Space Flights Mechanics

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Lecture No. # 11

Two Body Problem (Continued)

In the last lecture, we have been working with the orbit determination problem. It is a part of the two body problem. So, in this lecture we continue with that the orbit determination problem, it has many sub parts. So, for the last time what we did that given the initial position and the velocity, then we get the orbital elements or vice versa. If the orbital elements are given at certain instant of time, so at that instant of time you find out the position and the velocity vectors of the satellite.

So, there are certainly other components into these orbit determination problem. So, if we call this as the Kepler's equation. So, Kepler's equation basically it deals with the propagation of the true anomaly. So, under the assumption that the earth or any heavenly body of spherical mass. So, if the orbit of a satellite under such situations is spherical and homogenous mass. So, orbit of the orbit under such situations it remains in either an elliptical or as the circular orbit.

So, if it is like that or it may be depending on the initial position and velocity even for parabolic or as a hyperbolic, but the main thing is that there are no portability forces available at that time. So, considering only two body this two body problem. So, if the portability forces are there, then the orbital elements they change with time.

So, here in our situation portability forces are not present and therefore, only the true anomaly it changes with the time means, the position of the satellite in the orbit it changes with time and that position is here we indicate, in terms of the true anomaly that is theta. Other parameters etcetera, like semi measure axis, angle of inclination, nodal angle argument perigee and eccentricity they all remain constant.

So, under this assumption, that all these five elements are remaining constant and only the true anomaly is changing with time. So, if we work for the Kepler's equation which is just projecting this that the five elements remaining constant and then theta is varying.

So, at the time thow the present time is theta is known, so how at the next time the theta will be given.

This is part of our Kepler's equation and there after we will go into the Kepler's problems. So, Kepler's problem it is related to the Kepler's equation not much different, but little bit technically different. So, we will look into that afterwards, but we will not develop the Kepler's equation. Fine as in the full form, because that involves the propagation of the ephemeris completely means the given the position and the velocity of the satellite at time t0 then, find out the position and velocity of the satellite at any other timet.

And, there are different methods for that and obviously one method is doing the numerical integration. So, we will avoid that because of this in this particular course. All these details cannot be included, so we go with the Kepler's equation.

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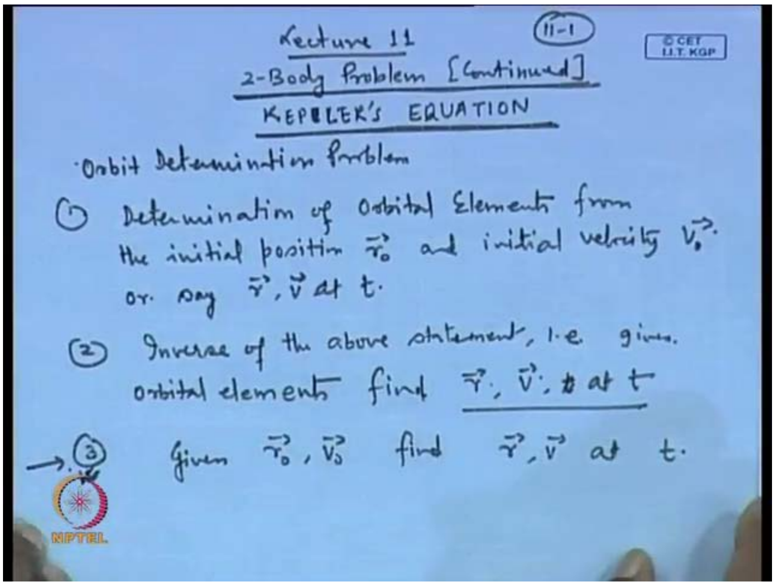

So, in the orbit determination problem till now, what we have done orbit determination problem it can be divided into determination of orbital elements from the initial position r0 and velocity v0 or say r v at t. The second problem is inverse of this, so the inverse of the above statement that is given orbital elements, find r v at t. The third is given r0, v0 find r v at t. So, this problem this is specially referred to as Kepler's problem the third one, the third statement here.

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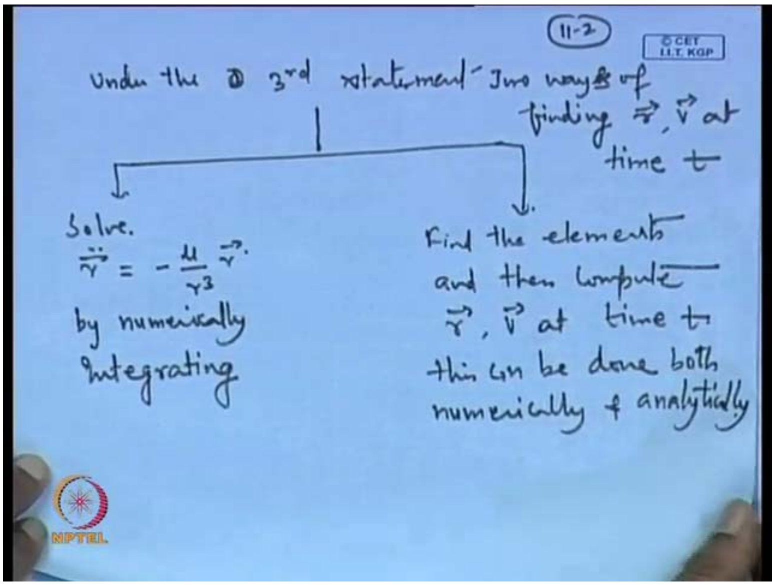

So, under the third statement under the third statement we have two ways of finding r v at time t. First is to solve, r double dot is equal to minus mu by r cube r, by numerically integrating, numerically integrating The second one is to find the elements and then computer v at time t. This can be done both numerically and analytically.

