High Speed Aerodynamics

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Module No. # 01
Lecture No. # 34
Similarity Rules for High Speed Flows

So, over the last few lectures, we have discussed linearized flow over two dimensional bodies; planar bodies and bodies of revolution. And we have noticed particularly where we have obtained the explicit analytic solution, as in case of subsonic and supersonic flow over wavy wall, and supersonic flow over cone, we have noticed that the pressure coefficient can be arranged in a special functional group so that, a single curve represents the solution for a whole family of set as well as arrange of mach numbers.

Particularly, we may recall, we may recall for supersonic flow over slender cone, (No audio from 01:31 to 01:44) slender cones, we had c p by delta square is function of delta root over m infinity square minus 1. In particular, for slender cone, this function is explicitly known which is log of 2 by delta root over m infinity square minus 1.

However, we can see that this can be arranged in these two functional groups; one is c p by delta square, the other is delta into root over m infinity square 1. And for all slender cones, if c p by delta square is plotted against delta by root over m infinity square 1, along the x axis, and this along the y axis, we will have a single curve representing all slender cones and covering a range of mach numbers.

Similarly, for wavy shaped wall, for wave shaped wall, we had c p, and if we combined both subsonic and supersonic flow by this, where alpha h or if we recall that h was the amplitude of the wave and alpha was the wave number becomes a function of alpha x alpha root over m in (( )) (No audio from 03:41 to 03:54) we are taking this module assigned. So, this applies for both subsonic and supersonic flow combined.

Once again, we have seen that we have been able to arrange the parameters in certain functional groups so that a single curve represents solution for a whole family of shapes over a range of mach numbers. In this case, the pressure coefficient over the body, and as you know that body; that means, this quantity remain fixed. If this parameter, this modified pressure coefficient parameter is plotted against this modified x co-ordinate, we will have a single curve for all wave shaped wall having different amplitude, different wave numbers and different free stream mach number.

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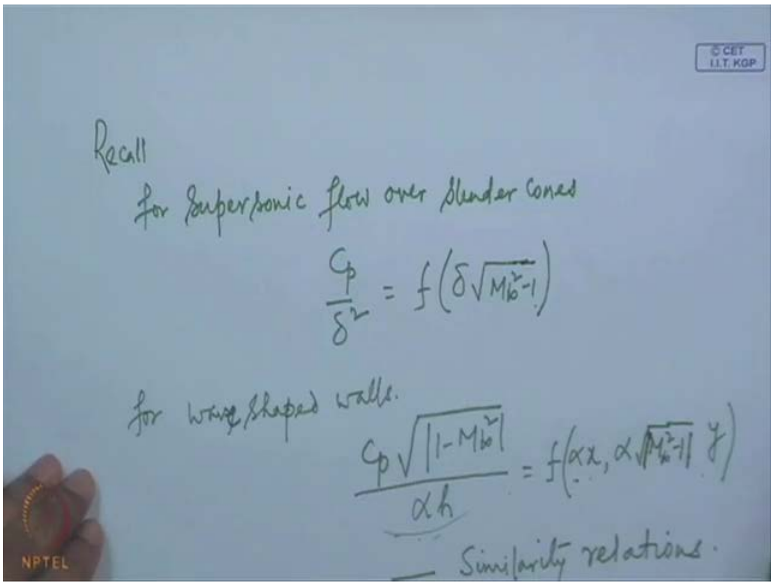

So, a single solution serves the purpose for all possible such cases. So, these are the similarity rules or similarity relations. (No audio from 04:59 to 05:18) So, similarity relations are obtaining some sort of relations or arranging the parameters involved in certain functional groups so that we can reduce the total number of parameters involved in the problem. Of course, these parameters are in terms of modified flow parameters and modified geometric parameters.

Now, in this case, of course, the solutions are explicitly known to us and we have been able to obtain these similarity relations from known relations; however, in many cases, the solutions are not obtainable particularly for non-linearized problems as in transonic flow, where even for the simplest geometry, the solution is extremely difficult. So, whenever such a situation arises, where solution cannot be obtained straight forward manner or a very simple manner, these similarity relations may prove highly useful.

However, in cases where explicit solutions in the closed form is available, even then similarity relations provides certain insight into the problem and of course, when less number of parameters are involved, it is much easier to see the importance of one parameter over the others.

