Space Flight Mechanics

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Lecture No. # 12

Two Body Problem (contd.) & Three Body Problem

In the last, we have been working with Kepler's equation so if and Kepler's problem we discussed about that also little bit. So, whatever the remaining portion is there with Kepler's equation and its solution, we will complete this and there after we will go into the three body problem from this lecture.

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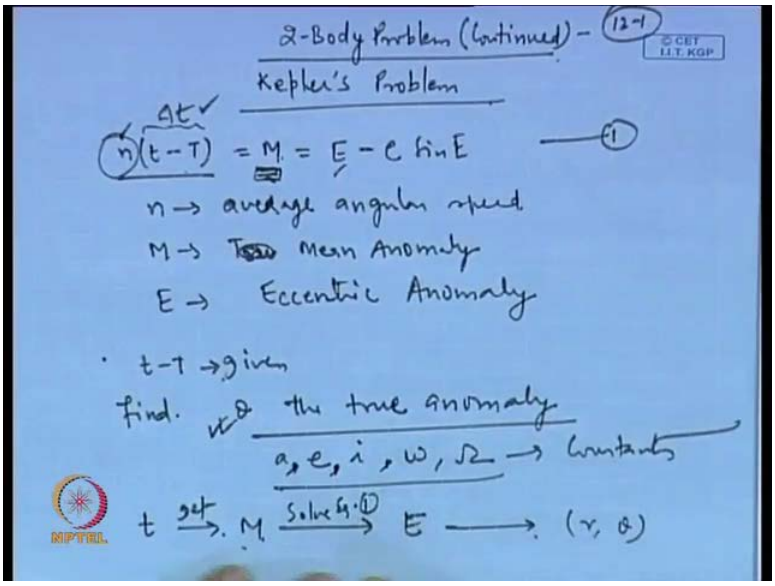

So continuing from the previous lecture, we saw that the Kepler's equation we wrote it as t minus T -n times t minus T is equal to M times, M is equal to e times, E minus e times sin E, where n is the average angular speed and M we wrote as the mean anomaly and E we wrote as the eccentric anomaly.

So, M is a quantity now if you look into this equation so n is the average angular velocity. So, if you multiply it by delta t, this is t minus T, this is nothing but delta t so if you multiply by this quantity, so this is your average angular velocity or your average angular speed. And this your multiplying by delta t, so the total quantity you get is an a angle.

So this angle, what we write as the M the mean anomaly, this describes the if the satellite is moving with certain mean angular speed, so this is the mean or the average angular speed. So, in time delta t how much angle it is going to cover. So therefore, this is a quantity which is just a mathematical description it does not have any physical interpretation just like the eccentric anomaly. We had the physical interpretation, for the true anomaly we have the physical interpretation but as such that kind of physical interpretation does not exists with M, it is a just a mathematical quantity.

So, now our problem can be that if the initial time and the final time it is a given, so find out what will be the true anomaly or the eccentric anomaly. So, we have start with we given this t minus T so this is given, so then we will try to find theta the true anomaly. So, once we find this and our a e i the small omega and capital omega these are constants. Therefore, if we find theta then the orbit determination process or of describing the satellite position at any other time that can be completed.

So here the, to find out the theta, what we have to do we get the time t so from t, we can express get M the mean anomaly. Mean anomaly can be used to solve the equation 1 so here, solve equation one and get eccentric anomaly. Once you have got the eccentric anomaly so there after we have already the equations for in the last lecture, we have seen that how the eccentric anomaly is related to the radius vector and the true anomaly. So using those equations, we can solve for the true anomaly. So, we can solve for r and theta so find theta the true anomaly and the r is also available and theta is also available.

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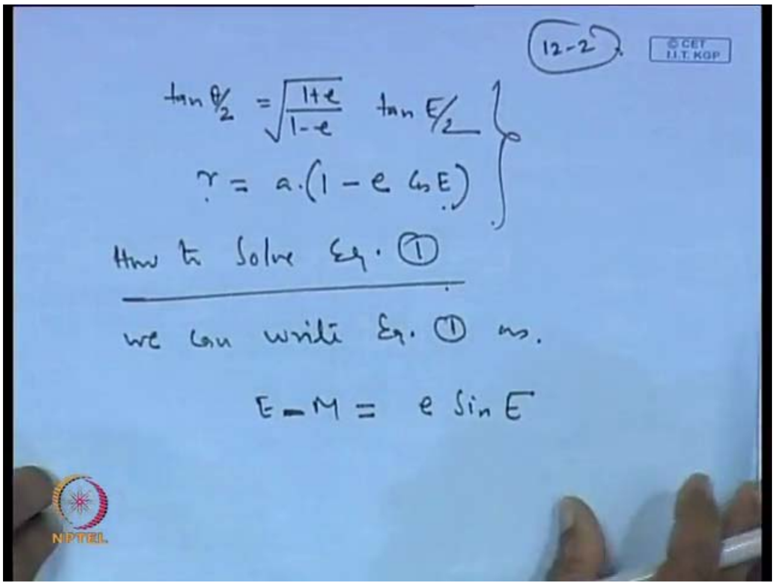

So for solving the theta, we use this equation tan theta by 2 this is equal to 1 plus e1 minus e under root times tan E by 2. So, this completes the process of calculating the true anomaly at any other instant of time and of course, we know the value of r can be written as a times1 minus e cos E so, if E is known so r can also be calculated. So, once we have done this then the question arises, how to solve equation 1. So, for solving the equation 1 we can do it by some numerical process, but in the beginning to a start with the numerical process we need some guess for the eccentric anomaly, so we can write equation 1 as E minus M is equal to e sin E.