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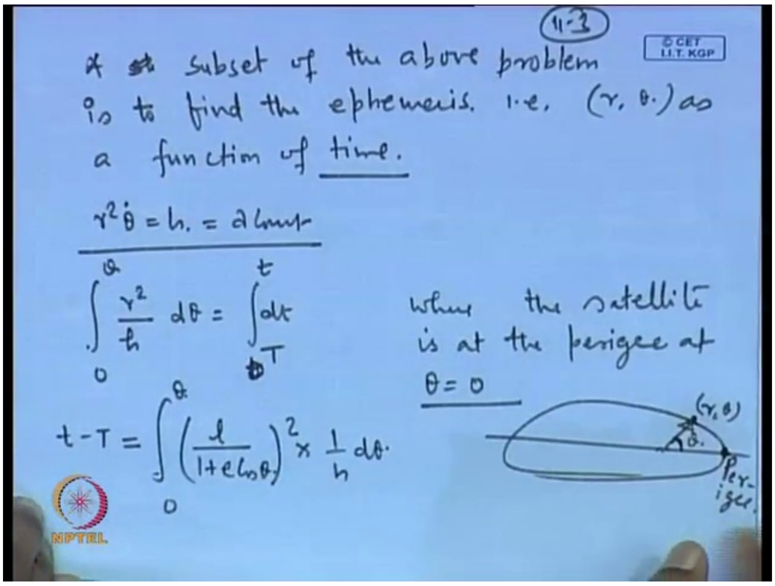

A subset of the above problem is to find the ephemeris, that is r and theta as a function of time. Now given the equations, we know that r square theta dot this is equal to h, this is a constant. Therefore, from here we can write r square by h, d theta is equal to d t we can integrate between say the starting time T to t and the theta from0 to theta, where the satellite at the perigee at theta is equal to zero. So this is theta, the coordinate of this point is r theta, so this point is the perigee point.

So, we are measuring theta from this place and t is also being measured from this place. So ,if we write this point as the capital T or we can put a equally zero here in this place, if we count time from this place so find out solve this equation. This we can write as t minus T and this becomes now l we know this r is nothing but 1 by 1 plus e cos theta. So, r square becomes this square and divided by h, d theta, 0 to theta.

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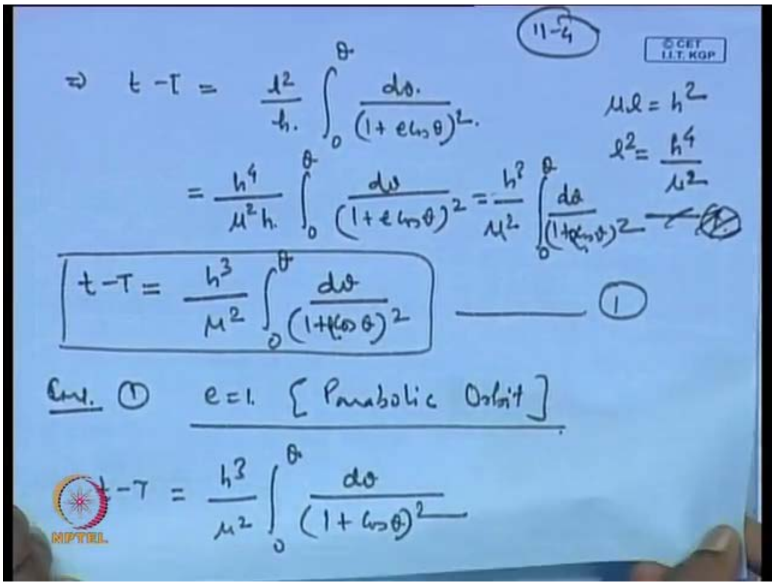

So, this implies t minus T, this will be equal to l square divide by h, 0 to theta, d theta by1 plus e cos theta whole square. Now we know that mu times l this is nothing but h square. So, this implies l square will be h to the power4 by mu square. So, we can put here in this place h to the power4 by mu square h times, 0 to theta the whole square. So, this is h cube divided by mu square, 0 to theta, d theta by1 plus cos theta whole square.

So, this is our equation number one. Now or either we write here, h cube by mu square0 to theta, d theta by1 plus cos theta whole square, this is equation number one. So, solve this equation; so solving this equation requires certain manipulations so, we first solve it for the case where e is equal to1.

So, here we have missed out e cos theta we put here e, e cos theta. So, we deal with the case1when e is equal to1.So, this is e is equal to1 for a standing for the parabolic orbit so for e is equal to1, t minus T becomes h cube by mu square, 0 to theta, d theta by1 plus cos theta whole square and this is easy to solve in integration problem.

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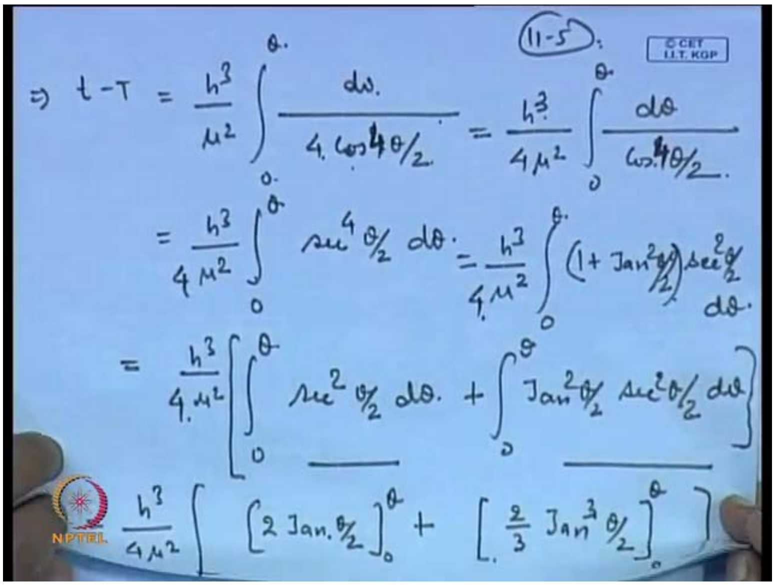

T minus t, h cube by mu square, 0 to theta, d theta by now1 plus cos theta we can write this as 4 cos square theta by2, 4 times cos square theta by 2. Now ultimately this will give h cube by 4 mu square, 0 to theta, d theta by cos square theta by 2 cos square theta by 2. So, this will become 4 because,1 plus cos theta is equal to 2 cos square theta by 2.

So therefore, this becomes 4 cos to the power 4 theta by 2 and here also we have to put cos to the power 4 theta by2 because, here we have1 plus cos theta square. So, here this quantity will be 4. Now, changing cos to the power 4 theta to sec theta term, so this is becoming sec to the power4 theta by2, d theta.