However, it may be remembered that when in a linear problem where we can obtain solution for some basic problem, we can use the principle of superposition to obtain solution for other problems and the similarity relation or special rule to derive similarity relations are not that useful; however, they serves the essential purpose that how to obtain similarity relations for problems where such solutions are not straight away available. Similarity relations can also be obtained from experimental data, and to be precise, in many cases, similarity relations are originally obtained from experimental data.

Now, since the basic purpose of similarity rules is to reduce the number of variables or number of parameters involved by arranging them in some functional groups, similarity rules or similarity transformations can also be used to obtain solution or simplify the solution of many problems.

One such famous example is the Blasius solution of boundary layer which we have discussed in earlier aerodynamic courses, where we know that a special (( )) taylored stream function is defined and a modified normal coordinate which depends on both the coordinates y and x as well as a Reynolds's number is obtained. And in terms of these parameters, the boundary layer equations become instead of a… become ordinary differential equation instead of the original partial differential equation. Of course, they still remain non-linear.

So, the basic purpose of similarity rules used to get some functional grouping of the parameters involved so that the number of parameters involved is reduced. Now from dimensional analysis, we know or we can say, from dimensional analysis, we can say that the pressure coefficient or the pressure distribution on an airfoil will be function of the flow speed in terms of mach number.

It will be function of the gas; gamma, and of course, the coordinates x by c and t by c and of course, if necessary, angle of attack. Alpha in this case is angle of attack. It is no longer the wave number.

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The alpha in this relation is wave number which was 2 pi by l, where l is the wave length of the wall. (No audio from 10:43 to 10:54)

Now, it may appear that even dimensional analysis and similarity rules are basically the same, that here also we have arranged certain the parameters in certain non dimensional group; however, the similarity rules are much more than dimensional analysis. In dimensional analysis, we have arranged certain parameters in dimensionless group; however, basically these parameters or the variables involved are obtained from guess or from other knowledge and then just based on a dimensional analysis, they are arranged.

It may so happen that if our original guesses are some extraneous, then we will be getting some extraneous groups which may not have any influence in the practical case. The similarity relations are not exactly our dimensional analysis.

Similarity analysis now try to group these parameters itself in certain functional group so that we can reduce the number of variables; like in this case, even including this angle of attack, we have six parameters involved. So, the aim of the similarity rules will be to group these six parameters or few of them in certain functional groups so that those functional parameters that are involved finally, are considerably less than this number six; with the aim that if it is possible to have a single curve representing say, the pressure coefficient by a single relationship; that is, can we group all these parameters so that the modified pressure coefficient parameter at a given modified station or at given modified x by c be represented by a single curve for all mach numbers, for all gases, for a family of set.

If this can be done, then the purpose of the similarity rule is achieved and as you have mentioned that for make similarity rules for linearised problem is quite simple and straight forward, because we have explicit solution for some problems and from which we can draw the general form with the similarity rule, and since any other problem are basically superposition of these simple problems, the similarity rules can be extended without much analysis.

However, the most important use of similarity rules are for non-linear problems as in case of transonic flow, where solution or in analytic form is not available easily so that if we can frame a similarity rule, so that if we know the solution for one particular transonic flow or for one particular airfoil in one particular gas, we may be able to obtain the solution for other airfoils in other gas at other mach number in transonic mach number.

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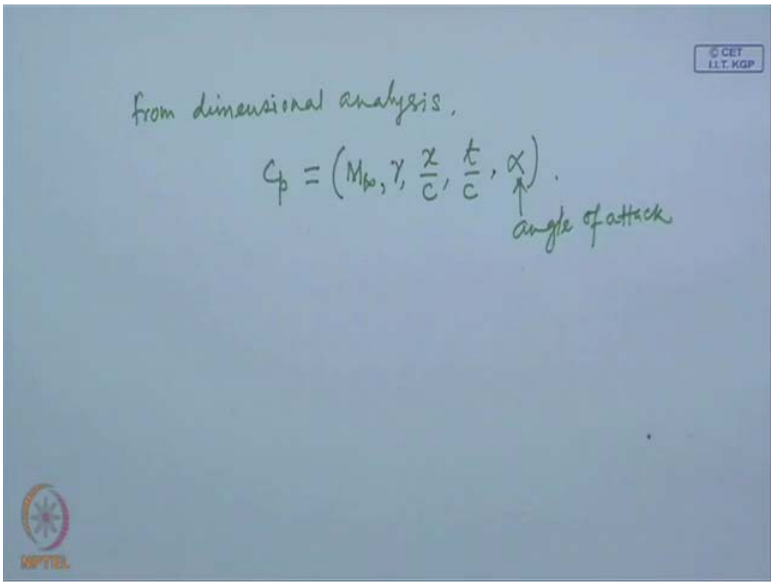