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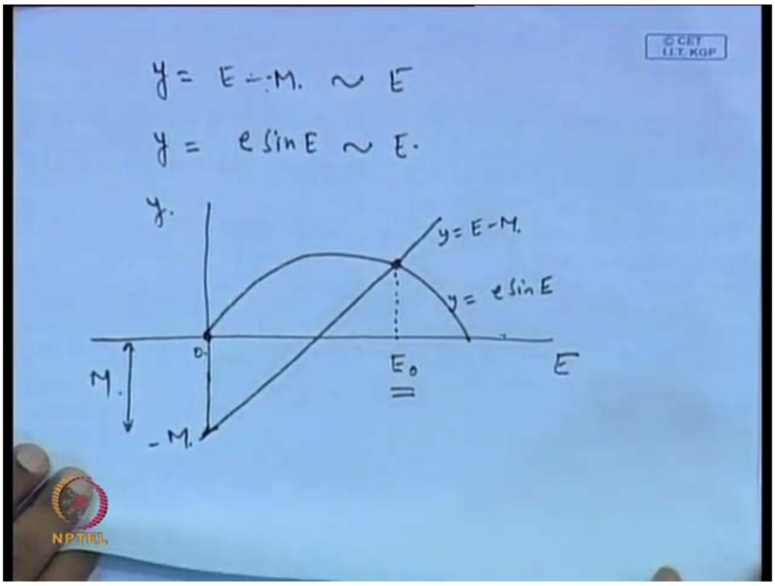

So on the solving it, through the graph so what we write, y is equal to E minus M and we write y is equal to e sin E and plot this verses E. So, this y is equal to E minus M and plot this verses E. So, if you do this plot we are plotting y on this axis, E on this axis. If E is equal to 0 so this is the origin here, E p is equal to 0 y is equal to minus M. So, this is the point, here minus M quantity from here to here this is M. And the slope of this curve is dou y by dou E this is positive therefore, this will come like this. Now we can plot y is equal to e sin E verses E so if we plot, y is equal to e sin E, this curve we gets something like this. So, the cutting point of this 2 intersection of this 2 curves so that will give us y, this is y is equal to E minus M and this is y is equal to e sin E. So, the intersection it gives us the starting value for the E this is nothing but E0.

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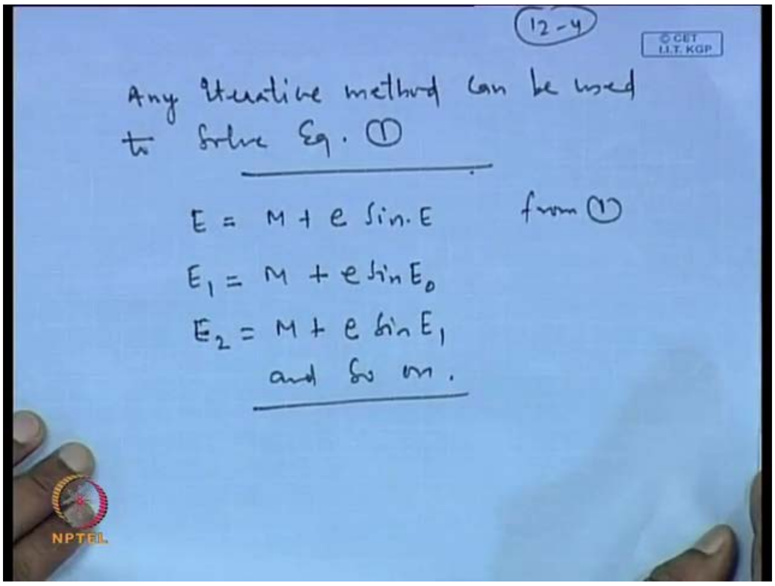

Now we can use any iterative method to solve this equation 1. So, suppose we write E is equal to M plus e sin E this is from1. Therefore, I starting with E is equal to E0, we can write E1 is equal to M plus e sin E0. Next time we can write E2 is equal to M plus once, we evaluate this numerically and there after we can find E 2 is equal to e sin E 1 and so on. So, if we iterate this process then we this will converge and we get the value for the E, which will nearly satisfy the equation number 1.

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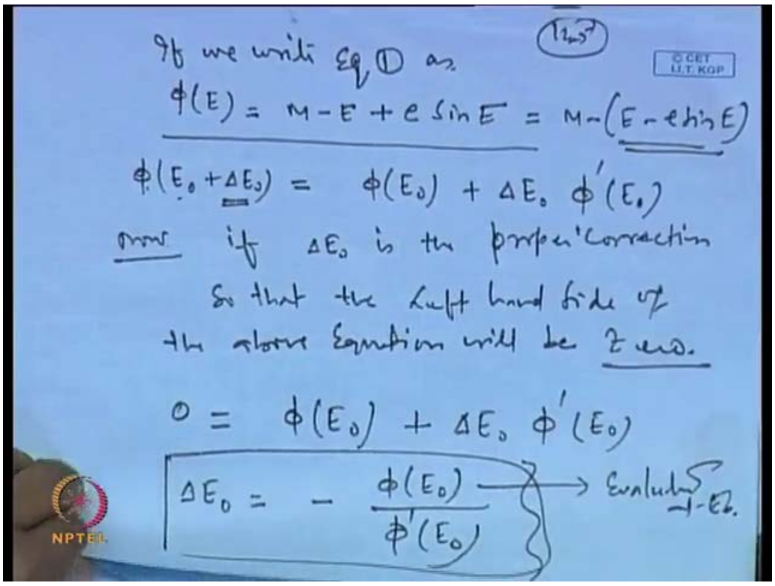

So, another way of doing solving this can be written as, if we write equation 1 as phi E is equal to M minus E plus e sin E and we do the Taylor expansion of these to the first order, then we can write E 0 plus delta E0. So, this can be written as phi E 0 plus delta E0 times phi prime 0.