So, this can be further written as 0 to theta 1 plus tan square theta times; theta by2 times sec square theta by 2, d theta. Now this can be easily integrated so we break this into two terms h cube by this is 4 mu square, 4 we have taken outside. So, this is 44 mu square, 0 to theta, sec square theta by2, d theta and plus tan square theta by 2 times sec square theta by 2, d theta.

Now, these are broken into two simple integration terms and therefore, it can be easily seen that this quantity can be written as sec square theta this is nothing but, if you take the tan term, so this can be written as 2 tan theta by2. So, we will put here between 0 and theta.

And plus if we differentiate, if we can check from here if we have tan 2 theta by 2 so if we differentiate that, so we get this term tan square theta by 2 times sec square theta by 2. So, from here we can write 2 by 3 tan to the power cube theta by 2, between 0 and theta. So, we can check it this will become 3 tan square theta by 2 times sec square theta by 2 and then differentiating theta by 2, this will give1 by 2 so, if we use for that terms2 by 3 cancels out the 3 and 2 and therefore, this 2 and 3 has appeared here 2 by 3 has appeared.

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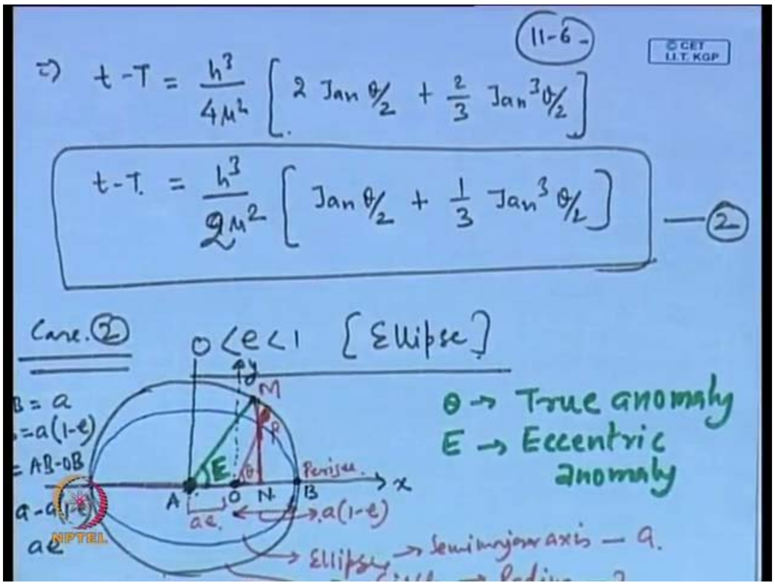

So this implies, t minus T this is equal to h cube by4 mu square. So, from this equation if we insert this boundary values 0 and theta. So, we get this as 2 tan theta by 2, plus 2 by 3 tan cube theta by 2, h cube by 4 mu square and this become 2; 2, 2 we cancel out. So, this is tan theta by 2 plus, 1 by 3 tan cube theta by 2. This is our equation number two. So, this equation with the relation for the theta so if you know the time t, so at that time t what will be the value of theta can calculated by solving this equation.

Now, the case2 when e is less than1,so this is the case of an ellipse. So, here we use certain results from our geometry this is theta suppose, this the point p and we extend this line above, this is the line perpendicular to this axis so extended it and this meets the this point here so the inner one is an ellipse; this is an ellipse and this is a circle so, ellipse semi measure axis.

Semi measure axis a and circle radius is also a. So, next we have the center of the circle here so we join this point from here to here, the angle theta here is true anomaly and this angle this is indicated by capital e and this angle is eccentric anomaly. This point we write as n, this is o this axis we write as x and here this axis we indicate as y, so the distance from the focus to the perigee point of this ellipse this is nothing but, this quantity is from here to here, this is a times 1 minus e.

A times 1 minus e; so this is our perigee point. Let us say, this is here we write it as perigee here, the distance then from here to here this will be a times e this can be easily calculated, we can do the calculation here itself. We have here let us, write this point as a and this point as b then a b is nothing but, a and ob is we note this is a times1 minus e. Therefore, u will be A B minus O B so, a b is a minus and o b is a times1 minus e. So, this cancels out and this leaves us with a e. So, this is the quantity we have shown here.

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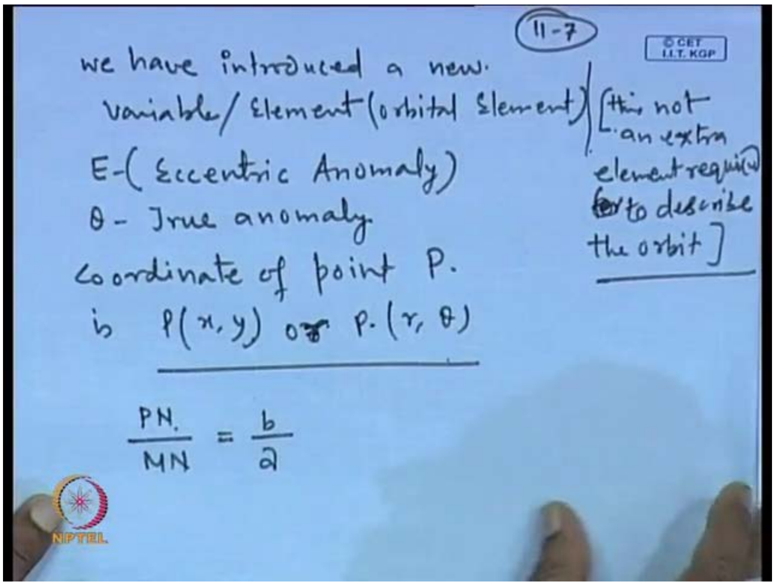

So, now from here so what we have done, we have this is sixteen. We have introduced a new variable -variables slash elements known as eccentric anomaly. So, new orbital elements this is orbital element. So, but one should remember that this is not an extra variable, this is not an extra element required for required to describe the orbit because the orbit is described always by total six number of elements.

Therefore, either we use the true anomaly or the eccentric anomaly just now, we have introduced both will work because they are related together. So here, we have introduced a new orbital, new variables plus element the orbital element, E which is called the eccentric anomaly and theta is our true anomaly so here, we try to find out the relation between E and theta first and moreover this can be done geometrically, but we will try to go here using the analytical method. So, the coordinate of point p- point p is p x y and this we can also write as p r theta or say or p r theta.