Now, as you mentioned that this non-dimensional grouping is done by dimensional analysis in which we need to guess only the parameters which might be involved or which might be responsible in a given problem. Then it is just a method of just a measure of the well known Buckingham pi theorem by which we can arrange certain number of non dimensional groups, and we can very easily obtain those non-dimensional groups by simple processes.

The similarity rules need much more than that. To obtain similarity rules, we need to have as we have shown here that explicit solutions or when explicit solutions are not available, may be the governing equation and the boundary conditions or even may be the experimental measurements which gives a complete description of the problem.

So, the governing partial differential equations and boundary conditions are essential to obtain these similarity rules or of course an experimental solution or experimental measurement can sometime be used to frame similarity rules. Now in the present context, we will mostly try to obtain similarity rules in the form that we have already shown, that is, a modified pressure coefficient or pressure distribution in terms of a modified geometry or modified some parameter representing the geometric set.

So, first we will see the similarity rule for two dimensional linearized flow. You have seen that, we have already mentioned that, constructing similar rules for linearized problems is straight forward, and no special analysis is really necessary. However, we will do the steps so that they can be used when you go for transonic flow problem.

So, first, to show the steps or how the similarity rule is formed so that a modified pressure coefficient is obtained in terms of a modified geometric parameter, the steps involved; we will discuss with the help of similarity rules for two dimensional linearized flow problems.

So, first, we will consider similarity rules for 2 D linearized flow. (No audio from 17:47 to 18:00) Say, let us say that phi x y is perturbation potential (No audio from 18:13 to 18:25) for a 2 D linear for a steady 2 D linearized flow, flow at free stream mach number of m 1. Remember that phi; we are treating as the perturbation potential, not the full potential, that is, the gradient of phi gives only the perturbation velocities, not the total velocity.

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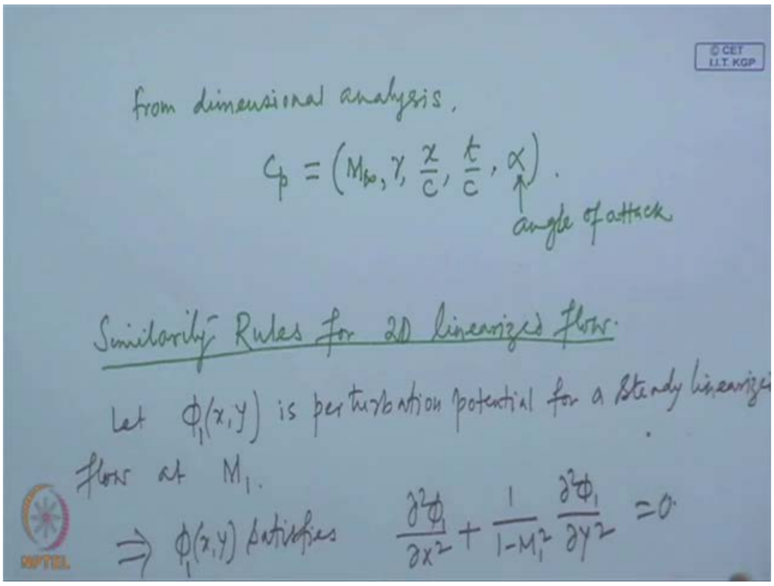

So, the undisturbed potential or the free stream potential is to be added with these to obtain the total potential. Anyway since phi x is the perturbation potential, then phi x y must satisfy (No audio from 19:18 to 19:40) or let us call it phi 1; phi 1 plus 1 by 1 minus m 1 square d 2 phi 1 d y square equal to 0. So, phi satisfies this boundary condition, phi x 1 phi 1 also satisfies the appropriate boundary condition, phi 1 also satisfies the appropriate boundary condition, also satisfies the appropriate linearized boundary condition. (No audio from 20:30 to 20:44)

Now, let us consider this body is, let the body shape be y equal to t 1 function of x by c, (()) where t 1 is the maximum thickness ratio, maximum thickness of body. And if we non-dimensionalise this side also, you have y by c equal to tau 1 f x by c. So, tau 1 equal to t 1 by c; the thickness ratio. We know the airfoils are usually characterized or described by their thickness ratio. So, we have this body shape also in terms of the thickness ratio, c is of course, the chord.