Now, if delta E0 is the proper correction so that here we are doing, we are minimizing the difference this is exactly this we can write as M minus E minus e sin E. So, the phi is a function which is a difference between the mean anomaly and this function. So, we are trying to if we E is the proper solution then the minimum of this should be 0. So, if we get the proper solution so minimum of this will be 0 so here, we have written this in the first order to the first order expansion Taylor series expansion. Therefore, if this quantity the delta is the proper correction given to E 0, then this must be a minimum and therefore, this must get reduced to 0 so that the left hand side of the above equation will be zero.

So, we said the above equation left hand side to be 0 so this becomes phi E0 plus delta E 0 phi prime E0. So, from here we can get the value of the correction delta E0 this will be nothing but minus phi E 0 divided by phi prime E0. So, these are evaluated at so this function evaluated at E0.

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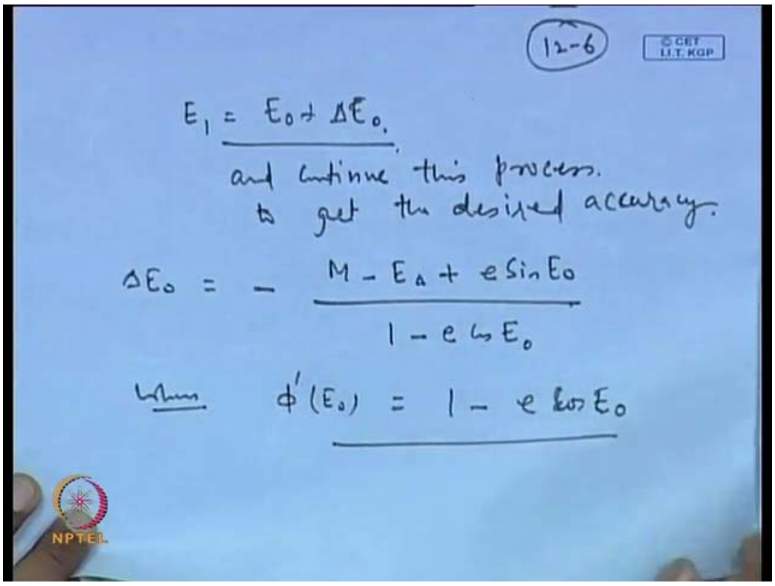

Now, once we know the correction then using that correction, we can write E1 is equal to E 0 plus delta E 0 and continue this process. This process, iterate it to get the desired accuracy as much you want. So, here we have delta E 0 this is in the previous step, we have written this is nothing but phi E 0.So, phi E0 is nothing but M minus E 0 plus e sin E0 and the derivative of this will be we are taking the derivative of that, so that will be nothing but1 minus e cos E we are taking the derivative with respect to E, where phi prime E 0 this we have written as 1 minus e cos E0. So, after iterating this process we get the proper value for E.

Now we come to the Kepler's problem, so as we have stated in the Kepler's problem that Kepler's problem can be precisely a stated as give the position in velocity of the satellite at time T0, you find the velocity and position of satellite at another time t. So, there are various methods for solving this we do not go into this, because the numerical integration precise numerical integration method are available in such as the Runge Kutta scheme or either the Gauss version method what is called the predictor corrector method. So for using any one of them we can propagate the position and velocity of the satellite with respect to time.

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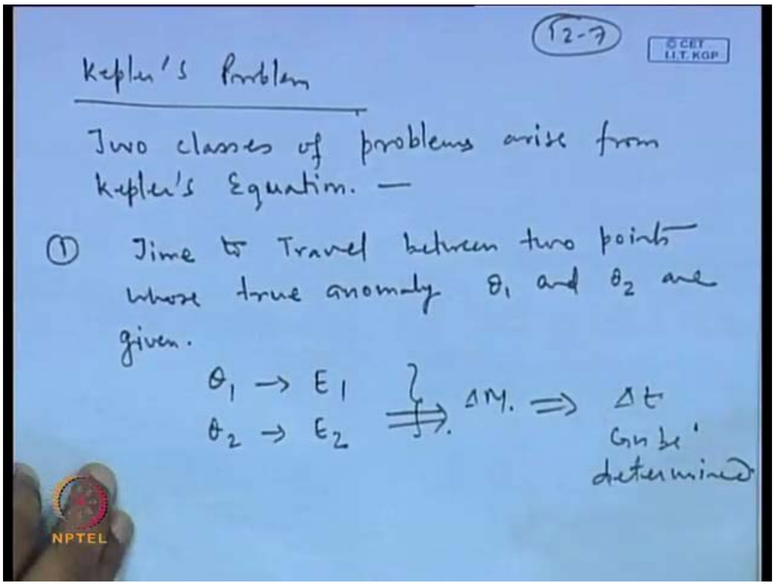

So, here we just look into the description of this problem, so we have the Kepler's problem. So, two classes of problem arise from Kepler's equation. One is time to travel between two points whose true anomaly theta 1 and theta 2 are given. So, here in this case theta 1 and theta 2 are known and then we need to solve find out the delta t. So, theta 1 and theta 2 are give so theta 1 will give us E1 and theta 2 will give us E2. So, you can find delta M from here and delta M the difference in theme an anomaly between this 2 positions and therefore, from here this delta t can be obtained, so delta t can be determined.

