Space Flight Mechanics

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Lecture No. # 13

Three Body Problem (Contd.)

In the last lecture, we have started with the three body problem, and then we constructed the equation of motion for the i eth body. So starting with that again today we derive the equation of motion in the inertial difference frame, and that relative motion and then the Bary centric formula. And thereafter once these things are over then we will go into the, because there is no generalized solution. So, we go into the 3 body restricted problem or restricted 3 body problem.

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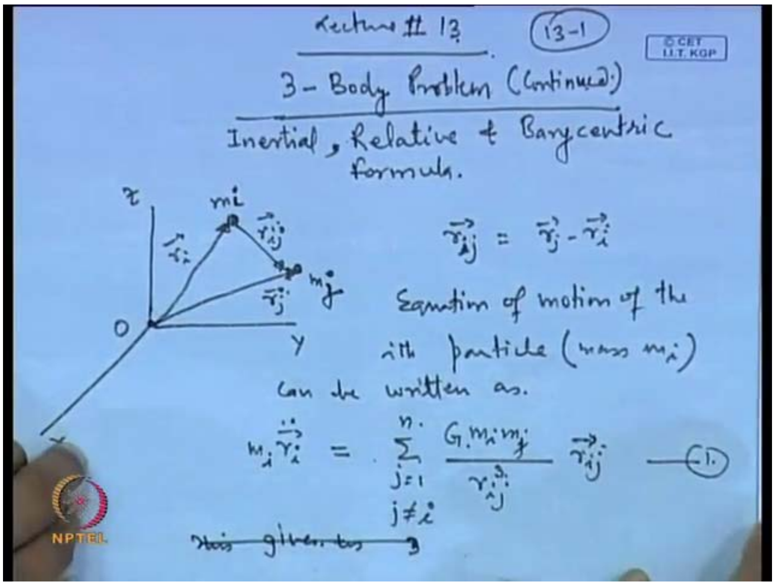

So, last time we had the this inertial reference frame x y and z over the origin r i and r j, where the position vectors of 2 masses, 2 a v masses and m 1 and m 2. r i j vector was defined which is r j minus r i, then the equation of motion of the i th particle. So here, we write this as m i and m 2, instead of writing m 2, we write it as m j. So, of the i th particle in mass m i can be written as m i times r i double dot, this is equal to summation j is equal to 1 2 n, where j not equal to i. And then m i times m j, multiplied by G, by r i j whole cube, then multiplied by r i j vector. Now this force on the i th particle, it is a directed along the i j; therefore, the positive sign has been taken here. So, this is our equation number 1.So, this will give us this gives us 3.

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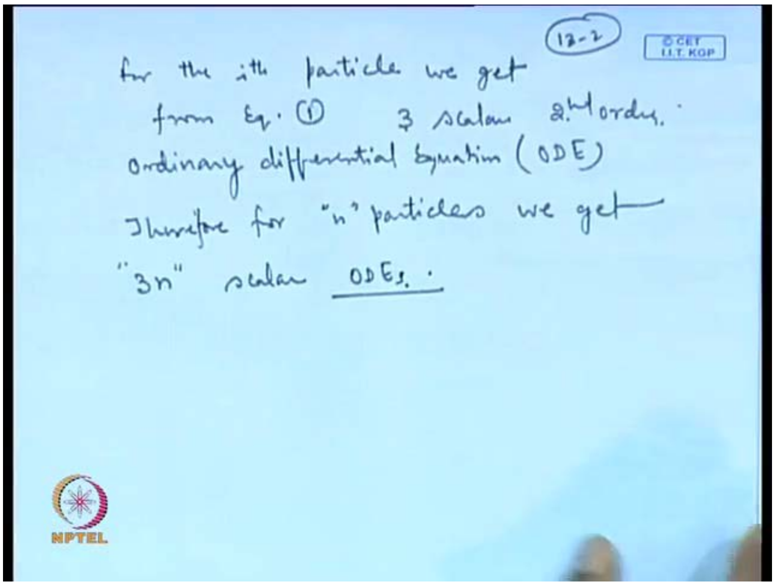

So, for the i eth particle, this gives us, we write here for the i eth particle, we get from equation 1. 3 scalar second order ordinary differential equation. So, if we write the equation of motion for all the n particles then all together we will get 3 n scalar of ordinary differential equation. Therefore, for n particles we get 3 n scalar ordinary differential equations. Now, the question obviously, we cannot solve for more than 2 particles system completely, therefore even that for the relative motion only and therefore, for the 3 body system, we restrict ourselves to the 3 body system.

So, here after for our further treatment for the inertial relative and bary centric formula, we restrict ourself to the 3 body problem and therefore, thereafter for finding out the integrals of motion for the n body problem, we treat it in a general way as we have earlier done using the cross product and the dot product finding the angular momenta of the system and then finding the total energy of the system. So, this is where we try to analyze the 3 body problem, and thereafter we look into the restricted 3 body problem, where we impose certain condition on the 3 body and then try to look into the solution what kind of solution it leads to.