So, now from our geometry you can look into your earlier higher up geometry book P N by M Nit can be written as b by a.

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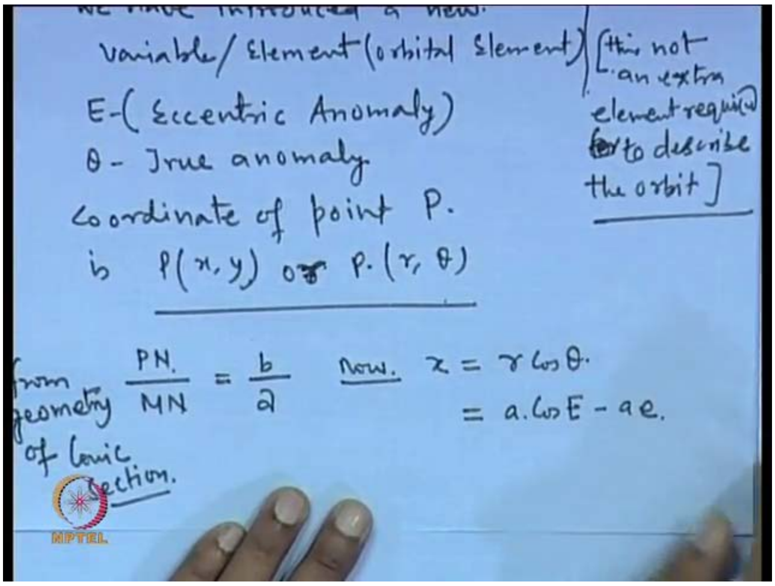

Now, x is equal to this is from geometry from geometry of conic section. So, x becomes equal to rcos theta and we can also express the same things in terms of the eccentric anomaly so, this can be written as a times cos E minus a e.

Here this quantity is this, is the radius of the circle is nothing but a. So, the projection of this on this axis from here to here is a cos e and from there then, we are subtracting the quantity a e which is the quantity from here to here and therefore, we are getting O N. So, this quantity x is equal to rcos theta this is nothing but, O N here, basically what we have written this is nothing but A N minus A O so, x is equal to.

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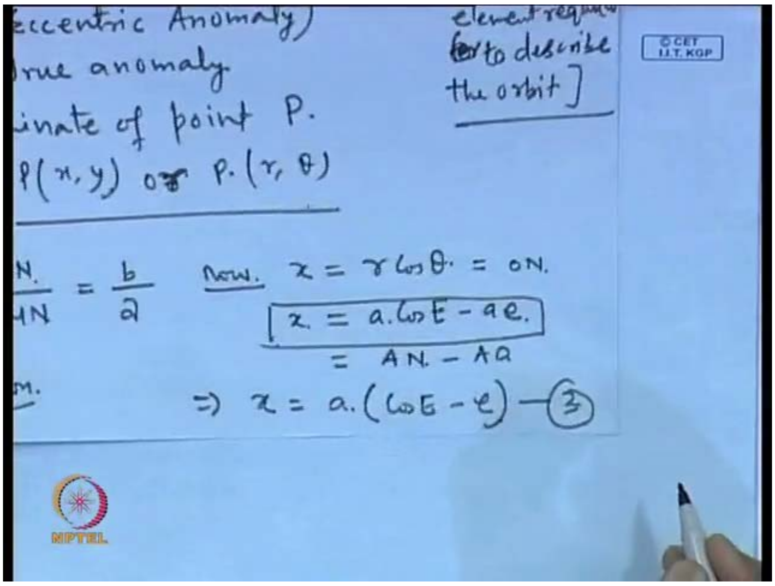

Now similarly, we can little bit modify this - this is make it compact a times cos E minus e. So, this is our equation number three.

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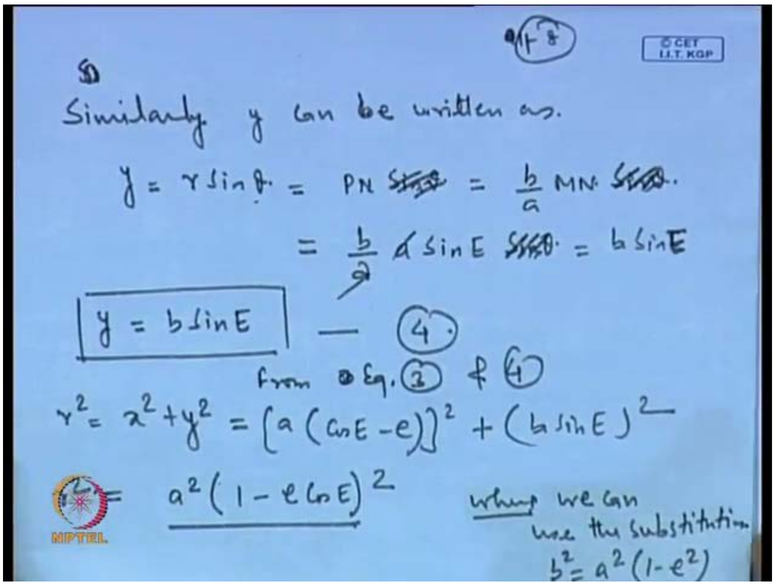

Similarly, y can be written as y is equal r sin theta. This is the point p, so this, the quantity r from here to here this vector is r therefore, this is our y and this is x, so y becomes r sin theta -r sin theta so this is nothing but, P N sin theta. And we can use the relationship, that we have just now written so we can us the relationship P N by M N is b by a. So, this is P Nis equal to b by a times M N sin theta and M N is the quantity here which is, you can see from here this is the quantity from here to here M N is nothing but a sin a. So, M N is a sin E this a cancels out this becomes b, r sin theta is equal to and this theta cancels out, this is equal to P N. We do one correction here, y is the quantity from here to here, so this is r sin theta so we have written r sin theta here and this is the quantity nothing but, equal to P N so this is not P N sin theta because r is not equal to P N so we delete this, so this is b sin E. So, y is equal to b sin E this is our equation number four.