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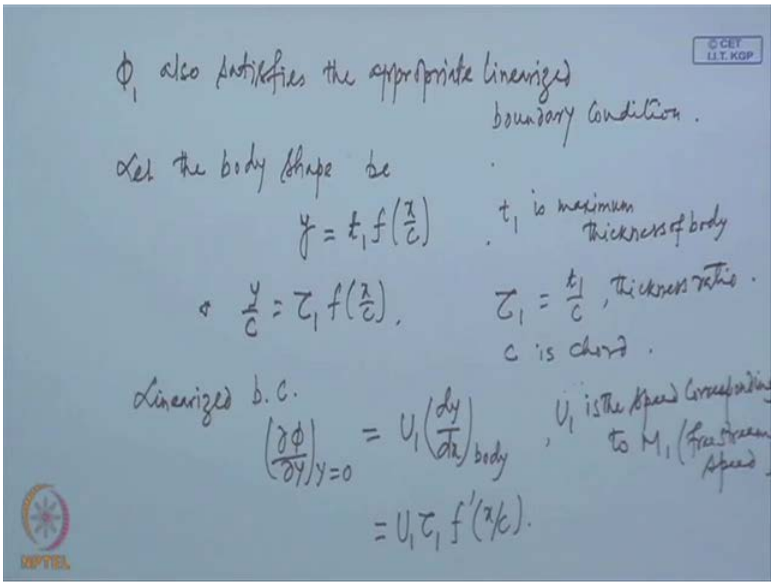

Now the boundary condition; the linearized boundary condition states that the perturbation, normal component of the perturbation velocity on the body surface which can be approximated to be the y equal to 0 is u 1 d y d x on the body surface. u 1 is the free stream speed corresponding to m 1, u 1 is the speed corresponding to m 1; that is, a free stream speed, and from this, this can be written as u 1 tau 1 function of x by c.

So, phi 1 satisfies this relation as well. Now we know the pressure coefficient, pressure coefficient on the boundary or on the body, (No audio from 24:21 to 24:35) c p you know as minus 2 u y equal to 0 by u 1 in this case. So, this becomes minus 2 by u 1 d phi 1 d x 1 y equal to 0.

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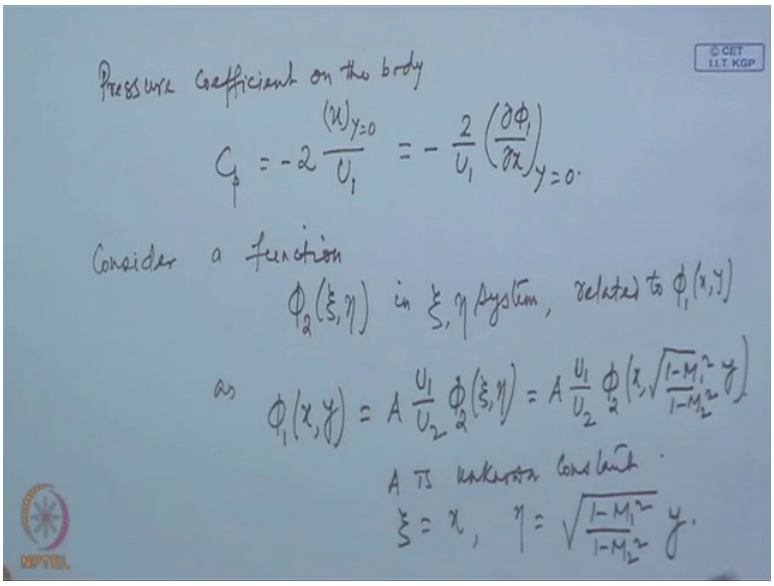

Now, consider a second function, ((No audio from 25:14 to 25:28)) consider a function phi 2 zeta eta in xi eta system, xi eta system related to this phi 1 x y as ((No audio from 26:10 to 26:23)) a u 1 by u 2 phi 2 xi eta a u 1 by u 2 phi 2 x root over 1 minus m 1 square by 1 minus m 2 square into y. a is an unknown constant, and we have these transformation xi equal to x and eta equal to root over 1 minus m 1 square by 1 minus m 2 square into y.