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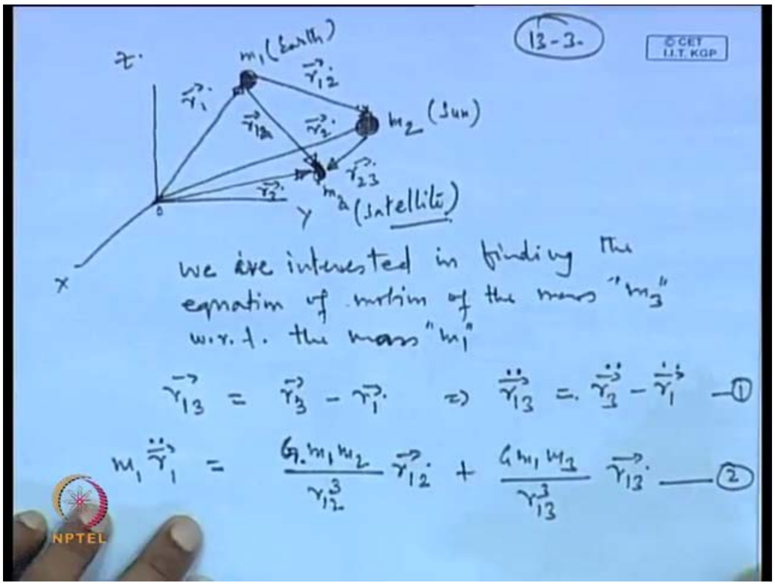

So, for the 3 body problem, so we have 3 particles here of mass m 1 m 2 and m 3 we assume the mass m 3 to be satellites say or and m 1 in this case suppose, this is the earth and m 2 this indicates the sun. So, for this kind of system, we can develop the equation of motion. So, m 2 let us suppose this is sun and m 1, this is earth and m 3 this is satellite. So, we are interested in finding the equation of motion of the mass m 3 with respect to the mass m 1. Now, we can write r 1 3, this is equal to r 3 minus r 1. So, this implies r 1 3 double dot the acceleration, then this will be r 3 double dot minus r 1 double dot. So, this is our equation number 1. Now, we can find out the acceleration of mass m 1. So, m 1 times r 1 double dot, this will be equal to m 1 times m 2 multiplied by G by r 1 2 whole cube r 1 2 and the force, this force will be directed.

So, we are considering the acceleration of this mass, the force is directed towards this vector r 1 2 vector therefore, this positive signs comes in and the another one, the force acting on mass m 1. This is due to m 3, so that will be given by G m1 m3 by r 1 3 whole cube r 1 3 this is our equation number 2.

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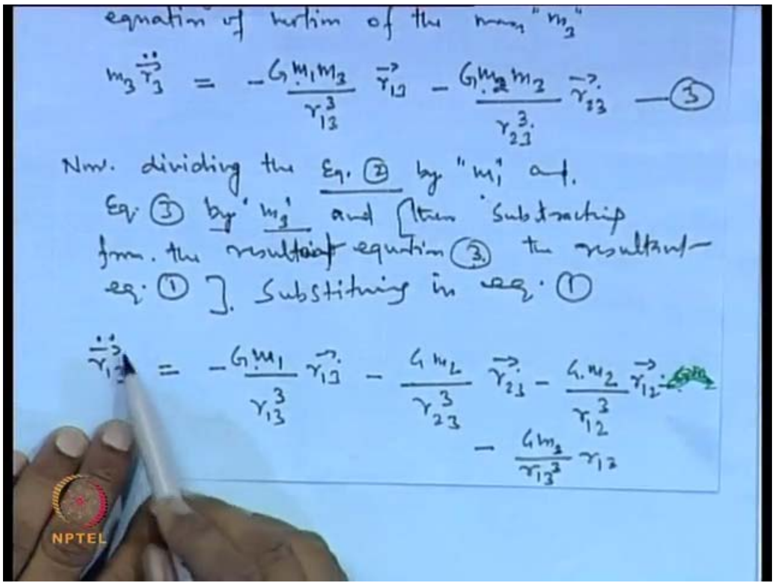

Similarly, we can write the acceleration or the equation of motion of the mass “m 3.” So, m 3 times r 3 double dot this will be equal to. So, we are now writing the equation of motion of m 3 therefore, the force acting on this is directed opposite to r 1 3 vector and also opposite to the r 2 3 vector due to the mass m 1 and m 2. Therefore, both of the terms the forces corresponding to these 2 masses will have negative sign. So, we will have here m 1 times m 3 divided by r 1 3 whole cube times r 1 3 with a negative sign and similarly, we will have m 2 times m 3 divided by r 2 3 whole cube times r 2 3. So, this is our equation number 3. Now, dividing the equation 2 by m1 and equation 3 by m3 and then subtracting from the resulting equation 3, the resultant equation 3, the resultant equation 1 so and substituting.

