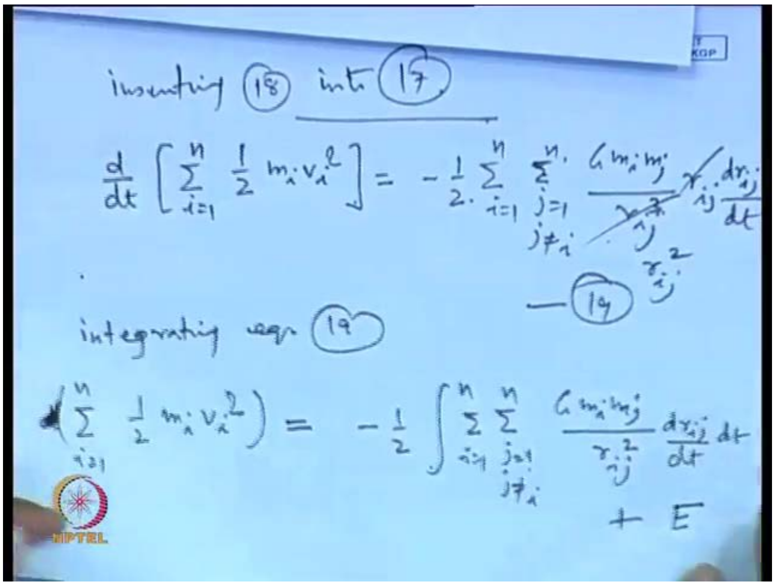

So, inserting; inserting 18 into 17, this gives us d by d t minus 1 by 2 G m i G m j by r i j whole cube, and multiplied by r i j times d r i j by d t. Now, this will cancel out, so we can cancel it out, and we can write r i j square. Now, integrating the above equation, so this is our equation number 19, integrating equation 19 on the left hand side what we get i is equal to 1 to n 1 by 2 m i v i square, so, directly we write like this. So, on the right hand side what we will get plus a constant say E; so we need to work out this quantity and it's a very easy to see this.

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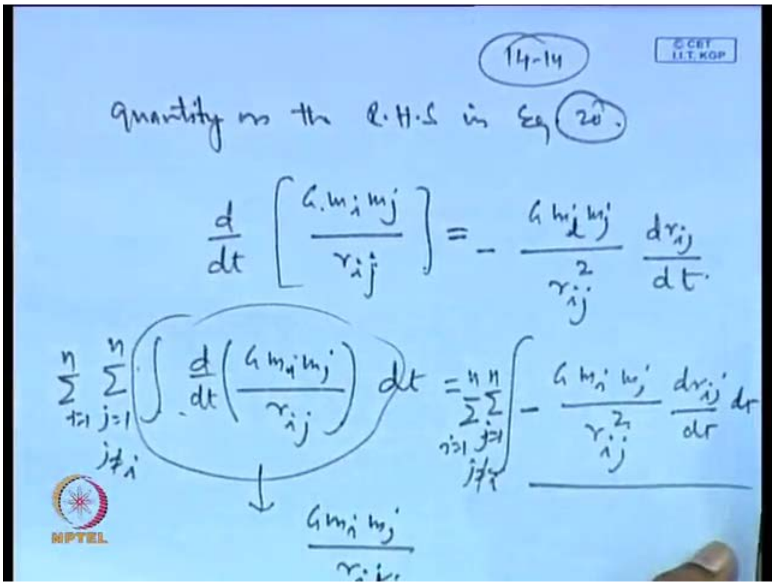

So, the quantity on the right hand side in equation number 20 quantity on the R.H.S in equation 20, this is nothing but G m i G m j d r i j, so, if we take the; let us take the derivative of this with respect to t. So, if we take the derivative of this quantity with respect to t so what we get this will give us G m i m j r i j whole square; this with a minus sign here and then d r i j by d t this is what it will get. So, we can directly, if we integrate this with respect to d t what we are going to get, so, the our integration the quantity if which is present here, if we take out this minus sign from here, and include it here in this place.

So, exactly this quantity is matching with this only thing is d t is appearing outside, if we now integrate both hand side this quantity, and taking out the summation sign outside, we can put it like this j is equal to 1 to n i is equal to 1 to n j not equal to i, summation sign we have taken inside integrate with d t, so this exactly appears as minus sign here also we can take it outside inside. So, this is G m i m j d r i j by d t and this whole summation we can write here, j is equal to 1 to n i is equal to 1 to n j not equal to i. So, exactly we can see that the quantity whatever we have written here is nothing but this quantity here, and the integration of the quantity, which is inside this integrand, this will be nothing but G m i m j by r i j that is a very simple.