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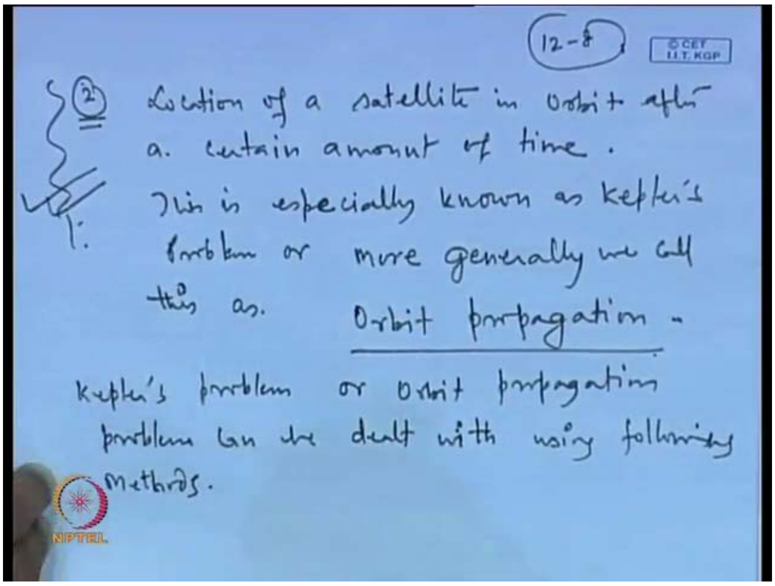

The second problem is the location of a satellite in orbit after a certain amount of time. So, this part this is especially known as Kepler’s problem or more generally we call this as orbit propagation or just the propagation. So, we can call this as the orbit propagation. So, the Kepler equation it Kepler’s problem it basically deals with the orbit propagation. So, for the orbit propagation as I told earlier that either the numerical method can be used or either the Kepler’s equation itself can be used, where we propagate the true anomaly and solve true anomaly and then the other orbital parameters are constant. Therefore, we can calculate the position and velocity at any other time.

But the propagating in terms of the orbital elements we are using, because here the orbital elements, the five orbital elements are constant only the true anomaly is varying. So, this case is easier that suppose the case where the all the orbital elements are varying and that happens in most of the cases in the this our solar system it happens, because the there are various perturbative forces it is not just the two body problem, where the central body is just a spherical one. But even in the two body problem if the central body is not a spherical one, it is bulged just like the earth and around this the one satellite. Most then satellite will not remain in the same orbit that is all the orbital parameters will change with time because of the bulged of the earth. Therefore, if we consider the solar system so in the solar system if the earth is moving around the sun, but simultaneously there are other planets also and those planets they perturb the orbit of the earth. So, the orbit keeps changing but obviously it happens over a long period of time.

So therefore, to get the general orbit solution forit in body problem the as we have stated earlier that even for two body problem in the true sense, we are not getting the absolute motion. But we are not solving for the absolute motion, but only for other relative motion and for the relative motion we got the Kepler's equation and we solved it.

But, if the other perturbative forces are available as we will see latter on for the 3 and n body problem then, we have to do this by numerical integration. There is no other way of doing this the analytical solution it is a only available in few restricted cases. So, we have the problem 2, this the Kepler's problem which is the statement number 2 here. This can be dealt with so Kepler's problem or the orbit propagation problem using following methods.

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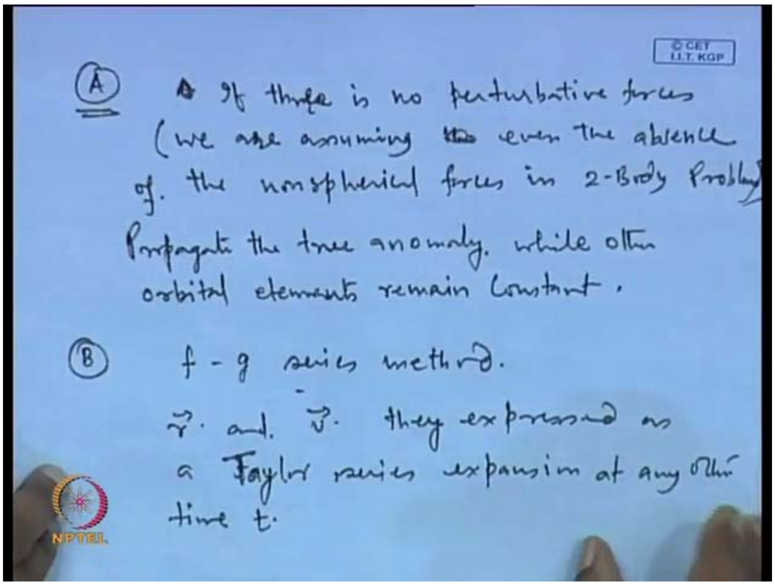

So, as I stated you earlier if there is no perturbative forces, we are assuming even the absence of the non-spherical forces in 2 - Body problem. Only the in the two body problem only the spherical forces are considered. So, if there is no perturbative forces then we can propagate the orbit so the orbital elements here the propagate the true anomaly, while other orbital elements remain constant. So, another method can be it isa called a f g series method. So, in the f g series method r and the velocity vector they are expressed as a Taylor series expansion at any other time t.

And can be therefore, r and v at time t can be obtained. So, we are not going into the details of this methods f and g series method or either the (( )) method. If you are interested for this particular orbit propagation problem so you can either look into the book by P R Escobal it is called the orbit determination, it is a book on orbit determination. So, by P R Escobal very classical book so you can refer to this and it greatly describes this f g series method.

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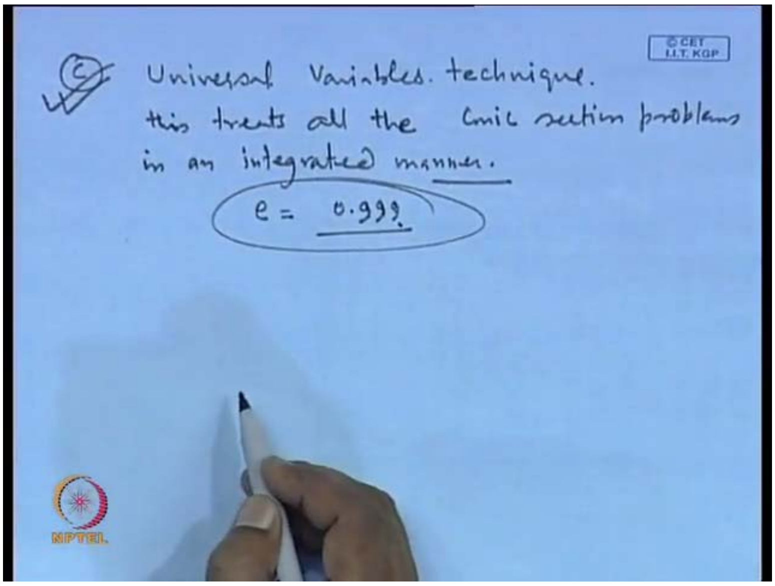

Then the third one is the universal variable technique, so as we have seen that we have to create the thesis for the parabolic or the elliptical or even the, for the hyperbolic orbit will have to treat them separately, the method which we have been following. So, if we follow that method then the separate treatment is required, but if we want to integrate them in one so this is called the universal variable technique, again which we are not going into the details of this method, you can refer to the standard text books which is mentioned in the syllabus.