So, basically dividing equation 2 by m 1 and equation 3 by m 3 and then subtracting from the resulting equation 3, the resulting equation 1 or either, we can say that substituting in equation 1. And substituting in equation 1, so what we get r 1 3 double dot. This will be equal to minus m 1, m G is missing, so here G. So, G times m 1 by r 1 3 whole cube, G m 2 r 2 3 whole cube and then from the equation 2, we get minus G, m 2 times here r 1 2 whole cube r 1 2 minus G m 3 r 1 3 whole cube. So here, can be fit m minus G. So, the resulting equation looks like this, we have r 1 3 double dot, so all these 4 terms are having negative sign. So, this is r 1 3 here. Now, this is our equation number 4.

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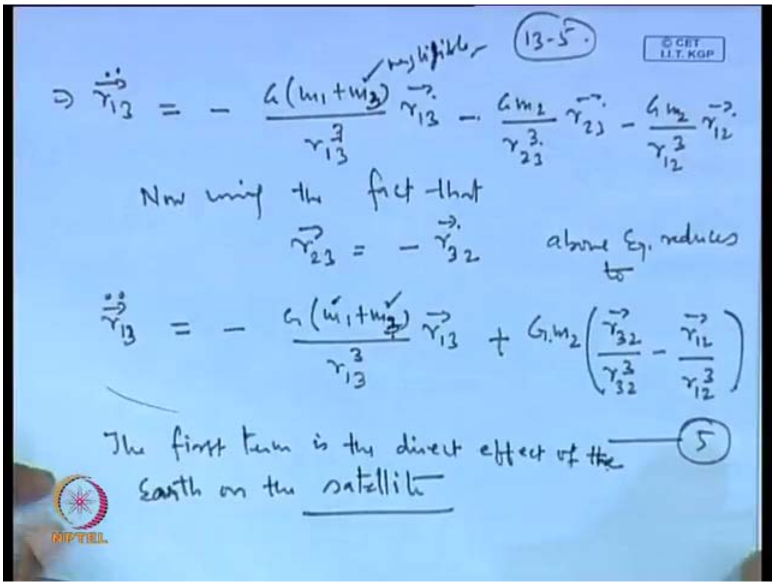

So, this implies r 1 3 double dot, this we can combine some of the terms together. So, we have this term and this term they are related to r 1 3. This r 1 3 vector, so they are related to r 1 3 vector, so we can combine them together, so if we combine them, we get here G times m 1 plus m 2 divided by r 1 3 whole cube r 1 3 and then we get minus G m 2 r 2 3 whole cube m 2 r 1 2 whole cube r 1 2. Now using the fact that r 2 3 is equal to minus r 3 2, so we can write here the above equation reduces to r 1 3 double dot. This will be equal to minus G m 1 plus m 2 r 1 3 and then this will become plus G. We can take outside m 2, we can take outside inside the bracket we will have r 3 2 divided by r 3 2 whole cube minus r 1 2 divided by r 1 2 whole cube so this is our equation number 5.So, now in this equation number 5, the first term what we see here, this is the direct effect of the see here, the terms are coming due to the m 1 and m 2. So, both the masses m 1 and m 2 are involved here.

So, the first term, the direct effect this m 1 plus this is m1 plus m 3, not m1 plus m2. we do the correction here, this is m1 plus m 3 and m 2 has gone outside, so this is m 3, so this is the direct effect of the this mass, this mass gets neglected almost, this is negligible. We consider the earth, sun and the satellite system. So m 3, this is negligible, so living out only the m 1 term and m1 term here. So, the first term in is the direct effect of the earth on the satellite, so this is basically reflecting the direct or the direct gravitational effect of the earth on the satellite just like in the 2 body system. We, if we remove this 2 term so, we can see that in the 2 body system we got this same kind of equation where, m3 was the mass of the body which was or writing the heavier body.

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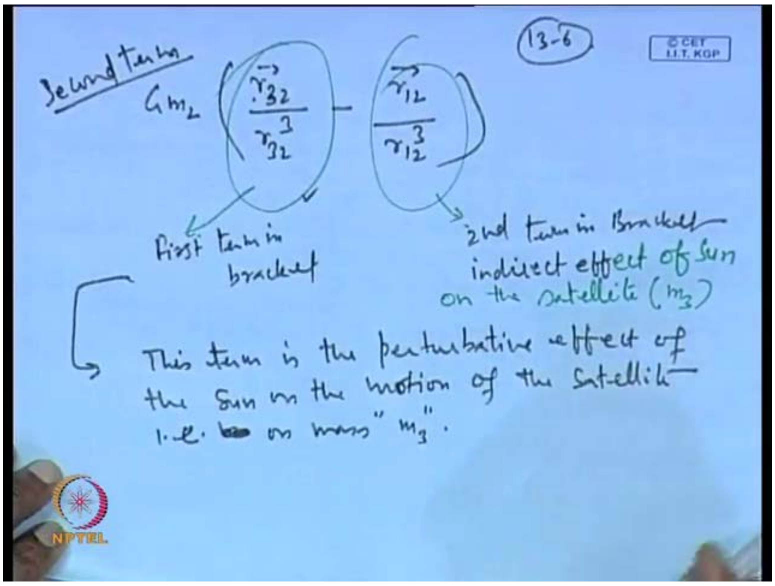

