Wind Energy

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Lecture 05: Fluid Mechanics- Dimensional Similarity

Continuity: $\frac{\partial \rho}{\partial t} + \nabla.\left(\rho \vec{u}\right) = 0$

Splitting the second term $\nabla.\left(\rho \vec{u}\right) = \rho \nabla.\vec{u} + \left(\vec{u}.\nabla\right)\rho$

$$\frac{\partial \rho}{\partial t} + \nabla.\left(\rho \vec{u}\right) = 0 \quad \Rightarrow \frac{\partial \rho}{\partial t} + \left(\vec{u}.\nabla\right)\rho + \rho \nabla.\vec{u} = 0$$

Rearranging $\frac{d\rho}{dt} + \rho \nabla.\vec{u} = 0 \qquad \frac{1}{\rho}\frac{d\rho}{dt} + \nabla.\vec{u} = 0$

Continuity for incompressible flow: $\nabla.\vec{u} = 0$ Steady or unsteady

Condition for incompressible flow: $\frac{1}{\rho}\frac{d\rho}{dt} = 0$

Welcome back. So, we'll continue our discussion on this fluid mechanics. So now, what we move on. So, essentially what we have talked so far like fluid statics. We have looked at the kinematics. Then we had two different kind of description.

Lagrangian and Eulerian. Then we looked at the Reynolds transport theorem, under the continuum hypothesis and derived all the integral form of mass conservation, momentum conservation, energy conservations and their applications. Some of the example of their application. Then, also we looked at difference between the Lagrangian and Eulerian approach.

So, now what we move on to discuss about the conservation equations in differential form. So, now this one we have already looked at it the continuity equation. This is the differential form of the continuity equation. This is the partial derivative or time derivative and this is the convective derivative part. If you expand the Second term, so one can write like this rho del dot u bar u dot del rho.

And then if I combine them together, this turns out to be like this. If you rearrange this goes like this. So, for incompressible flow, (this d rho by dt) or d rho by dt that is zero it turns out to be del dot b is zero, so whether it's a steady or unsteady that is what so in continuity for incompressible flow we are very commonly used del dot u bar or velocity product is zero but that comes that the material derivative of density is zero which is this one so, what happens here is that density is constant everywhere although this is not necessary but, the density of the fluid particle remains constant but density may vary point to point. So this is called the variable density flow. Most of the incompressible cases density is considered to be constant everywhere okay, now one can look at the same thing in terms of thermodynamics.

So density is function of pressure and temperature. If I write down the partial derivative of density then del rho by del P at constant temperature del T by P and, C is the speed of sound which is defined as del Y by del T. From here we can define the Mach number which is again a number between the velocity scale, pristine velocity scale or the flow scale with the speed of sound and typically Mach number less than 0.3 is considered to be usually the flow is considered to be incompressible. Typically the speed of sound is high in liquids, liquids flows are therefore always incompressible.

From thermodynamics: $\rho = \rho(p, T)$

$$d\rho = \left(\frac{\partial \rho}{\partial p}\right)_T dp + \left(\frac{\partial \rho}{\partial T}\right)_p dT$$

speed of sound $c = \sqrt{\left(\frac{\partial p}{\partial \rho}\right)_T}$

Mach number $\mathrm{Ma} = \frac{u}{c}$

Incompressibility requires high values of c (small Ma)

Ma < 0.3 is usually considered incompressibe

Now, if I try to compare the speed of sound also that is quite high in gases and most or majority of the engineering applications are involving air which is in compressible regime. So that gives an idea about the speed that usain bolt who runs at 44 kilometer per hour, f1 cars go 300 kilometer per hour, category 5 hurricane gives 300 kilometer per hour, an example of compressible flow quick speed of passenger aircraft is 270 meter per second, ak-47 bullet is 700 meter per second which is supersonic and re-entry vehicle is 2200 to 700 so which is But what we would be talking mostly incompressible flow in this particular course because even in the context of wind turbine applications, so the application is primarily incompressible, so we would restrict our discussion to the incompressible flow and then we take things from there. So, now if you re-look back incompressible continuity equation, Then one can write in Cartesian system, this is 1D, this is 2D, this is 3-dimensional, this is steady or unsteady continuum equation. So, now for 2-dimensional flow field, if you have a flow field given U, that would be constant D0, then I can find out the, then streamlines also I can find it out. So, different value of C provides different streamlines.