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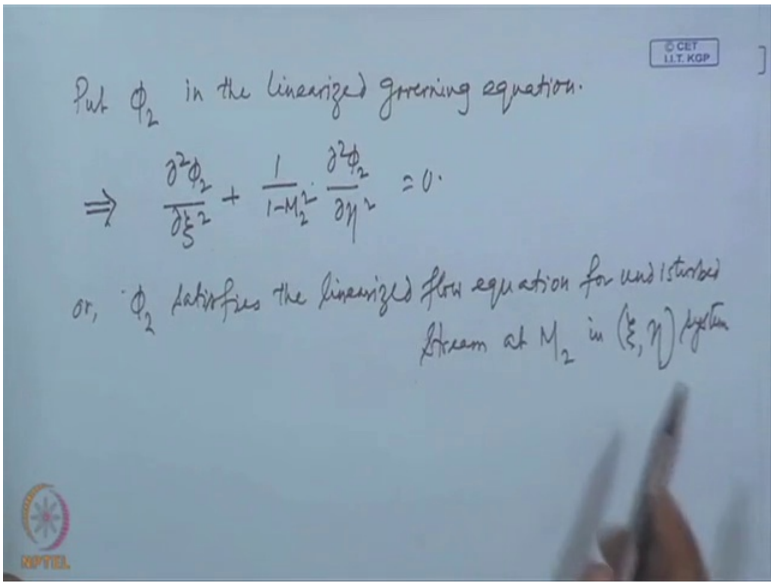

Now, let us substitute this phi 2 in the linearized differential equation. ((No audio from 27:43 to 27:55)) Put phi 2 in the linearized governing equation, and if we do that, we see that d square to d xi square plus 1 by 1 minus m 2 square d 2 phi 2 d eta square equal to 0; that is, phi 2 satisfies the linearized governing equation for a flow at free stream mach number m 2; that is, phi 2 satisfies the linearized flow equation for undisturbed stream at undisturbed, for undisturbed stream at m 2 in xi eta system. ((No audio from 29:57 to 30:10))

Now, each phi solution, then over a certain body, phi satisfy this governing equation, but you know that if a function is to be considered as a varied potential flow solution over a certain body, then it must also satisfy the boundary condition; appropriate boundary condition. So, does this phi 2 satisfy the appropriate boundary condition for over a certain body at free stream m 2? And let us check that. What happens to this d phi 1 d y at y equal to 0. We can express this phi 1 in terms of phi 2 to show that this becomes A u 1 by u 2 root over 1 minus m 1 square by 1 minus m 2 square into d phi 2 d theta, at theta equal to zero.

And; however, we know that d phi 1 d y at y equal to 0 is u 1 tau 1 f prime x by c, and hence this gives (No audio 32:08 to 32:45) d phi 2 u 2 tau 1 f prime; this x of course, we can replace by xi, (No audio from 33:04 to 33:17) sorry u 2 tau 2, (No audio from 33:24 to 33:37) where a root 1 minus m 2 square by 1 minus m 1 square into tau 1.

We see then that phi 2 satisfies the governing equation for a undisturbed stream at m 2, corresponding to speed u 2, and also satisfies the boundary condition in the form of u 2 tau 2 f prime xi by c, which is exactly the same form as in the earlier case for flow over body of same shape because this is the same f. So, phi 2 satisfies the boundary condition for bodies of same shape at free stream speed of u 2 with thickness 2. So, phi 2 satisfies the boundary condition (No audio from 34:59 to 35:10) over a body of same shape, but with thickness tau 2, (No audio from 35:22 to 35:40) when the undisturbed stream velocity is u 2 or in terms mach number; m 2.