Now, the second term which consist of G, m 2 times r 3 2 by r 3 2 whole cube minus r 1 2. Second term, so this term here and this term here we analyze them, so if we look into this term, this is the r 3 2 so, this is the term which related to the this is the first term in bracket and this is the second term in bracket so, the this term is the perturbative effect of the sun on the motion of the satellite that is on mass m3. So, this directly tells that is, we have only 2 body m 1 and m 3 present and m 3 is moving around m 1. So, if we bring in the third body which the second body which is the m 2. Here, in this case so, which is this is nothing but sun in this case so, if we bring in the sun, then the sun acts as a perturbation on the motion of the third body around the one first body. So, here the satellite is moving around the primary body, which is the earth in this case and the sun is trying to perturb the orbit of the satellite around the primary body.

So, this is the significance of this term, while the second term, we can see from here, this is the term which is related to 1 and 2. So, this is connected to the earth and the sun, so in this case what happens that the sun it affects the motion of the earth. So, this is affecting the motion of earth and inturn, because earth is affecting the motion of the satellite therefore, ultimately the motion of the satellite gets affected. So, if the suppose the distance between the sun and the earth or the m 1 and m 2 is increased, so because of this effect, so ultimately what will happen, the distance between the satellite and the earth will also change. So therefore, the effect of the force of this mass m 2 on the mass m 1, it also affects the motion of the satellite where, which is in this case mass m 3, so if it effects the mass m3 therefore, this is called the indirect effect of sun on the satellite which is here the mass m3.

So, thus we see that the equation of motion of the mass m 3, it consist of altogether. Two terms, the first term, which is the direct effect of the primary body on the satellite and the second term, which consists of the perturbation term. Due to the sun, one is directly and other one is indirectly. So, in both way this is arising due to the sun and therefore, this is called the second term, is called the indirect term. And the first is, the direct effect of the sun on the satellite and this acts basically as a perturbative term on the motion of the satellite around the primary body which is related to this equation.

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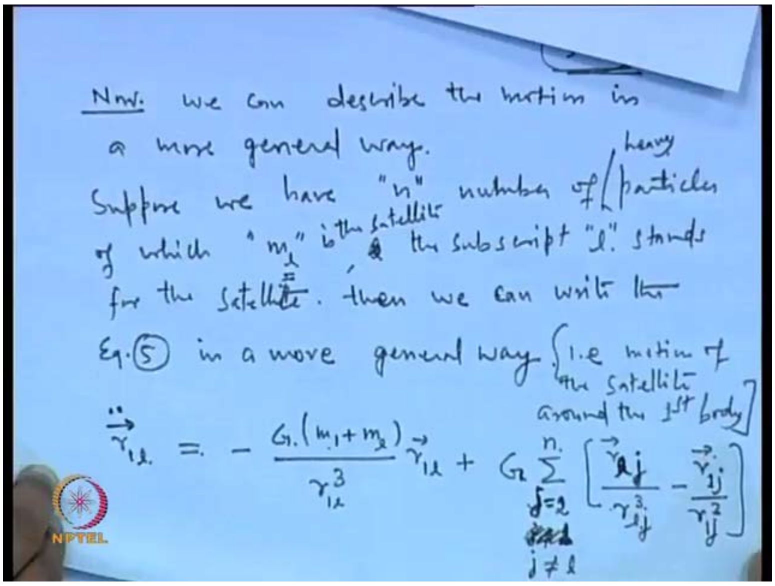

Now, we can describe the motion in a more general way, suppose we have n number of particles, n number of heavy particles basically or heavenly body of which “m l” where, the subscript l is indicating the satellite, so the subscript l stands for the satellite. So, suppose we have n number of particles or which the which m l is the satellite, so the subscript l will be standing for the satellite, then we can write the equation 5, equation 5 in a more general way. So here, let us write this as r 1 l. This is the motion of the satellite, so motion of the satellite around the first body, that is motion of these satellite around the first body. So, r1 l double dot, this will be equal to minus G time m1 plus m l divided by r1 l whole cube r 1 l, so here 3 we have replaced by l.