In Cartesian coordinate system

1-D: $$\frac{\partial u}{\partial x} = 0$$

2-D: $$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

3-D: $$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

So, also in 2-dimensional flow field, we can define a function, which is called the stream function. So the stream function is such that, u is derivative of that function with respect to y, where the v is derivative with respect to x. So once I put it back in the continuity equation, so, you get this del two psi by. del x del y so it satisfied the continuity equation. Now along a streamline i can write the equation of the streamline and if i put it in terms of psi so psi remains constant so that means along a streamline steam function is constant, okay.

For a 2-D incompressible flow field: $u = bx,\ b = \text{constant}$

Find v, when $v(y=0)=0$

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0 \quad \Rightarrow \frac{\partial v}{\partial y}=-b \quad \Rightarrow v=-by+f(t,x)$$

$f(t,x) = 0$ since $v(y=0)=0$

How the streamlines should look like?

$$\frac{dx}{u}=\frac{dy}{v} \quad \Rightarrow \frac{dx}{bx}=\frac{dy}{-by} \quad \Rightarrow xy=c \ (\text{constant})$$

For **2-D incompressible flow,** there exists a function, called **streamfunction**

$$\psi(x,y,t) \text{ such that } u=\frac{\partial \psi}{\partial y}, v=-\frac{\partial \psi}{\partial x}$$

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=\frac{\partial^2 \psi}{\partial x \partial y}-\frac{\partial^2 \psi}{\partial x \partial y}=0$$ Continuity satisfied!!

Along a streamline $\quad \frac{dx}{u}=\frac{dy}{v} \quad \Rightarrow udy-vdx=0$

$$\Rightarrow \frac{\partial \psi}{\partial y}dy+\frac{\partial \psi}{\partial x}dx=0 \quad \Rightarrow d\psi=0 \ \Rightarrow \psi=\text{constant}$$

So one can find out the steam function and the equation of the streamline for a two-dimensional flow-field which is given equals to bx and v equals to by so i write u then you integrate you get bx equals to and by equals to like this. So if you integrate, the final you get this. So the stream function is given like that and stream line is xy equals to c. So for different values of c, we provide you different stream lines. So given a flow, you can find out the equation of stream line and the stream function as well.

$$u = bx, v = -by; \; b = \text{constant}$$

$$u = \frac{\partial \psi}{\partial y}, v = -\frac{\partial \psi}{\partial x} \qquad \Rightarrow bx = \frac{\partial \psi}{\partial y}, by = \frac{\partial \psi}{\partial x}$$

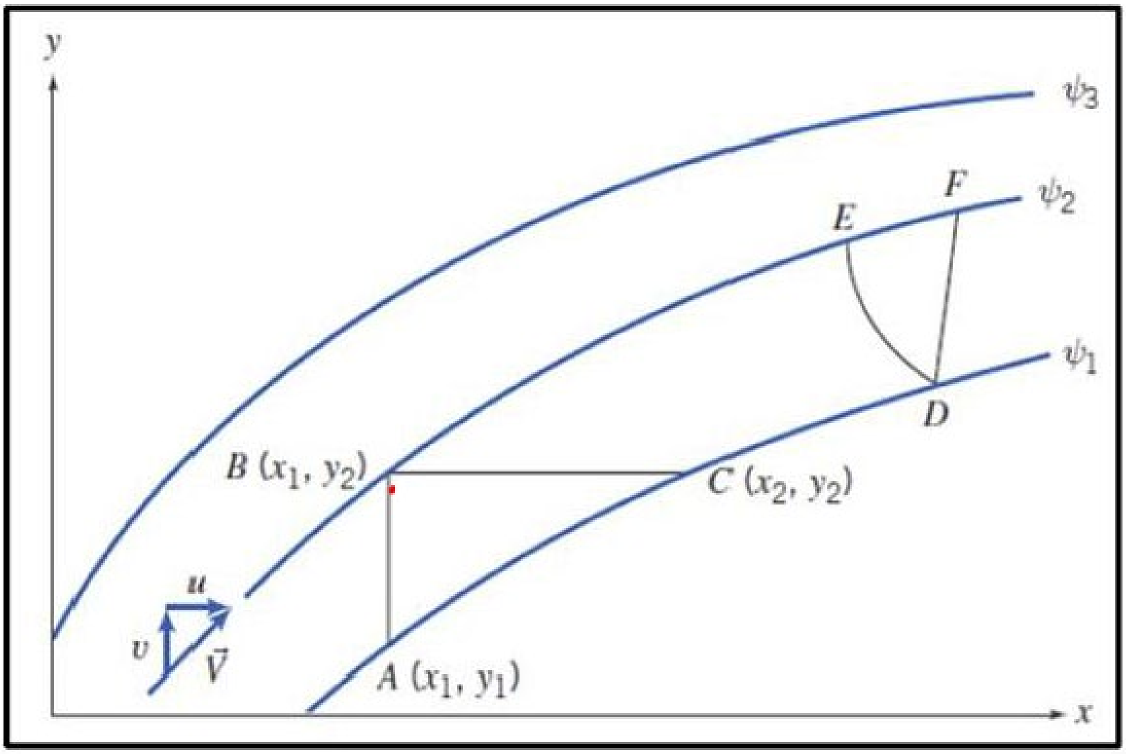
$$\Rightarrow \psi = bxy + f_1(x), \psi = bxy + f_2(y) \qquad f_1(x) = f_2(y) = c_1 \; (\text{constant})$$