So, universal variable techniques this treats all the conic section problems in an integrated manner. So, in this technique in the first one, if there is no perturbative forces the propagate the true anomaly so in this what we have done that given the position r and v. So from here, we can get the classical orbital elements we get a e i a small omega, capital omega and theta. Then from here propagate theta, so propagating theta then again we get the value of theta at time t, propagate theta 2 at time t and then again convert from a e to and then you get it back to suppose, you are start with r 0 v 0 so can you get r v at time t.

While the v that we have considered this f g series method so this is only applicable in the spherical forces at present. So, any perturbation this cannot be applied or if the say the we have a satellite for that, we want to get the position and velocity at any other time t. Then in that case in a f g series it will be difficult to apply it, because it will be difficult to apply this f g series method when the satellite this thrusting. Because in that case the extra terms are available beside the spherical terms, where will be the thrusting terms which will be accelerating the satellite, so again this method will not be applicable.

While the universal variable techniques, this is mostly used in the boundary cases where the e is equal to 0.999. So, it says whether we can write this as the this can be written as the say the for a parabolic orbit also because this nearly equal to1 and this can be also worked as an elliptical orbit. So, in such boundary cases this universal variable method is very useful. So, we conclude our two body problem here and proceed to the three body problem.

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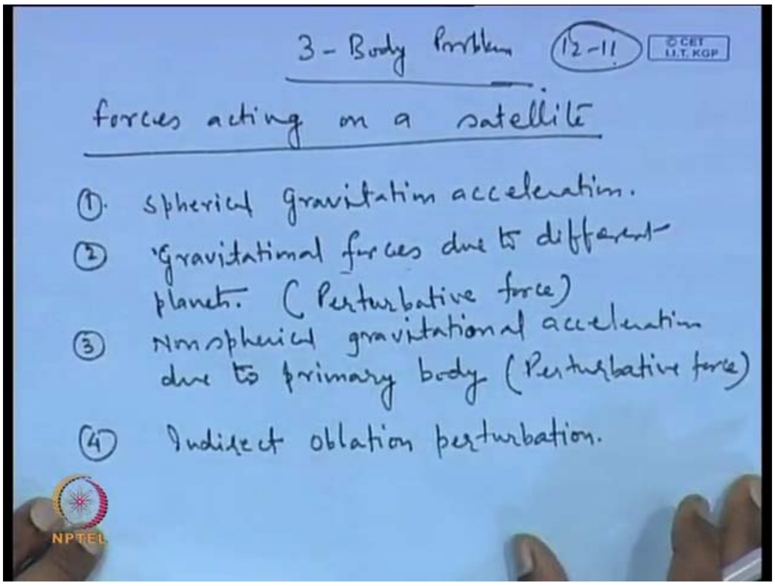

So, already we have been working with the two body problem, so we looked into the equation of motion of the two body; equation of motion of two body, is the way we worked out the initial things for the two body problem. In the same way, the three body problem proceeds but as obviously we have told earlier that for the two body problem, even we cannot solve the absolute motion in absolute motion or the two body problem.

So, similarly here also (( )) what we cannot do it, but in the case of two body problem it happen that at least, we had the solution for the relative motion and we got a lot of insight into how the orbit, will look like and how to propagate the orbit. So, it is a very useful in the case of suppose we have a satellite around the earth and we want to look into this problem.

So, obviously in some preliminary analysis can be done using the two body method and but for getting very useful result, very reliable result then we have to go through the three body problem. And the three body problem solution obviously as we have stated in that even the two body case, we cannot solve now in even in the case of the three body problem. It is still no solution exist, but some solution I have been attempted so we go through that for a restricted most (( )) solutions are available, once we impose certain conditions then we can solve the three body problem.

So let us start with this write forces, so here we are in a space (( )) we are specially interested not in the heavenly bodies, what we are interested in the satellite motion. Therefore, we consider what are the forces acting on a satellite, so this forces are the spherical gravitational acceleration that is we are considering the mean body, what is the satellite is moving the what we call as the primary body around which the satellite is moving to a spherical, then this force will be present only.

Then, we can have gravitational forces due to different planets so this we can assume that the planets to be the point mass, then this will be the gravitational force on the satellite will be acting and truly it affects the orbit of the satellite. Then the third one we can write as non-spherical gravitational acceleration due to primary body. So, all these three they are falling under the gravitational attraction term. Then the fourth one we can write as indirect oblation perturbation so these are act as, they act as perturbation. So, this is perturbative force, this is also perturbative force, so indirect oblation perturbation so this also is perturbative force.

So, from where it arises it arises the due to the non-spherical shape of the other planets and it is a indirect force, because say the here the non-spherical shape of the planets affect the acceleration of the primary body. So, in this case it is earth so the primary body or what we call as the reference body.