Now therefore, r square can be written as x square plus y square. So, from equation three and four, we can write r square is equal to x square plus y square. So, x we have written as a times cos E minus e so, this whole square and plus y square is b sin E, this whole square. So, you can reduce it to a square times1 minus e cos E whole square. Where we can use the substitution b square is equal to a square times 1 minus e square

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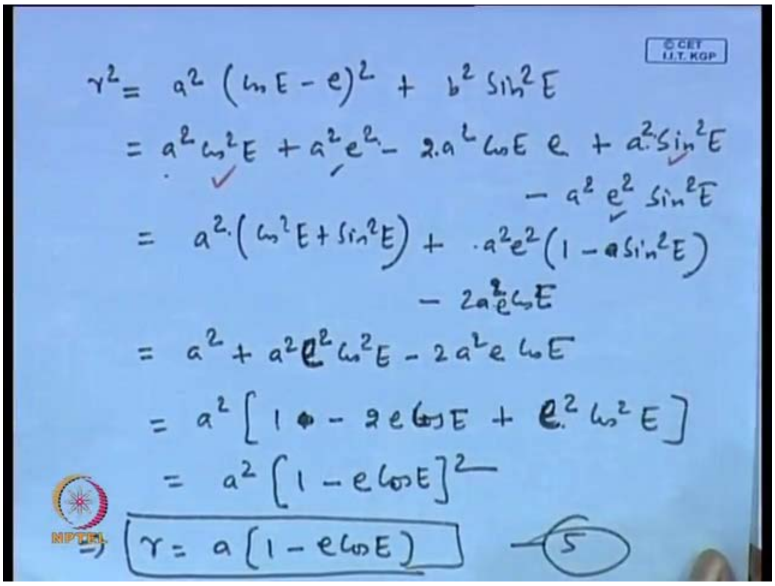

So, from here we can write r square let us expand it and write it so, a square times cos E minus e whole square plus b square sin square E. So, this becomes a square times cos square E plus, a square times e square minus, 2 times a square cos E times e. And this can be written as a square so, b square we can replace in terms of a square times 1 minus a square.

So, this will become a square times sin square E minus, a square times E square times sin square E. So, this term- this term - this term we can add so that will give us a square cos square E plus sin square E. So, for these two terms we have accounted for and then we have a square, e square taking common1 minus sin square E taking this term, this term into account and then we have last term 2 a square cos square cos E - 2 a square e times cos E, this becomes a square plus, a square E square cos square E minus, 2 a square E cos E.

So, can we write this as taking a square outside1 plus or 1 minus, 2 e times cos E plus, E square cos square, this is small e square - small e square cos square E. Therefore, the quantity this becomes a square times1 minus e cos E whole square. So, this implies r is equal to a times1 minus e cos E, this is our equation number five. So once, we have got the relationship between r and cos E so, next remains to find out the theta in terms of e so we can see from these two equations we have written earlier.

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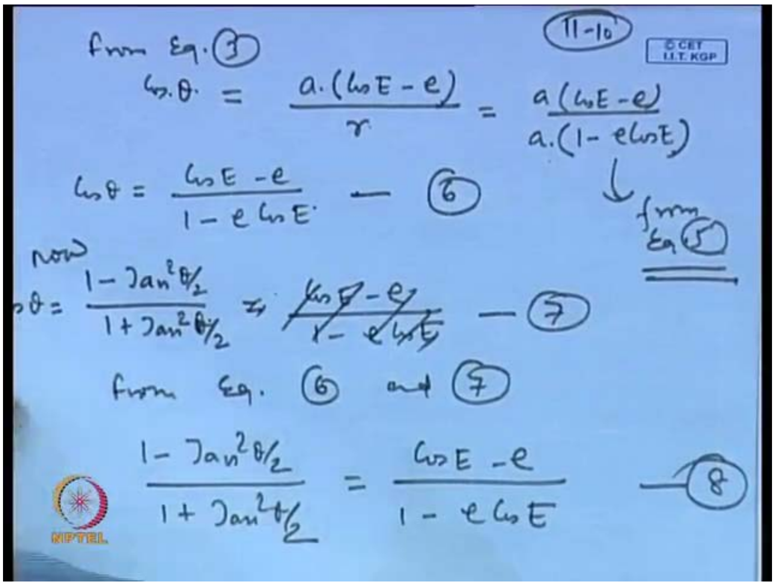

So, cos theta we can write as equation number three we can look into, so cos theta from equation number three, this is page number one nine, eleven nine, eleven ten so from equation three cos theta can be written as a cos E minus e divided by r. This is nothing but, a times cos E minus e and r is equal to we know, from this place equation number five a times1 minus e cos E so, this is a times 1 minus e cos E. So, this substitution we have done for r from, this is from equation five. So thus, we have cos theta is equal to cos E minus e divided by1 minus e cos E.

Now, doing little bit of manipulation so cos theta can be written as 1 minus tan square theta by 2 divided by 1 plus tan square theta by 2. Once we do this, so we can write now cos theta is equal to; this implies cos theta is equal to 1 minus tan square theta by 2. So, this is the quantity we get or we remove this first (( )).

Now, cos theta is equal to this quantity and this equation, let us term this as equation number six and this is the equation number seven, from equation number six and seven we can write1 minus tan square theta by 2 by 1 plus, tan square theta by 2. This will be cos E minus e divided by 1 minus e cos E, this is our equation number eight.

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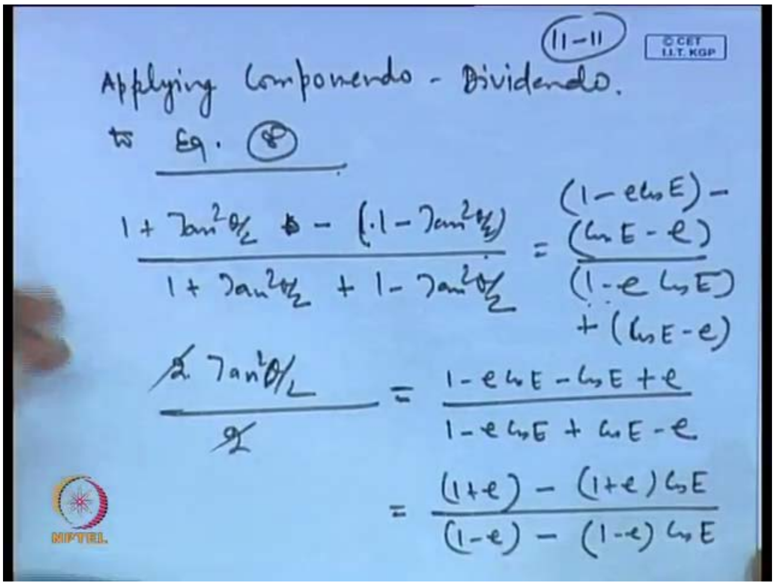

Now applying, componentdo and dividend to equation number eight to equation eight we can write this as, 1 plus tan square theta by 2, plus or minus let us write it minus so 1, so this becomes minus 1 minus tan square theta by 2 and 1 minus, 1 plus tan square theta by 2 plus, 1 minus tan square theta by 2 this becomes cos E minus e by1 minus e cos E.