Streamfunction: $\psi = bxy + c_1 \Rightarrow xy = \frac{\psi - c_1}{b} = c$

Streamline: $xy = c$ Different values of c will provide different streamlines

So along a streamline, the d psi is zero. That means psi remains constant. Okay. So essentially, let's say QAB, if I calculate A and B, this would be y = y1 to y2 udy. So I write del psi by del psi 2 minus psi 1.

And Cd psi 2 minus psi 1. So del psi... which is the volume flow rate between two steam lines that is remains constant.



Along a streamline

$$d\psi = 0 \;\Rightarrow \psi = \text{constant}$$

$$Q_{\text{AB}} = \int_{y=y_1}^{y=y_2} u\,dy = \int_{y=y_1}^{y=y_2} \frac{\partial \psi}{\partial y}\,dy$$

$$= \psi_2 - \psi_1$$

$$Q_{CD} = \int_{x=x_1}^{x=x_2} v dx = \int_{x=x_1}^{x=x_2} -\frac{\partial \psi}{\partial x} dx = \psi_2 - \psi_1$$

So, that one can show that. So, this is an application of the kind of an continuity equations to stream lines and all these things. Now, similarly I write the momentum conservation which is a change of momentum equals to all the forces, body forces and surface forces. From RTT I write that, If I put that back, so this is what you get and then these are related to the normal and shear stresses. So, one can have a normal stress like this and then the shear stress which is find out that.

Momentum conservation: $\frac{d(m\vec{u})}{dt} = \vec{F} = \vec{F}_B + \vec{F}_S$

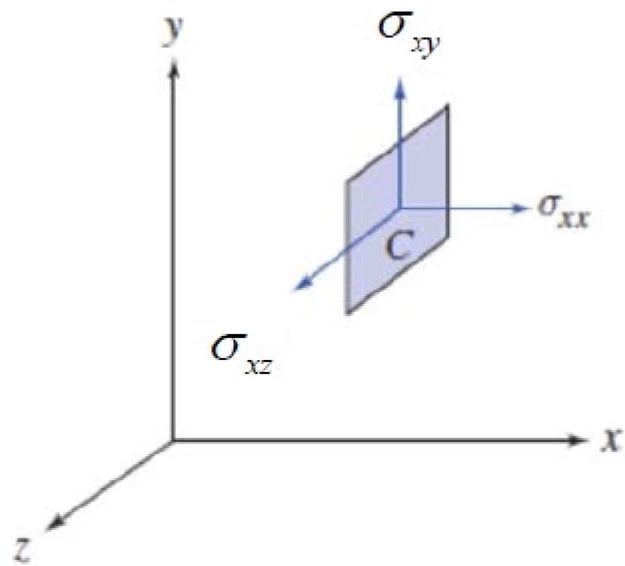
RTT: $\frac{dB_{sys}}{dt} = \int_{CV} \rho \left[\frac{\partial \beta}{\partial t} + (\vec{u}.\nabla)\beta \right] d\forall$

using $B_{sys} = m\vec{u}$ we have $\beta = \vec{u}$ and $\frac{dB_{sys}}{dt} = \vec{F}_B + \vec{F}_S$

$$\vec{F}_B + \vec{F}_S = \int_{CV} \rho \left[\frac{\partial \vec{u}}{\partial t} + (\vec{u}.\nabla)\vec{u} \right] d\forall$$

$$\int_{CV} \rho \vec{g} d\forall + \vec{F}_S = \int_{CV} \rho \left[\frac{\partial \vec{u}}{\partial t} + (\vec{u}.\nabla)\vec{u} \right] d\forall$$