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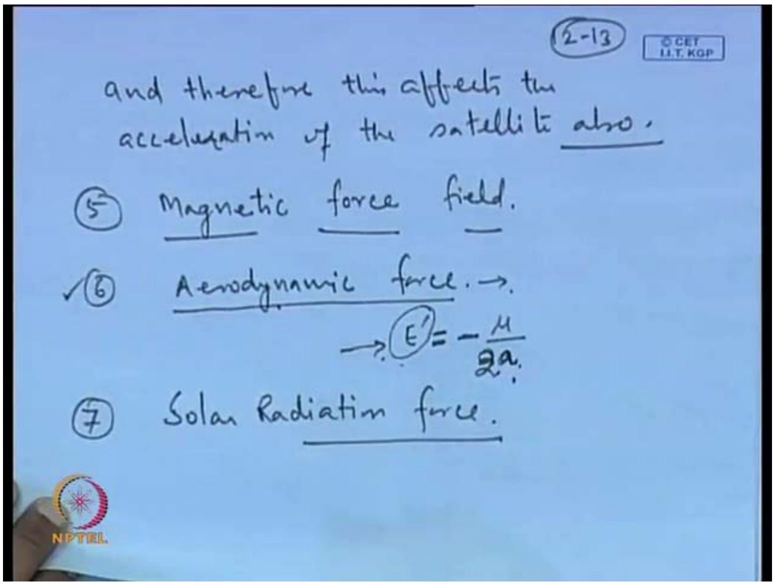

And therefore, this affects the acceleration of the satellite also. So, this is the indirect force which they affect the non-spherical shape of the planets, they affect the primary body which is very large in shape, large in size and therefore, once the primary body gets affected so the satellite moving around this, it will be affected also. So, therefore it is a called the indirect oblation perturbation.

So, then we have the fifth one magnetic force field so in the case of our the earth is having the magnetic field around, this is because of the in the core of the earth, there is a liquid molten iron is there and because of the some motion it is a rotating the inside and therefore, it generates the magnetic force.

So, if we have a satellite on which if it carries current or any charge, so if it passes through this satellite is passing through the magnetic field in the orbit, then obviously it faces the magnetic force because of that motion. Then we can have the aerodynamic force, so if the satellite is in the low orbit like in the case of a if the satellite is in the 300 kilometers orbit or of that so not very high altitude because the aerodynamic force it depends on the air, so as the air becomes thinner and thinner, as the air gets (( )) so the aerodynamic force will go down.

So it goes down drastically with a altitude therefore, that high altitude the aerodynamic forces are negligible, but for the 300 kilometers orbit this is very high and in fact the aerodynamic forces they are responsible for decay in the total energy of the satellite. As, we have seen earlier this is the total energy of the satellite of per unit mass this is written as mu by 2 a.

So, if the energy of the satellite this decays with time, due to the aerodynamic force and therefore, they will change and so it (( )) the satellite to fall towards the earth. So, if you live a satellite at low altitude and you do not have any propulsion device, then over a period of time it will enter into the earth atmosphere, where atmosphere is thick and ultimately it will burn.

So, the seventh force we can mention as magnetic force, aerodynamic force and the last one we can write as the solar radiation force. So, the solar these are tangible forces and they can be used even for propagating the orbit of the satellite and sometimes this also refer to as the devices, which are used to solar as the solar sail as on the boat, we we can have sail so similarly, we can have sail on the satellites and it can move the satellites over a period of time.

So though not very large, but still it has because of the protons they have in a momentum and once this momentum if it is absorbed by the satellite, totally then that momentum is imparted to the satellite and therefore, that changes the velocity of the satellite. So, this altogether the seven forces they act on the satellite to change its velocity and position and some of them can be used very actively to model to change the orbit of the satellite like the aerodynamic forces, they can be used actively to change the orbit of the satellites. If we have flats on the satellites, so we can put it in proper direction on the proper angle to induce more drag or less drag and accordingly we can maneuver the orbit.

So similarly, the this the magnetic forces are there so magnetic forces can be used nowadays, the (( )) forces also can be used (( )) forces they implied that we have the a charger there are number of satellites in the orbit they are making a formation. So, if we have the charges on different satellites, so they apply force on each other.

And therefore, because of this force is acting on them so if be control, we will be able to control the formation, we can change the formation of the or the shape of the formation. Therefore, there are number of devices as the technology is improving so number of ways of maneuvering or maneuvering the satellite it is a becoming available. So, this is a new proposed thing is still it is not implemented but now in theory it exist.

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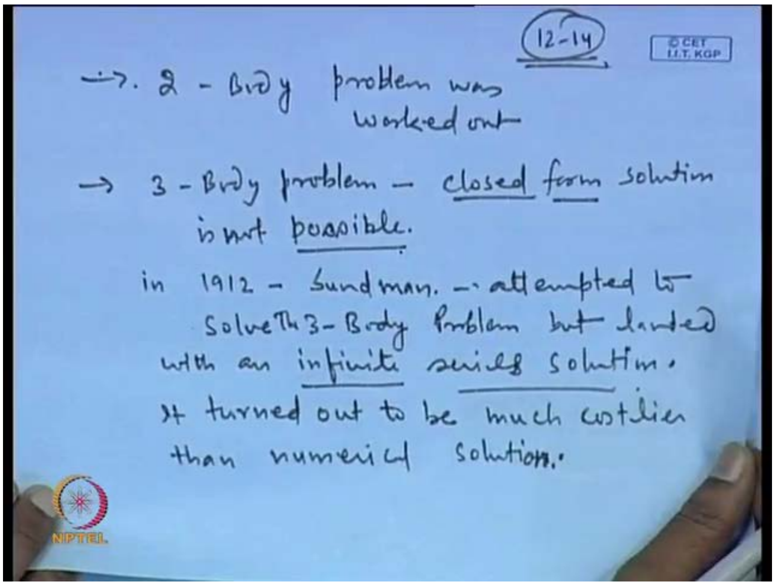

So, we have till now the 2 - Body problem was worked out. For 3 - Body problem closed form solution is not possible. So, in 1912 Sundman, attempted to solve the 3 - Body problem but landed with an infinite series solution. So, no closed form solution, closed form solution has very big advantage that it requires very less computation and it gives a lot of insight into the motion of the body. So, here the infinite series solution is there so it cannot give insight into the motion of the body. The second problem it requires a lot of computation. So, in this case it happened that even it was much more costlier than the numerical solutions, so it turned out to be much costlier than numerical solution. So as of today, we do not have any general solution for the three body problem but we have the restricted solution for the three body problem.