And then we will have terms, where the instead of only sun being present there are number of other massive particles present, so all of them will be acting as a perturbation on the motion of the satellite around the primary body, which is the mass m1, so we can write here G times summation j is equal to the first body. We have already taken so we here, write j is equal 2 to n and j is not equal to 1, so that is very obvious, because one we have already taken. So, j and here, we are starting from j is equal to 2, so j is equal to, we write j is not equal to l in this case. So, we can write here following the formulation here, so, we can follow the same notation and here we can write r 3, instead of 3 we put here l so r l and 2, then becomes it varies from j is equal 2 to n. So, here we put r l j divided by r l j whole cube minus r1 j divided by r1 j whole cube. So, in a more general way, we can write the, even if we do not want to continue with the notation for the timely body as 1, but other we want to write in terms of i.

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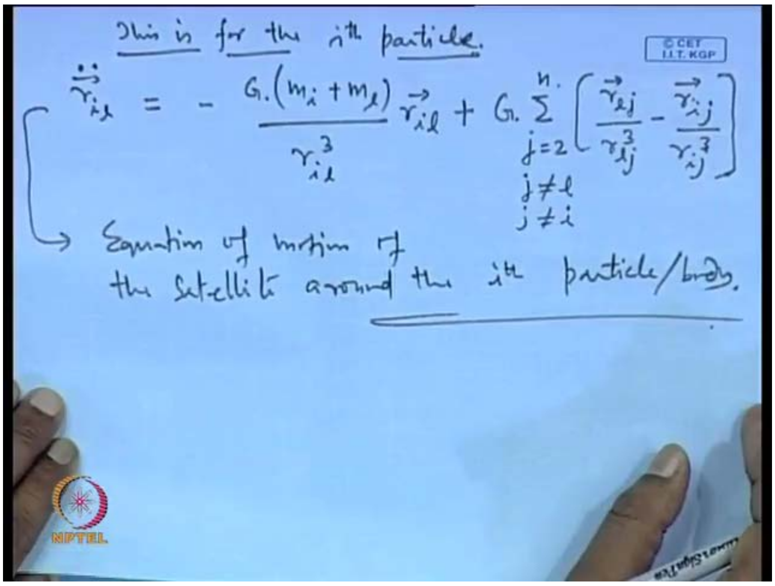

So, we can write it as r i l double dot, this is equal to minus G m i plus m l divided by r i l whole cube plus G. j is equal to 2 to n and j is not equal to l and also j is not equal to i and r l j by r l j whole cube minus r i j divided by r l j whole cube. This is for the i eth particle. So, equation of motion, so this indicates the equation of motion of the satellite around the i eth particle or body. Once we have got this now, we go into the bary centric formula, we look in to, we try to look into the motion of the particle in the bary center. Bary center is nothing but the center of mass of the all of the particles. So, let us see what the result we get from that, so this is our 13-8.

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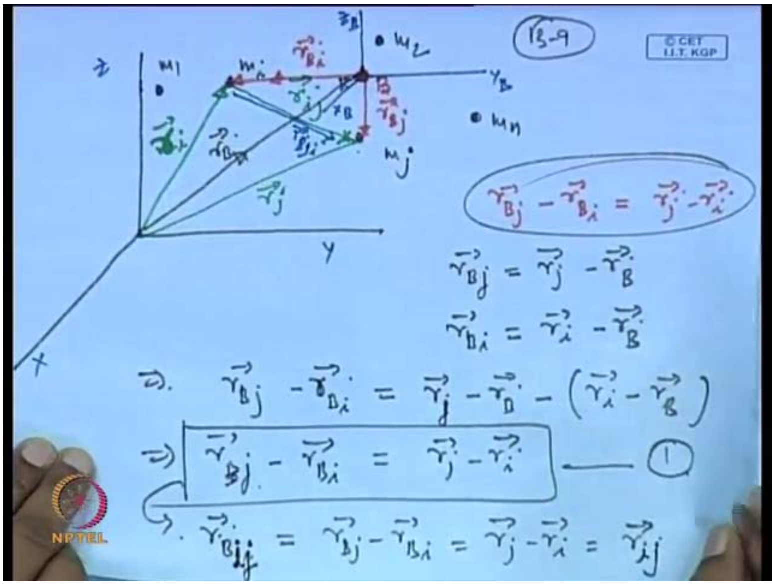

Let us consider that, these are number of particles available here and somewhere this is m n. This is r i j, this is r i, this is r j and then we consider that the bary center of all the particles. It is located somewhere here, so this is the bary center which is indicated by b and we fix a frame at the bary center, which is parallel to this frame x y z. So, we fix a frame here, so this is x b y b and z b and we know the bary center property that and we will also prove latter on for the n body system that the bary center velocity it remains constant that is it does not accelerate. So, if it does not accelerate, so we can fix a reference frame about this point and use it for the analysis of the motion. Now, we connect from this mass to this mass this we write as r B i, this is the vector directed from bary center to mass i and this is a vector directed from bary center to mass j, so r B j. so from here, we can see that r B j minus r B i this is nothing but r j minus r i.