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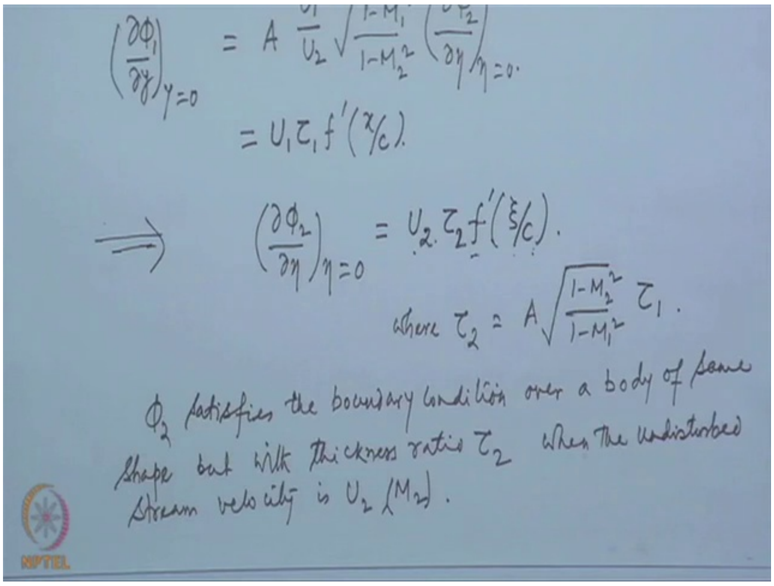

So, we see again that if phi 1 is a solution of the linearized equation corresponding to a undisturbed stream of m 1 over a body of a given set with thickness ratio tau 1, then this phi 2 is also a solution of same linearized equation; that is, of a similar linearized flow at undisturbed stream of m 2 of similar shape of body, but with different thickness and how the two thicknesses are related by this.

So, we see that we are getting some sort of similarity that if we have a certain shape of body and we know the solution over it, then in the xi eta; which is basically a transformed geometry, we get the solution over a transformed geometry, but of the same shape with different thickness from the known first solution. So, this also shows that bodies of same family can be compared; geometric shape of same family can be compared. That is an example. if we are considering NACA four digit airfoils, N A C A followed by four digits, they can be compared because they belong to the same family, but they cannot be compared with any super critical air foils or any air foils of six series. Now consider now the pressure coefficient, consider the pressure coefficient.

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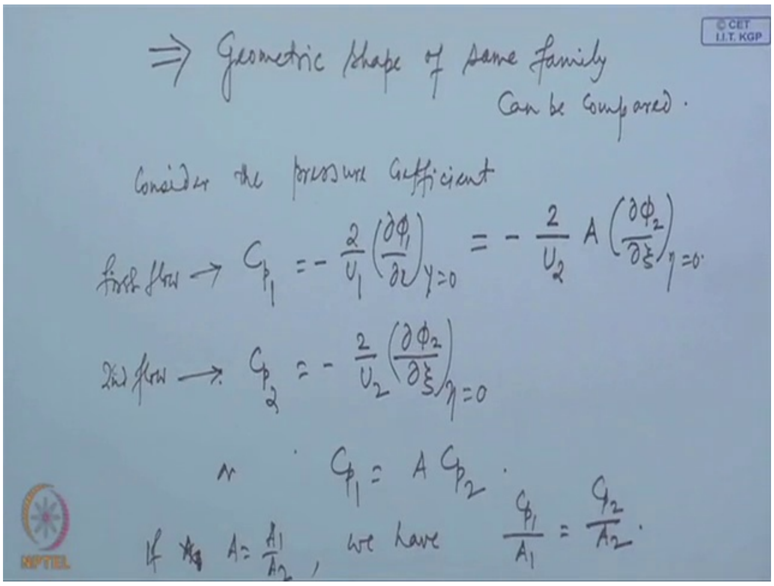

As you have said earlier, we want to express our similarity rules in terms of the pressure coefficient. Now for the first flow; that is phi 1 at m 1 over the body of thickness tau 1 or pressure coefficient c p 1 is minus 2 by u 1 d phi 1 d x at y equal to 0. If we substitute this, this then become minus 2 by u 2 into a d phi 2 d xi at theta equal to 0. Now the pressure coefficient on the second flow or over the second body is simply minus 2 by u 2 d phi 2 d xi eta equal to 0. (No audio from 40:21 to 40:44)

So, comparing these, what we are getting is c p 1 equal to A c p 2 or we can think that A is A 1 by A 2, and we can write it as c p 1 by A 1 equal to c p 2 by A 2, (No audio from 41:05 to 41:15) if A equal to sorry if A equal to A 1 by A 2, we have… (No audio from 41:26 to 41:42) So, what we are getting that two members of a particular family of shape, which are characterized by the thickness ratios tau 1 and tau 2, will have same pressure distribution given by the coefficient c p 1 and c p 2.