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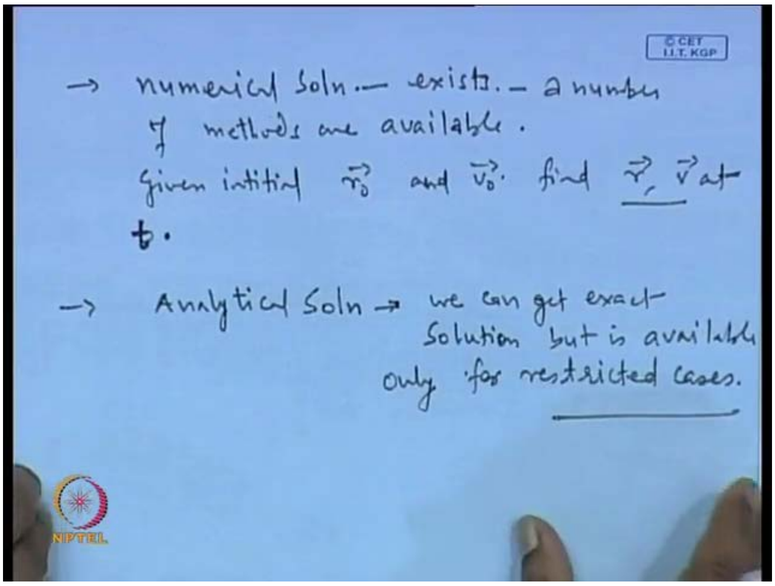

So, then we have the numerical solution exist, numerical solution it exist and number of methods are available. So, given initial position r 0 and v 0, find r v at t. So, the equation of motion can be written and under the action of various perturbative forces and can be numerically integrated to give r v at any other instant of time. So, obviously under it depends on the accuracy of the method, how accurate your method is if the method is not very accurate, then error will build up over a short period of time. If method is very good, then error will be less over a long period of time.

So and moreover the more precise your formulation is means if you take into account, all the forces acting on the satellite then precisely you will be able to predict the orbit of the satellite means it is the position and velocity in the future. The more you do the approximation the in formulating the problem, in writing the equation of the motion, suppose we delete the magnetic forces or the aerodynamic forces and some other the indirect oblation forces. So this is pretending towards the approximation, we are trying to approximate the motion using the less number of forces then obviously if you try to propagate the orbit over a long period of time, so it will produce very large over, a large error over a long period.

So, those kind of things are for interplanetary motion this kind of things are not accepted then we have to be very precise in formulating the problem and also in integrating. So, for integration also we require a very precise method, so that we get better precision and better position of the satellite in the orbit. So, then we have the analytical solution but this is available for we can get exact solution, but is available only for restricted cases means the general solution analytical solution is not there.

However, without solving the equation, you can study the property of motion just like earlier we have in the two body problem, even we saw that the total angular momentum or the 2 satellites that is the angular momentum of the system, it remains as a constant. So, this gives insight into the problem that the plane of the orbit of the 2 satellites, it will remain fixed because the h is in that case constant and the h is the angular momentum vector which is perpendicular to the plane of rotation of the 2 satellites or the 2 particles of the two body.

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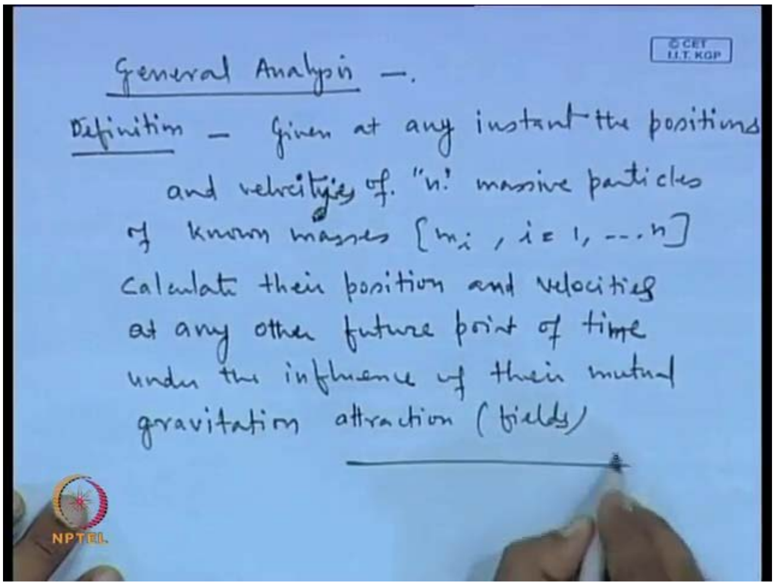

So, general analysis definition has given at any instant the position and velocity of n massive particles of known masses, i 1 to n so for in particles system given at any instant the position and velocity of n massive particles of known masses, calculate their position and velocities, the positions and velocities, position and velocities. Here, future point of time under the influence of their mutual gravitation attraction.