So, this will give us 2 tan square theta by 2 divided by 2, so we can cancel it out and similarly, on the right hand side also we have to apply componendo, dividend so here we are missing those terms, so I will write it here this will be1 minus ecos E, 1 minus e cos E minus this quantity and then this quantity and plus so we had in denominator this quantity so this quantity and plus this quantity cos E minus so, what we get on right hand sideso1 minus e cos E and this we equate it.

So, here we expand and write it1 minus e cos E and this is minus cos E plus e and here this becomes 1 minus e cos E plus, cos E minus e so, the right hand side will become equal to 1 plus e minus, 1 plus e cos E divided by 1 minus e and then we take minus e from here 1 minus e cos E.

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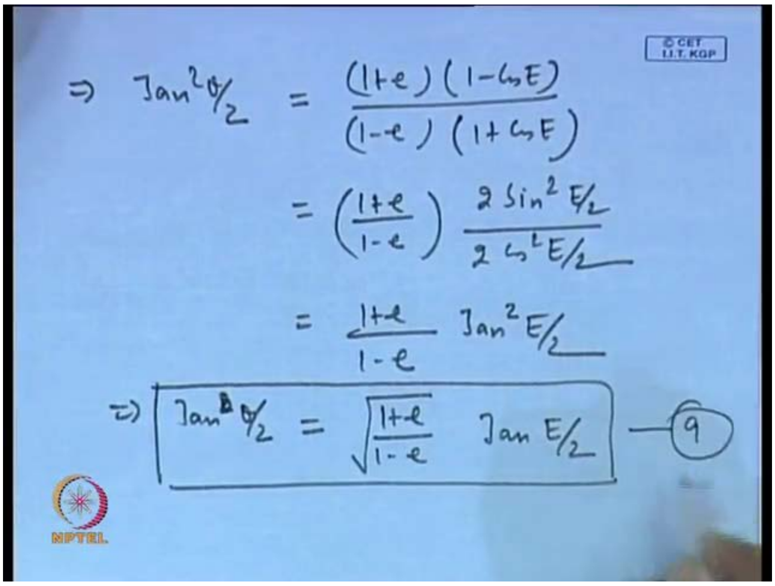

So, this implies tan square theta by 2 so this will be equal to 1 plus e times, 1 minus cos E divided by they missed the terms here this will be 1 plus, 1 minus e and cos E we are writing like this. So, cos E minus, e cos E so this term is plus here, so this becomes1 plus cos E and this term can be further simplified so this will become1 plus e, 1 minus e and this can be written as 2 sin square E by 2,divided by 2 cos square E by 2.

So, this becomes1 plus e by 1 minus e and tan square E by 2. So, this implies tan theta by 2 this is equal to1 plus e by1 minus e under root times, tan E by 2. So, this is the relationship between the true anomaly and the eccentric anomaly and the equation we number this as this is equation number nine.

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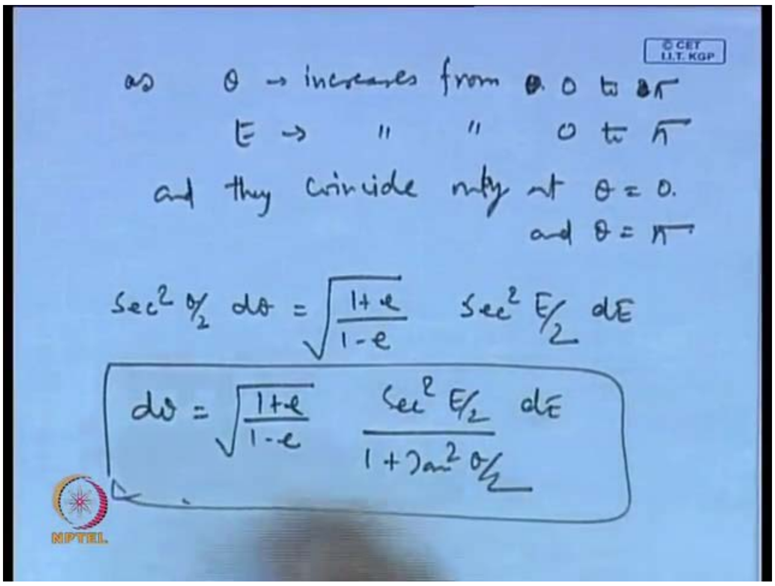

So, here as theta increases from 0 top i so, E also increases from0 to pi and they coincide only at theta is equal to0 and theta is equal to pi. So, once we have got this we have started with solving the equation for eccentricity lying between e is equal to 0 and 1.

So, for solving this case we need to go back into our original equation and then work out the whole thing so here, what we require d theta we need to replace d theta in terms of d therefore, we can write sec square theta by 2 times d theta is equal to1 plus e divided by 1 minus e under root times, sec square E by 2 and this substitution we can by 2 times d E so this substitution we can do in our original equation.

So, d theta can be further written as1 plus e by 1 minus e under root times, sec square E by2, d E divided by1 plus tan square theta by 2, making further substitution this equation can be simplified so this is our equation number eleven twelve.