And other way also, let us see, we draw a line from here to here and this is let us say, this is r B vector. The position of the bary center with respect to the reference frame x y z, so other way also, we can write that this r B j, we can express as r B j this will be nothing but r j minus r B. And similarly, we can write r B i this is nothing but r i minus r B and therefore, this implies r B j minus r B i, this will be equal to r j minus r B minus r i minus r B. So, this results in r j minus r i, so r B j minus r B i. So, the same result we get as we have directly written here, so this is our equation number 1. So, we can write this as to say r B j i. r B j i is a vector, so the r B j i vector, it will be directed from here to here. So, this is also the vector r B j i, which is directed from the i eth mass to the j eth mass, so this can be written as r B j i this is equal to r B j minus r B i is equal to r j minus r i is equal to r j, this we write as r i j directed from i to j therefore, we put here i to j not B j therefore, r j minus r i is equal to r i j.

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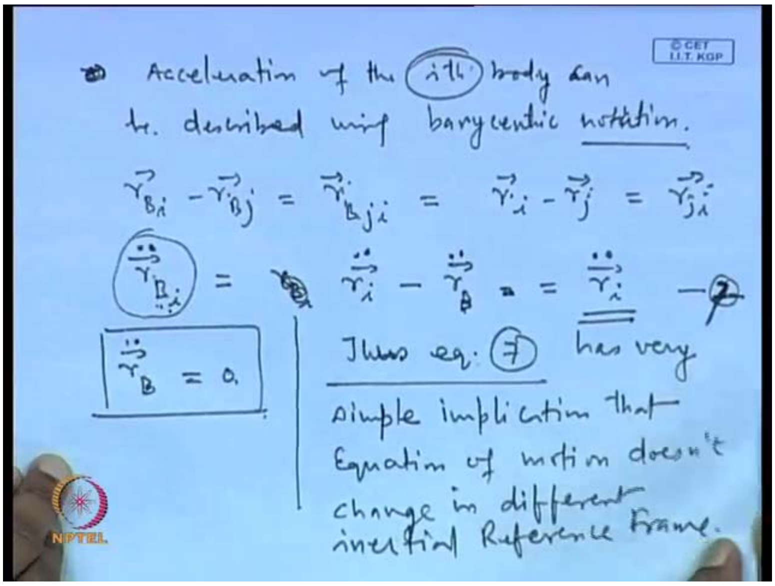

Acceleration i eth body, now the acceleration of the i th body can be described using bary centric terms notation. So similarly, following the same type of notation, we can write r B i minus r B j, this will be equal to r B j i and this will nothing be equal to r i minus r j, this will be equal to r j i. Now, if you look into the acceleration of i eth body, so from here, we can write the acceleration of the i eth body. So, r B i double dot, this can be written as r B r i minus r i double dot minus r B double dot. Now, we can check from the above equations that, we have developed this is r i minus r B. So, this we have written as r B i. Now, we know that the bary center of the system, this moves with a constant velocity therefore, r B double dot the acceleration will be 0. Therefore, this gets reduced to r i double dot. So, this has a very simple implication and this says that, thus equation 2, this is this we write as equation 6 and this is equation 7.

Thus, equation 7 has very simple implication that, equation of motion does not change in different inertial reference frame. So, whatever the acceleration in is in the x y z reference frame, for the i eth particle, this is indicated bar r i double dot. So, the same acceleration, it is noted it is experienced in the bary centric reference frame which is moving with a constant velocity.

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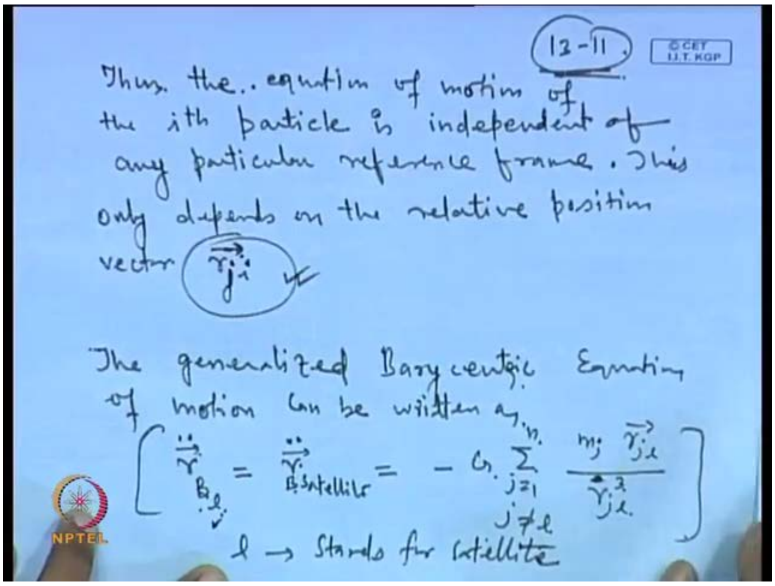

Thus, we can say that thus, the equation of motion of the i eth particle is independent of any particular reference frame. This only depends on relative position vector r j i. So, if we are considering the motion of the i eth particle. So and there are n number of other particles present, so it will just depend on the distance of the i eth particle from the j eth particle. And if are considering obviously last, as we have considered for, we have derived the formula that, if we are considering about any particular mass, so if we are considering a motion of a particle about any particular mass, then that mass constitutes the primary body and rest other they constitute as the perturbation body or what we call the acceleration due to other particles they act basically as a perturbation.