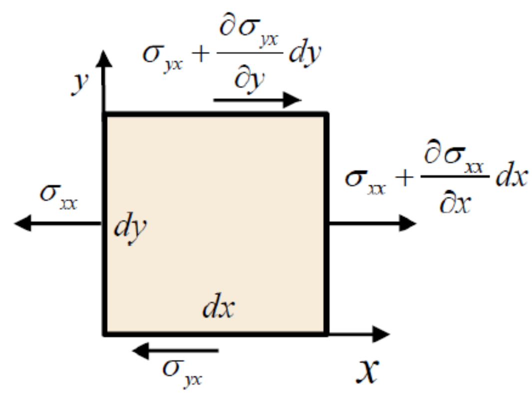
So, if you write sigma ij which is a surface normal direction i and the force direction j or surface normal direction minus i or minus j. If I take this kind of a section and write all the surface forces, so finally I get the surface forces in terms of the control volume. So recall I had this body forces and the surface forces is going to be. So, once I put them back and put it back in this x-momentum equation, so further simplification get me this. So this equation is true for an arbitrary control volume.



Surface force component in x-direction:

$$
\begin{aligned}
dF_{sx} &= \left(\sigma_{xx} + \frac{\partial \sigma_{xx}}{\partial x} dx\right) dy - \sigma_{xx} dy \\
&\quad + \left(\sigma_{yx} + \frac{\partial \sigma_{yx}}{\partial y} dy\right) dx - \sigma_{yx} dx \\
&= \left(\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{yx}}{\partial y}\right) d\mathcal{V}
\end{aligned}
$$

$$
F_{sx} = \int_{CV} \left(\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{yx}}{\partial y}\right) d\mathcal{V}
$$



Recall: $\int_{CV} \rho \vec{g} d\mathcal{V} + \vec{F}_S = \int_{CV} \rho \left[\frac{\partial \vec{u}}{\partial t} + (\vec{u}.\nabla)\vec{u}\right] d\mathcal{V}$

So, what I can write in differential form, this is the left hand side which is essentially, this is the body force and the surface force. Now, here to correlate the stress, we again use the Newton's law of viscosity where the stress is proportional to the strain and the viscosity is the constant which brings the stress and strain together.

x-mom Eq. $$\int_{CV} \rho g_x dV + F_{sx} = \int_{CV} \rho \left[\frac{\partial \vec{u}}{\partial t} + (\vec{u}.\nabla)\vec{u} \right]_x dV$$

$$F_{sx} = \int_{CV} \left(\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{yx}}{\partial y} \right) dV$$

$$\int_{CV} \rho g_x dV + \int_{CV} \left(\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{yx}}{\partial y} \right) dV = \int_{CV} \rho \left[\frac{\partial \vec{u}}{\partial t} + (\vec{u}.\nabla)\vec{u} \right]_x dV$$

Since the above relation is true for arbitrary CV

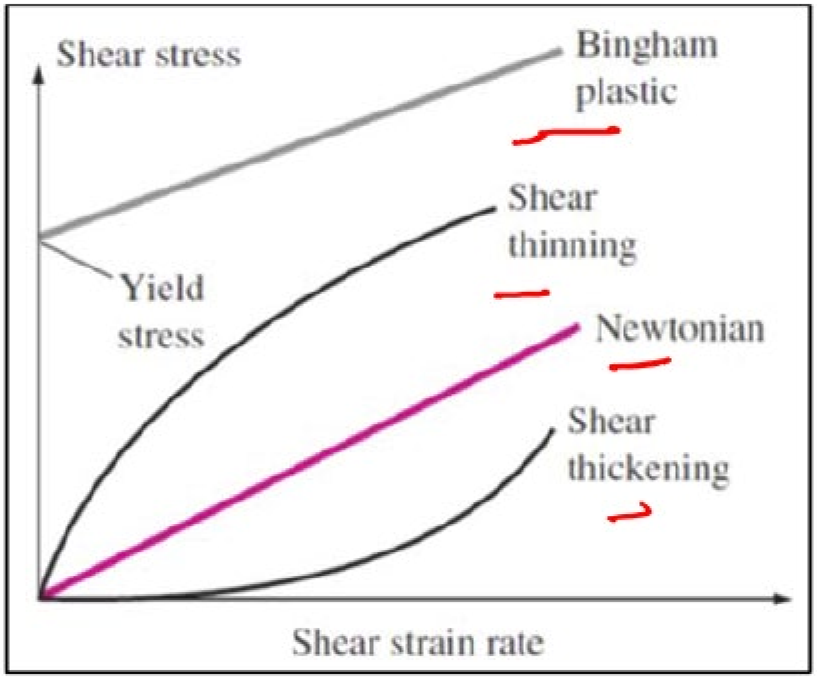
$$\rho \left[\frac{\partial u}{\partial t} + (\vec{u}.\nabla)u \right] = \rho g_x + \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{yx}}{\partial y}$$