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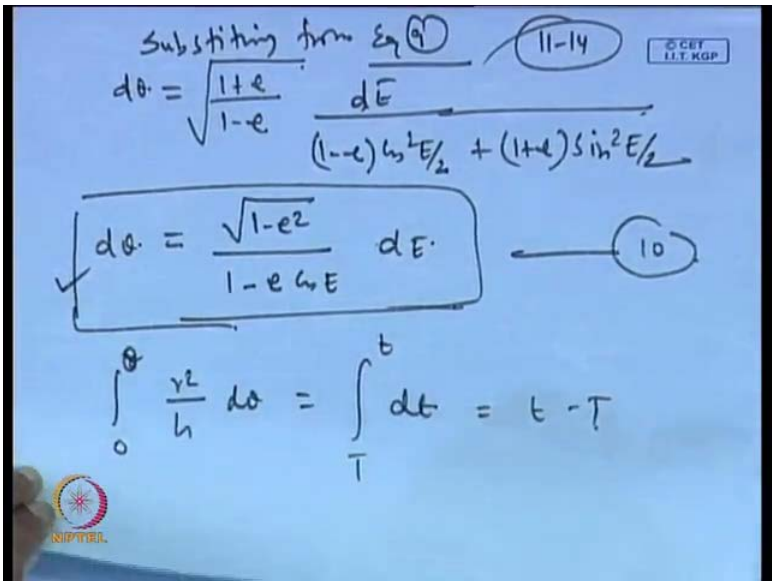

So finally, we can write this as d theta is equal to1 plus e by1 minus e under root when we need to because we want to express every term, everything in terms of eccentric anomaly and therefore, we need to replace in the previous equation – in this equation tan theta by2in terms of the here theta by2 in terms of eccentric anomaly.

So if we do this, we have to make the substitution from equation number nine. So, substituting from equation number nine substituting from nine so, d theta can be written as 1 plus e divided by1 minus e, square root times1 minus e cos square E by 2 plus, 1 plus e sin square E by 2.

So after little simplification, we can write this as 1 minus e square under root divided by1 minus e cos E dE, this is very easy we can reduce it to this quantity therefore, this is our d theta. So, this is our equation number ten now going back to our original equation, which is 0 to theta times r square divided by h d theta is equal to T to t, d t is equal to t minus T.

So here, we can replace r and d theta in terms of the r equation we have written in terms of eccentric anomaly and d theta also just now, we have expressed in terms of eccentric anomaly.

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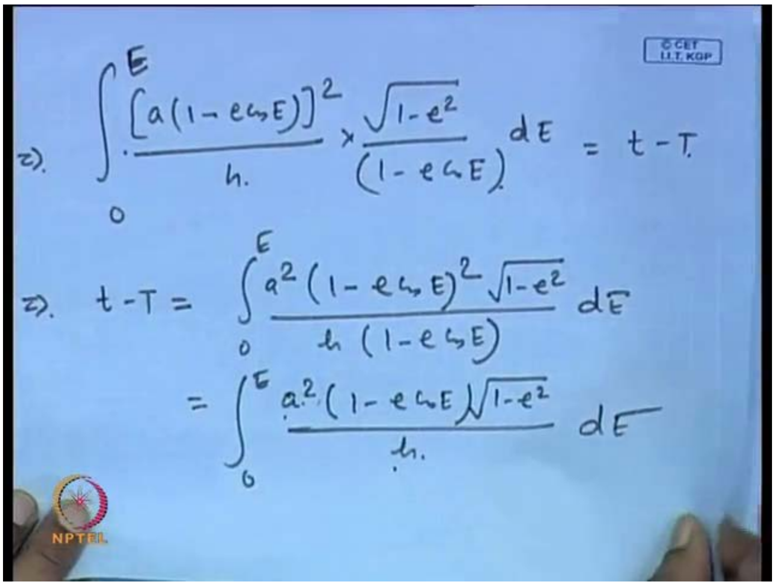

So if we do this substitution, we get a times1 minus e cos E whole square divided by h and d theta is1 minus e square under root divided by1 minus e cos E d E integrating between so, the limits changes from theta it becomes E and on the right hand side this is t minus T.

Or we can write t minus T this is equal to a square times1 minus e cos E whole square and then1 minus e square under root divided by h times1 minus e cos E. So, this will cancel out here giving us a square times1 minus e cos E times, 1 minus e square under root divided by h put it between0 and E;0 and E this is d E.

So, little bit more simplification is required we can do here, but here it is very clear that this equation is because a and e these are constant, which is constant and therefore, the integrand it appears very simple here1 minus e cos E times d E. We can integrate it and solve this equation.

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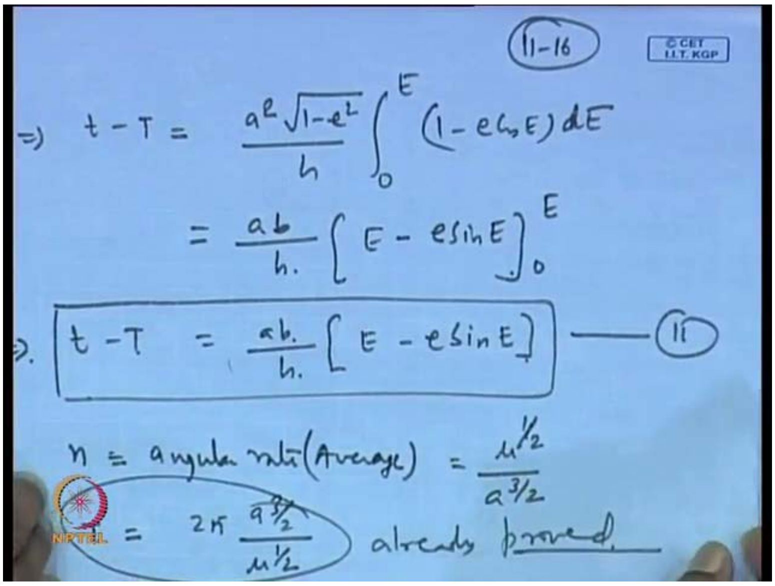

Therefore, this implies t minus T is equal to we can take outside a square1 minus e square under root divided by h, 0 to E and this becomes1 minus e cos E times d E. So, integrating now this can be written as a times 1 minus e square under root so this becomes b, a b divided by h and the integration of this will be e minus, e sin E, 0 to E. So this is nothing but, a b times E minus e sin E. So, this is the final equation we have got here and this equation and we write as equation number eleven.

Now this equation can be simplified little bit more, and the because we know here this term can be replaced in terms of the mean orbital period so if, we do that simplification. So, we define one quantity what we call as the mean anomaly so, we define another quantity let us say we know the n or the angular rate average; this is the average is nothing but, mu to the power1 by 2 divided by a to the power 3 by 2. And, time period is therefore, given as 2 pi, a to the power 3 by 2 times, mu to the power 1 by 2 already this has been proved.