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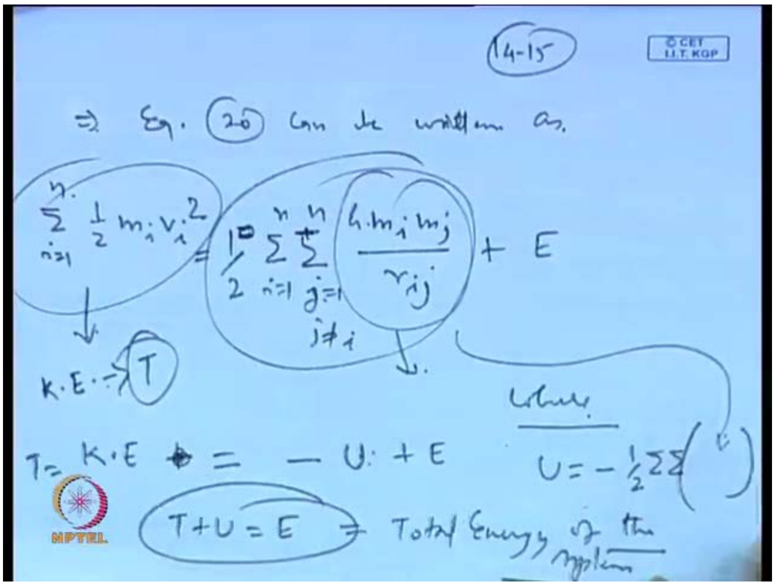

Therefore our equation number twenty this implies equation twenty can be writ 10 as1 by 2 m i v i square equal minus sign gets absorbed inside so once we integrate this minus sign disappears and what we are getting here is just this quantity and over this we have the summation i is equal to j is equal to 1 to n i is equal to 1 to n j not equal to i, and besides we have divided by 1 by 2 and this plus E. So, the quantity which is present here, we can write this quantity as, let us define the quantity as let us first define this quantity; this quantity is nothing but our total kinetic energy of the system, and for this therefore; this equation we can write it as kinetic energy, and this energy is nothing but the potential energy of the system.

So, kinetic energy this is we indicate it by T and potential energy, so potential energy of the system, this will be given with a negative sign, so this we can write as minus U and plus E, where we have written U is equal to minus 1 by 2, this whole summation times the quantity which is present here. So, this gives us T plus U this is equal to E, means the here, if this is the total energy of the system is the total energy so this result is exactly the same as we did for the 2 body system only; only difference here it is that in the case of the 2 body system the total it was the relative energy in the relative sense it was not the absolute energy as we are getting here

Here this is the absolute energy of the three body system or the n body system as we have done it, as we have not done anything relative, so this really indicates the total energy of the system. So, all together what we are getting here 10 scalar constants or the 10 constants of integration, therefore even for a three body system, we will be left out with 8 number of constant of integration to be determined; means we would not be able to find it out attempts have been made earlier by many people, but people have landed up at some mean finite series, which we are of no use. Because it took more time to compute then doing simple integration, so that was also aborted so those solutions they exist in the infinite series, but they are not meaning full for any purpose and neither they give any because it is any in terms of infinite series is also they do not give any insight into the problem, but for what we have done here it proves that this kind of system it is not solved simply.

But if there are certain quantities for the n body system, even which remain as a constant, which are the angular momenta of the system, the total energy of the system, and the centre of mass of the system, it moves with a constant velocity, so, if it is initially at rest the centre of mass. So, it remains at rest or if it is initially moving with the constant velocity, so, it continues to move with the constant initial velocity. So, after finishing this we go into the three body problem.

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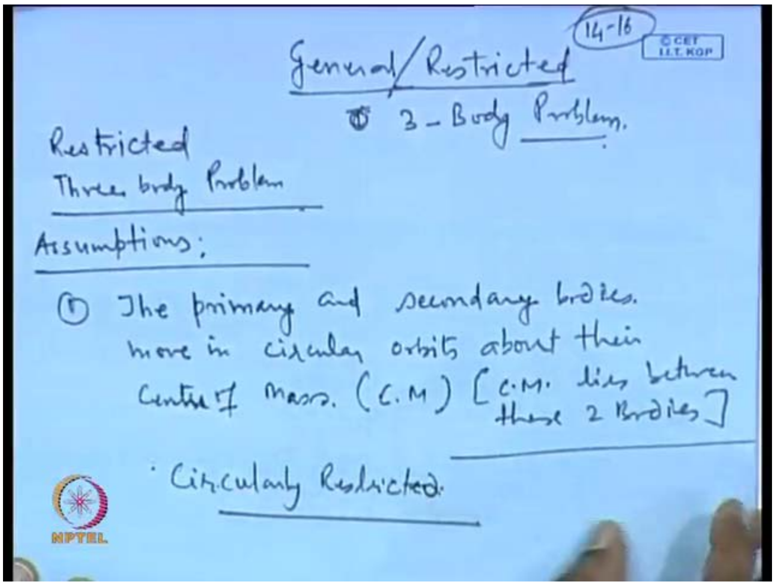