If the mach number of the flows are m 1 and m 2, then we have c p 2 equal to A c p; c p 1 equal to A c p 2 or c p 1 by A 1 equal to c p 2 by A 2; if we express A in terms of A 1 by A 2. You can write it explicitly that two members of a family of shapes with thickness ratios tau 1 and tau 2 will have pressure distributions, ((No audio from 43:25 to 43:37)) pressure distributions given by c p 1 and c p 2 such that c p 1 equal to A c p 2; if tau 1 and tau 2 satisfies a particular relationship, if tau 1 equal to A into root over 1 minus m 1 square by 1 minus m 2 square tau 2. ((No audio from 44:25 to 44:47))

This in the of course, it is not mentioned here explicitly that if the first member is at a free stream mach number of m 1, and the second member is at free stream mach number m 2; first member at free stream m 1 and the second member second member at m 2. Now these relation; we can combined as c p by A, ((No audio from 45:45 to 45:56)) if you combine these relation, c p by A is function of tau by A root 1 minus m infinity square. We are going back to the standard notation of m infinity as free stream mach number instead of two different free streams; m 1 and m 2.

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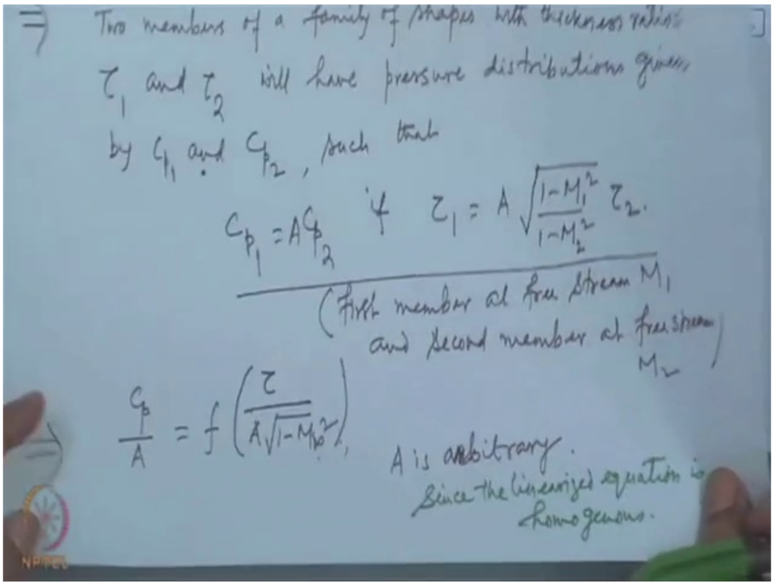

Now, A; we have never not been able to find A or rather we have seen that A is arbitrary. Whatever the choice for A is, it is valid. So, A is arbitrary, A is arbitrary. We can choose anything for A, and all the relationships remain valid. And this is because that the governing equation is homogeneous in phi and see if we multiply the equation by any constant or multiply phi by any arbitrary constant, the equation remain unchanged since the linearized equation is homogeneous. ((No audio from 47:45 to 48:01)) Consequently, if we multiply phi by any arbitrary constant, that also satisfies the equation. So, this constant remains arbitrary.

Now we can have different choices for different rules, and there are few particular choices which are quite popular. So, we have, A can be chosen arbitrarily and different choices give different rules. ((No audio from 48:55 to 49:20)) There are few very popular choices which includes Prandtl Glauert rule ((No audio from 49:36 to 49:48)) and Gothert rule and we will mention this rules; say choose A equal to 1, what we get is c p is function of tau by root over 1 minus m infinity square. This is of course, obvious choice that A equal to 1.

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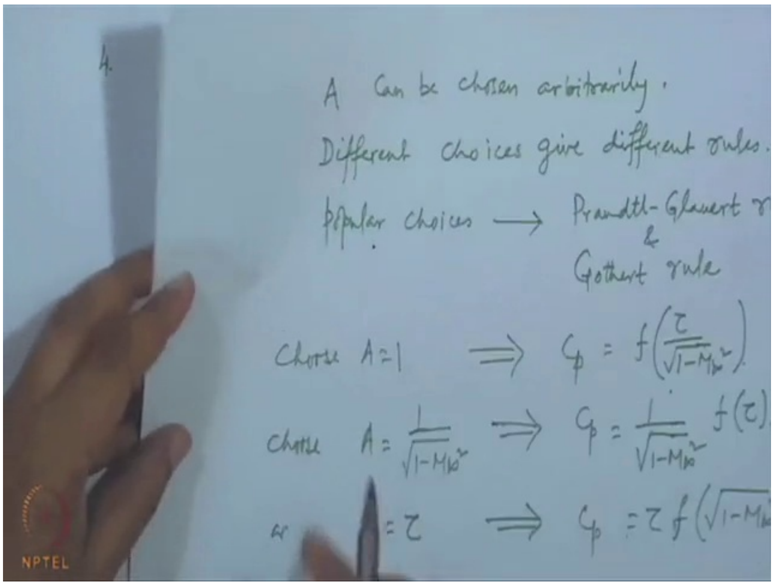