So, once we follow that, then there are, so these are called the linear variation of the stress and strain, which is most of the cases that follow that particular variation, we consider them as a Newtonian. But, I mean, in fact, the air that we are going to deal with for the wind turbine calculation and things like that, that's going to be a Newtonian fluid. But there are cases where there are some non-Newtonian fluid also, for example, ketchup, blood, toothpaste.

Newton's law of viscosity

stress $\propto$ strain-rate $\quad \sigma_{yx} = \mu \dfrac{\partial u}{\partial y}$

μ: viscosity



Also in non-Newtonian, there are... So Newtonian is a linear variation. Then there are shear thinning, there are shear thickening.

There are plastic... So, all these are different non-neutral fluids where their stress, the variation of the stress with strain are kind of different. Now, once we try to write down the general form of Navier-Stokes equation, we can write this stress component, expand that. and for incompressible flow this is coming from the continuity. So, this boils down to the pressure gradient term and the derivative of that. So, if I put them back in x momentum two dimensional, so left hand side I have end of this derivative of the momentum, body force, pressure and this.

Generalized form of Newton's law for incompressible flow (Stokes)

$$\sigma_{yx} = \mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) \qquad \sigma_{xx} = -p + 2\mu\frac{\partial u}{\partial x}$$

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{yx}}{\partial y} = \frac{\partial}{\partial x}\left(-p + 2\mu\frac{\partial u}{\partial x}\right) + \mu\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)$$

$= 0$, incompressible

$$= -\frac{\partial p}{\partial x} + \mu\frac{\partial^2 u}{\partial x^2} + \mu\frac{\partial^2 u}{\partial y^2} + \mu\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)$$

$$= -\frac{\partial p}{\partial x} + \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$$

x-mom. Eq. for Newtonian fluid, 2-D incompressible flow

$$\rho\left[\frac{\partial u}{\partial t}+(\vec{u}.\nabla)u\right]=\rho g_x-\frac{\partial p}{\partial x}+\mu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)$$

$$\frac{\partial u}{\partial t}+(\vec{u}.\nabla)u=g_x-\frac{1}{\rho}\frac{\partial p}{\partial x}+\nu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right);\nu=\mu/\rho$$

μ : viscosity (dynamic viscosity), Pa-s

ν : kinematic viscosity, m^2/s

In 3-D

$$\frac{\partial u}{\partial t}+(\vec{u}.\nabla)u=g_x-\frac{1}{\rho}\frac{\partial p}{\partial x}+\nu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right)$$

$$=g_x-\frac{1}{\rho}\frac{\partial p}{\partial x}+\nu\nabla^2 u$$

So, this is what one can write. where mu is the dynamic viscosity and nu is the kinetic viscosity. In three-dimensional, this one can write in this vector form, so this would include the other terms here, but essentially, you get this particular form where you kind of get the momentum equations and similarly you expand these things in the other two direction that is y direction and z direction and we can write the momentum equation so essentially if you write everything in vector form then this is what will be in general okay this is known as navier stokes equation by two scientists navier 1825 and stokes 1850 which is one of the major dark horse or the system that allows out to solve the free flow problem.

$$\frac{\partial \vec{u}}{\partial t} + (\vec{u}.\nabla)\vec{u} = \vec{g} - \frac{1}{\rho}\nabla p + \nu \nabla^2 \vec{u} \qquad \vec{u}.\nabla \equiv u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z}$$

3-D, N-S eq., scalar form, Cartesian coordinate:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} = g_x - \frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right)$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} = g_y - \frac{1}{\rho}\frac{\partial p}{\partial y} + \nu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}\right)$$

$$\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} = g_z - \frac{1}{\rho}\frac{\partial p}{\partial z} + \nu\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}\right)$$

Incompressible continuity: $$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

If we consider the inviscid form, that means in my previous system, so this is the viscous term. If I drop out the viscous term, then the rest of the equation known as the inviscid form, which is also known as Euler equation that was given by Euler in 1757.