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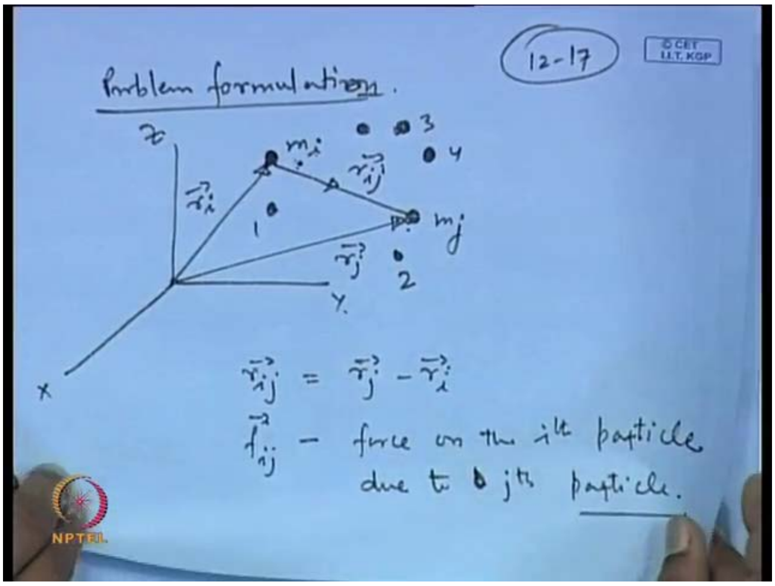

A problem formulation, so already we have discussed about the two body system and for the three body system it is a just a generalization of that, so then there is not much difference only if you mathematical techniques or the mathematical simplification required, just other things remain as it is. So, let us consider any inertial reference frame X Y and Z and we have a number of particles there; so we have a number of particles and we want to formulate the equation of motion for the ith body. So, here the number of particles say 1234 and so on these are the particles so under the their mutual gravitational attraction, how the particle the ith particle will move.

So obviously, we will have a lot of insight into this problem through the not a general solution, but general solution obviously we cannot get but some particular methods as we have done last time, that we take a cross product then we get the angular momentum and we get the dot product then, we get the total energy. So, if we apply those kind of techniques so similarly, we can get the similar kind of equation which can we used to derive some insight into this problem. Otherwise in general form, we cannot solve it. So, we can write here let us say, r i j this is equal to r j minus r i and let us also write f i j is the force on ith particle due to jth particle.

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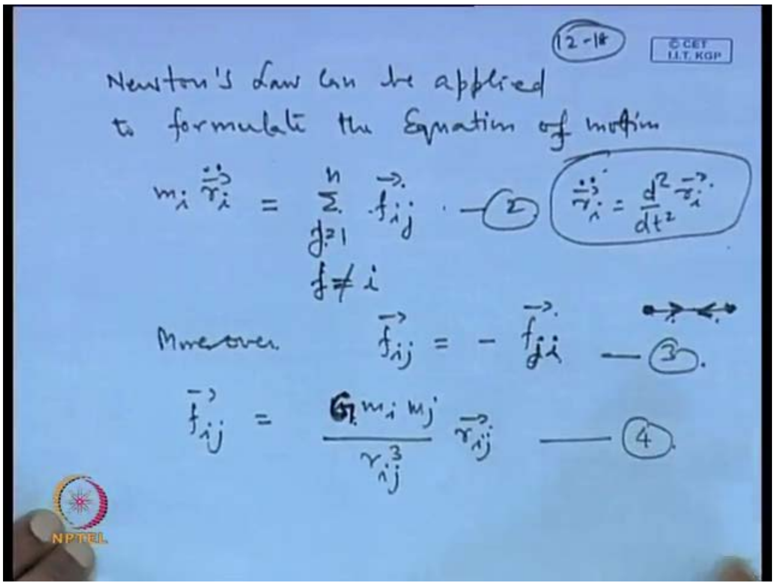

Newton's law can be applied to formulate the equation of motion. So, the motion of the ith particle it can be written as m i r i double dot, this is equal to j is equal to1to n f i j where j is not equal to I; here r i double dot implies d by d t squarer i. So, this is our first equation, this is the second equation. Moreover, f i j this will be equal to minus f j i that is force acting on the ith body due to the jth body, will be equal to the force acting on the jth body on the due to the ith body, but they are acting in the opposite direction because the forces they act in the opposite direction.

This is our equation number 3. f i j can be written as k m i times m j r i j whole q times r i j, you can see the force acting on this particle is nothing but m i times m j product of these two masses and then it is a distance between and it is a directed from here to here from i t h to jth particle, so the direction is r i j and then we divide it by r i j whole cube so this is nothing but f i j the force acting on the i t h particle.

So, the same thing we have written here and we multiply it by the gravitational constant G or we can write it as a K, so here let us write it as G, because most of the times we have using that notation for in your gravitational constant. So this is G times

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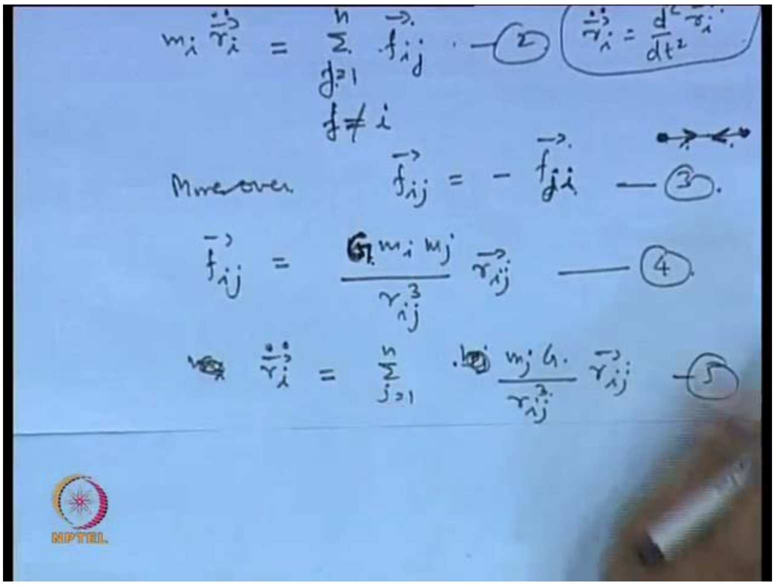

So, the equation of motion of the i t h particle, the acceleration can be written as summation j is equal to1 to n m j times G divided by r i j whole cube times r i j. So, our time is getting over, so we continue with this lecture next time.

Thank you very much.