So, it is not the general, because the general problem is anyway for the three body is not solvable, we what we do under the general three body problem, we just work for the restricted three body problem. So, for the restricted three body problem, we first do certain assumptions, we make certain assumptions, so, under that assumptions that is making the assumption is equal to putting some constraint on the equation of motion. So, say some particle is moving in a space, so it can move anywhere, but if constrain the system to move on the in the space itself. But on the surface of a sphere that appears as a constant, where the radius of the particle from the centre of the sphere or which we are assuming here to be the inertial frame it remains as a constant.

So it simplifies the condition if the particle is free to move all the particles are free to move in the initials space, so it may go anywhere. But if we constraint them on a surface of a sphere then that is one particular constraint we put here. So, here we in this case of the three body problem that, we are going to discuss, so, first assumption that we are going to make is that the 2 primary bodies; there are 2 primary bodies one primary body, and another secondary body, which are heavy as compared to the third body, which is quite negligible in mass, so, it is almost zero mass we can say, as compared to the this primary, and the secondary body just like in the moon earth system or the sun earth system so in the sun earth system if the various small asteroid is there so how it will move so that constitutes one three body problem.

So, moreover the second assumption will make that the 2 the primary body and the secondary body they move in a they have circular motion obviously they move revolving the centre of mass so they move in circular orbits and so under this assumption so our problem will get simplified. So, we write here the restricted three body problem; assumptions; the first assumption, we make the primary and secondary bodies move in circular orbits about their centre of mass, that is centre of mass lies between the 2 bodies the C.M lies between these 2 bodies. So, this makes the once we do this kind of assumption, so this makes little what we call the circularly restricted this will be called circularly restricted.

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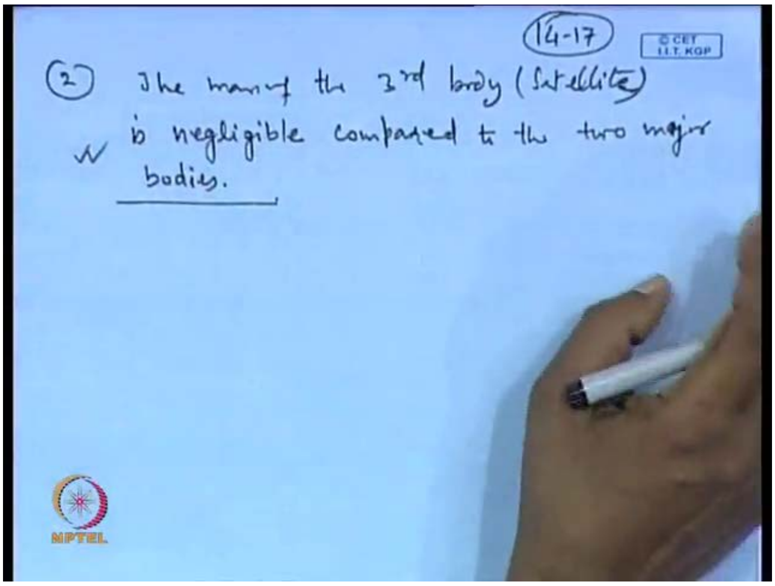

The second assumption, that we are making that the mass of the third body; that is the satellite is negligible compared to the two major bodies, so once under this assumption it will the equation of motion for the three body system it will look very simple, and basically what will we doing, we will describe the motion of the; we will fix one reference frame at the centre of the mass, which will be rotating at the angular rate of the rotation of the two bodies.

Therefore, the rotating reference frame, we will call as the senodic reference frame; that senodic reference frame, then will try to look at the how the satellite is moving. So, it is a basically a moving this reference frame, and rotating reference frame in that the satellite how it is a behaving. So, this is also some kind of relative motion, we are trying to look into, because the two bodies; are the two primary bodies are rotating, and at the same rate we are rotating the reference frame. And, with respect to that reference frame, we are trying to locate the third body it is a trying to describe its motion that will be very interesting, but it can also be expressed in the initial reference frame as well.

So, property of this motion is very interesting as we continue to grow, and we see, we will see that there are some points, which are called the lag ranges points, which come into picture, when some particular property. So, we continue in the next lecture. Thank you very much.