So, that is the relative motion described here. But what we have just now looked into this is the bary centric motion, means the points chosen or reference frame chosen at the bary center. So and this is an inertial reference frame, because it is a not accelerating, bary center does not accelerate, so it is a perfectly all right to choose the reference frame at the bary center. And more over, the if the bary center the difference frame fixed is a not accelerating and there more over, if we keep it parallel to the original reference frame, so if and all the time means the original reference frame is not rotating, so the bary center the frame fixed at the bary center that is also not rotating. Then, it constitutes an inertial reference of reference frame and therefore, the motion of the i eth particle described in the original reference frame is as go as the motion in the bary centric reference frame, so there is no difference as we have concluded from this equation here equation number 7.

The generalized bary centric equation of motion can be written as, r B l double dot. So, this is with respect to the bary center the motion of the i eth particle. In this case, we are taking this as the satellite, so we can write this as r B s. This is the satellite is equal to minus G times j is equal to1 to n j is not equal to l and m j times r j l divided by r j l whole cube, so this is the motion of the satellite which we are have indicated by l, l stands for satellite, so this gives the motion of the satellite about the bary center. So now, to solve the equation of motion some approaches can be, we can choose some of the approaches.

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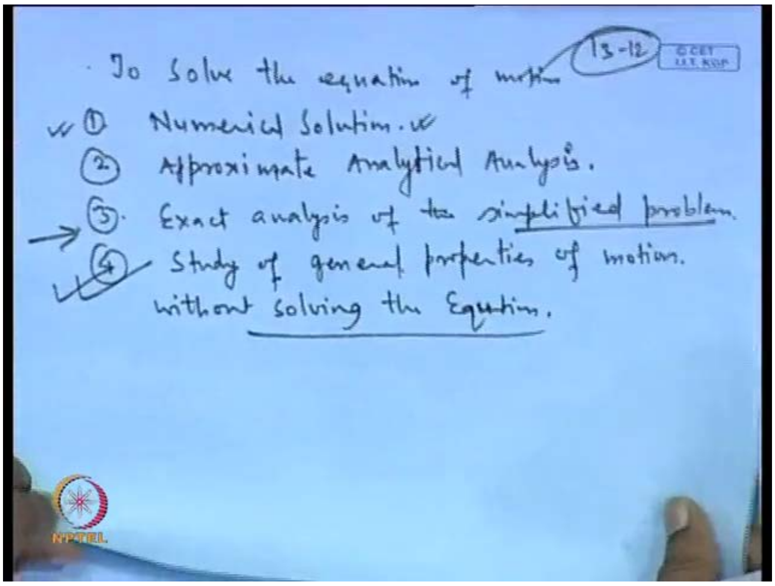

So, we have suppose, the first approach to solve the equation of motion, we have the following approach we can opt for numerical solution or we can do approximate analytical analysis. Why approximate because we reject solution any way we cannot do therefore, if we can do certain approximation. So, the it may be possible that we get certain kind of solution and this gives a lot of insight into the a motion of the system just like we have the earth and sun system and or the earth moon system and then we can have a satellite which is a moving about the earth moon system, so where earth is a primary body acting as the primary body, so we can get a lot of insight into the motion of this satellite. If we consider the approximate equation instead of just going through the numerical solution, because numerical solution it does not give us the generalized overview or generalized picture of the motion.

So, whatever way we integrate, so it is a going to only describe the how the particle trajectory changing, so the third one, we can have exact analysis or exact treatment of the simplified problem. And fourth one, study of general properties without solving the equation, so these are the four approaches we can choose. So, out of this we first do the fourth approach, we do the study of general properties of motion without solving the equation. And therefore, there after we will go into the exact analysis of the simplified problem, so we will simplify the problem and then try to solve it.