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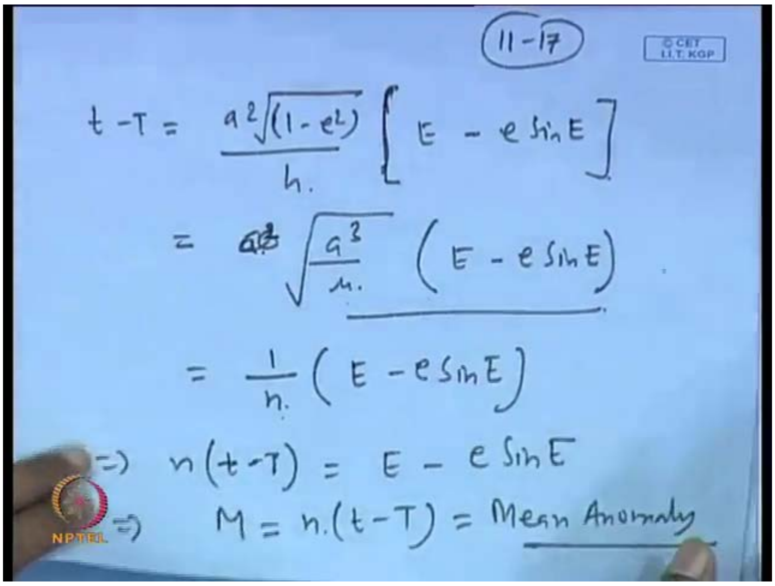

So, we have t minus T this we can write as a square times1 minus, e square earlier we have derived this so times e minus E sin E. So, this we can further express as a cube by mu under root times E minus e sin E. So how to get this, so we will look into this a cube by mu this is nothing but, the quantity a cube by mu this is related to our quantity here that we have written earlier the angular rate so this is just the inverse of this, so this quantity can be written as1 by n, where n is the average angular momentum. So, this is E minus e sin E.

Therefore, we can write n times t minus T this is equal to E minus e sin E and this quantity n times t minus T this is written as M. So, M is equal to n times t minus T, this is called mean anomaly.

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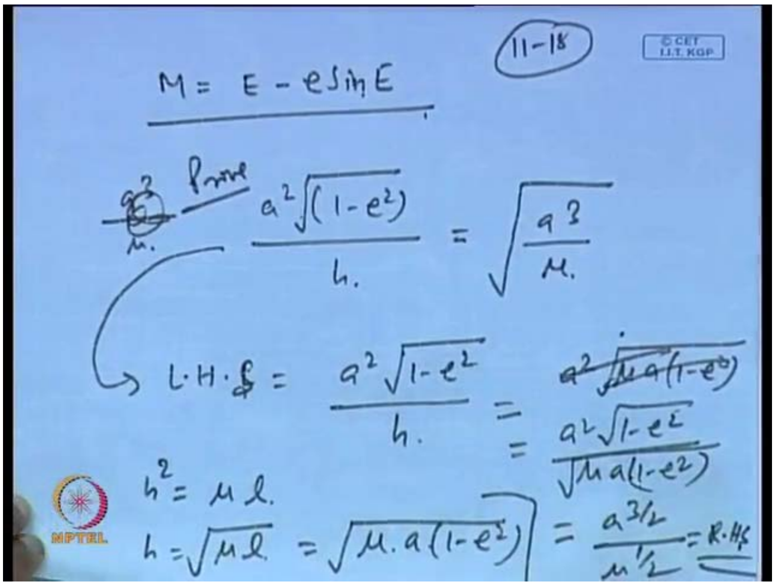

So, what we get the final equation in the form M is equal to E minus e sin E so, what we need to just prove this is a square times1 minus e square under root divided by h. This quantity is equal to prove a cube by mu under root so, it is very easy to see here the l h s the quantity is a square 1 minus, e square under root divided by h. So, we know that h square is nothing but, mu times l therefore, h will be mu times l under root and l we have written as a times1 minus, e square under root so this quantity becomes equal to this so here, we have l h s becomes a square times, mu times a1 minus e square under root, a square this quantity becomes equal to this a square times1 minus e square under root divided by mu times, a times1 minus, e square under root therefore, this becomes equal to the a power 3 by2, 1 minus e square cancel out leaving out1to the power of 2. So, the l h s is equal to r h s. So this completes our derivation for this (( ))

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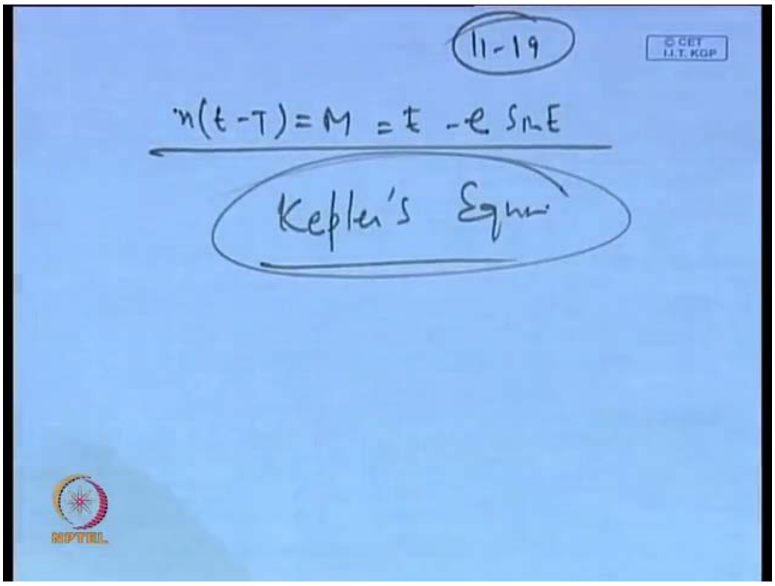

So, what we got ultimately this was our equation n times t minus T this is equal to M is equal to E times e minus sin E and this is what is called Kepler's equation. So, in the next lecture we will cover it little bit more about this but, as a whole this topic is complete. So, I will describe little bit in the next lecture about the Kepler's equation problem. Thank you very much.