Now, choose A equal to 1 by root over 1 minus m infinity square. ((No audio from 50:57 to 51:11)) This gives c p is 1 by root over 1 minus m infinity square into function of tau or choose A equal to tau, this gives c p equal to tau into function of root over 1 minus m infinity square. All three are known as Prandtl Glauert rules. All three of these are known as Prandtl Glauert rules.

And a fourth choice, so, let us say these choice we will number as 1, 2, 3 and a fourth choice we make which is A equal to 1 by 1 minus m infinity square which gives us c p 1 by 1 minus m infinity square into function of… which is called the Gothert rule. We will subsequently try to see what the meaning of these rules is, and what these rules particularly say, and how they are important or why are they important, but before going to these, we should mention that in this analysis, we have implicitly assumed the subsonic flow. If we have taken a supersonic flow where one of the term in the governing equation or the term representing that what derivative would have been negative, 1 by 1 minus m infinity square would have been replaced by 1 by m infinity square minus 1, making the sign of the term negative; however, as far as these similarity rules are concerned, we could see that nothing would have been changed. So, if we make this parameter sign independent, then the rule applies to subsonic and linearized subsonic and supersonic flow.

So, the rules applies to linearized subsonic and supersonic flows, if 1 by root over 1 minus m infinity square is replaced by 1 by root over 1 minus m infinity square. ((No audio from 55:20 to 55:32)). So, we can straight away replace those Prandtl glauert rules and gothert rule by this replacement, and the rule that we obtained; they are valid for both subsonic and supersonic flow as long as they belong to linearized case, that is, of course, they are not valid in the transonic range.

So, not valid in transonic range, and also not valid in hypersonic range. That is because, we have already discussed in both these cases, the governing equation is no longer remain linear, they become non-linear. So, this analysis is not valid in neither of these case and of course, then it implies that accuracy of all these rules, accuracy of all these rules decreases. So, rules become less accurate, less accurate as the transonic range is approached. ((No audio from 57:30 to 57:46))

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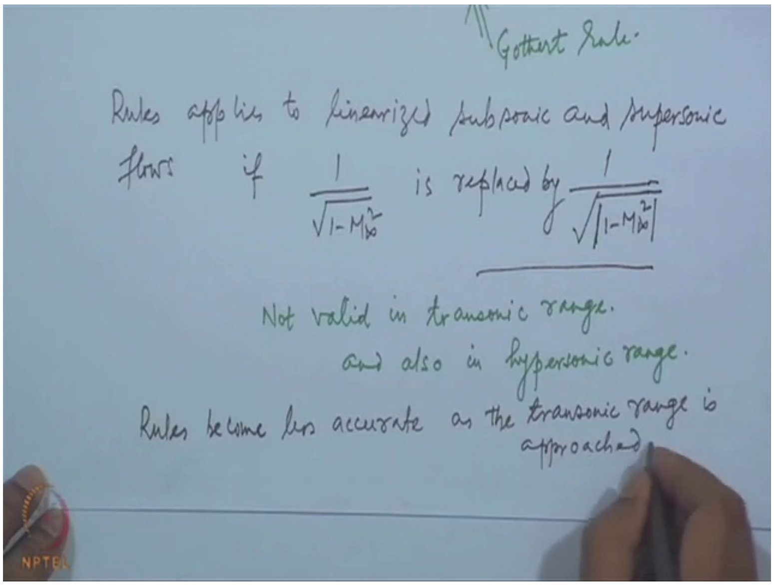

So, we will close here and discuss about these or implications of this Prandtl Glauert rules and Gothert rule in our next lecture, and also we will try to get other similarity rules.