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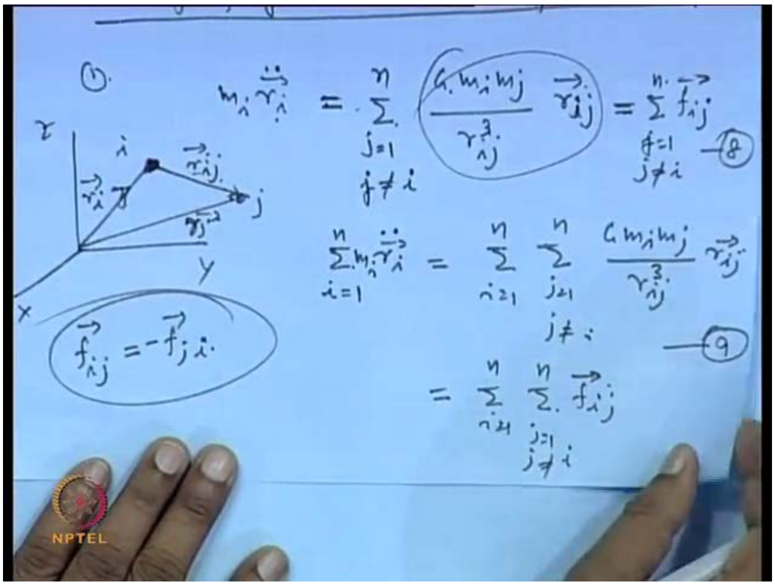

Study of so study of general properties of motion, so in a studying this general properties of motion, as we have seen for the 2 body case, if we take the gross product, then we get the angular momentum terms that was turn out to be a constant. If we get the dot product, then the if we got the that, the result that total energy it is constant. So, we apply the same approach for this system, so analysis, it differs only in mathematical treatment, but the approach is the same and we get also the same kind of result and but we will have little bit more insight into the problem for the 3 body or the multiple body problem, we will analyze it and see what really it reflects.

So, going back into the our original equation, where we wrote the equation of motion as a m i r i double dot for the i eth particle as G times m i times m j divided by r i j whole cube r i j. So, this is the equation of motion of the i eth particle, due to the j eth particle and with in some over j is equal to1 to n, where j is not equal to i. So, we are considering the motion of the i eth particle, this is r i, this is r j, so the force of the i eth particle is directed along the r i j. Therefore, here the minus sign is not appearing. Now, if we take summation of all the particles, so we taking summation over all the particles, so i varying from 1 to n, m i times r i double dot, this we can write as summation i is equal to1 to n, into 1 to n, j not equal to i.

So, this also we can write this term as f i j that is, force acting on the i eth particle and then submit up, j is equal to 1 to n and j not equal to i, so we write this as equation number 8. And this is our equation number 9. So, this is nothing but i is equal to 1 to n and summation f i j, j is equal to1 to n, j is not equal to i. So, if we look into the right hand side and here we should give a vector notation to this, because this is a force which is a vector, so here the property of this vector is f i j this is nothing but equal to minus f j i. So, if we look into this summation, it is a very easy to say that, they will exist in pair. Suppose for that case where we have only 2 particles, so 1 and 2. So, here if we expand it here, if we expand it, so we get f 1 2 and then we get minus f 2 1, so this two terms will appear and because both of them are equal in magnitude in a positive sign. So, this indicates f i j plus f j i this is equal to 0. So, right hand side this will be equal to 0.

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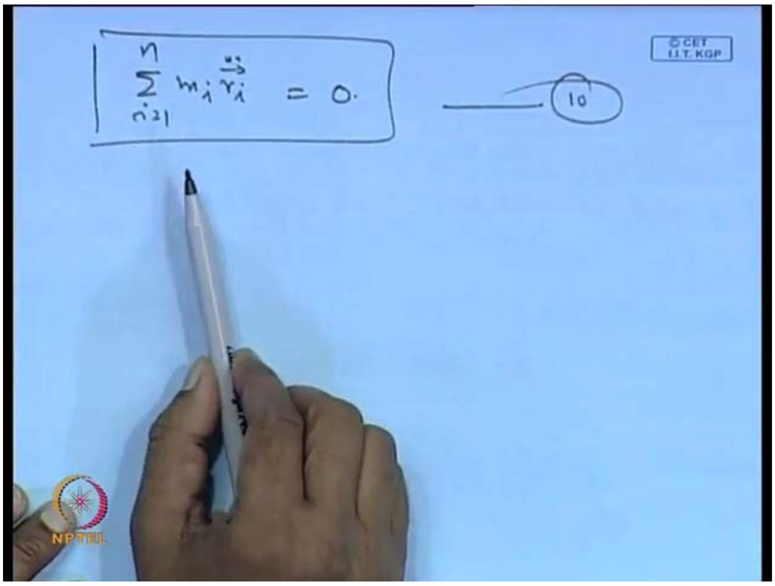

So, this simply implies that, m i r i double dot summation i is equal to 1 to n, this will be equal to 0. So, this is our equation number 10. So from here, we can get to know that the, we can reduce it to the equation for the center of the mass, and we will see that the center of mass of the system it remains it most reaches a constant velocity that it does not accelerate. So, we continue in the next lecture. Thank you very much.