There are applications of this Euler equation, geophysical flows, aerodynamics, flow from the wall. So, if you look at these great scientists, I mean, their lifespan and the time when they were doing the scientific experiments and things like Bernoulli to Euler to Navier and Stokes and what we use today Navier Stokes equation that is contribution of so coming back to the differential form so this is what applicable to any three-dimensional equation system so one can write this is x momentum this is y-momentum, this is z-momentum and incompressible continuity equation because so that's right, i have continuity equation, i have the momentum equations using this what we can do, we can look at some of the applications like laminar incompressible discuss flows their exact solutions so these are again giving you an idea how to apply. So this is where we are talking about the kind of an application of the Navier-Stokes equation. So, we have looked at the both the integral form and the differential form of the Navier-Stokes equations here, what we are trying to look at the application of this one. So the applications specifically here, what we are looking at would do some laminar incompressible viscous flows where we have some exact solution so we can compare the data and or the analytical solutions with our mathematical solution.

So couette flow is one of that so when you look at the couette flow this is the velocity profile along the solid surface with zero pressure constant you find that the skin friction coefficient so finally you find out wall shear states where you have applications, you have lubrications, geological system, painting, cleaning, this is where you have to go. How does the flow looks? This particular plate here is the fixed and the top plate is moving. So then you get to see a velocity profile like this here. And this is what you can derive by starting with the continuity equations and write down the momentum equation and simplify that using your assumptions and then solve the equations. And then you finally get this velocity profile.

You get the shear stress, wall shear stress, then finally coefficient of friction. And Poisson flow is both the upper and lower surfaces are fixed. So obviously this gives you a parabolic kind of profile and then again you find out CF and other information from this. So these are simple example of the application of Navier-Stokes equation in incompressible laminar zone. So, I mean obviously when you go to turbulent zone then you may not be able to find out the exact solution and that is why that's still an open classical problem.

Couette Flow

$$u = \frac{u_0}{2}\left(1+\frac{y}{h}\right) = u_{\text{av}}\left(1+\frac{y}{h}\right)$$

$$\tau_w = \mu\left[\frac{du}{dy}\right]_{y=-h} = \frac{\mu u_0}{2h} = \frac{\mu u_{\text{av}}}{h}$$

$$\frac{\tau_w}{\frac{1}{2}\rho u_{\text{av}}^2} = \frac{2\mu}{\rho u_{\text{av}} h} \qquad C_f = \frac{2}{\text{Re}_\text{h}}$$

$$\text{Po} = C_f \,\text{Re} = 2$$

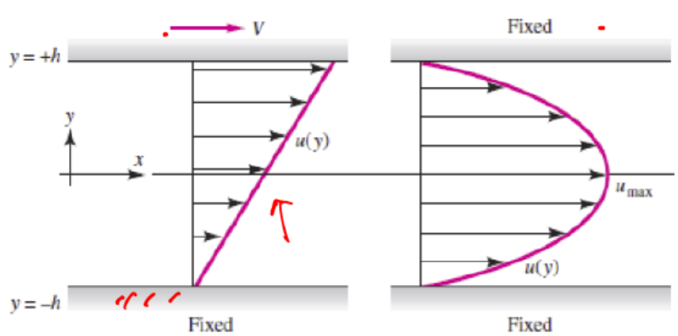
Poiseuille Flow: Summary

$$u = u_1\left(1-\frac{y^2}{h^2}\right)$$

$$u_{\text{av}} = \frac{1}{2h}\int_{-h}^{h} u\,dy = \frac{1}{2h}\int_{-h}^{h} u_1\left(1-\frac{y^2}{h^2}\right)dy = \frac{2}{3}u_1$$

$$\frac{du}{dy} = -\frac{2y}{h^2}u_1 = -\frac{3y}{h^2}u_{\text{av}} \Rightarrow \tau_w = \mu\left[\frac{du}{dy}\right]_{y=-h} = \frac{3\mu u_{\text{av}}}{h}$$

$$\frac{\tau_w}{\frac{1}{2}\rho u_{\text{av}}^2} = \frac{6\mu}{\rho u_{\text{av}} h} \Rightarrow C_f = \frac{6}{\text{Re}_\text{h}} \Rightarrow \text{Po} = C_f\,\text{Re} = 6$$



Now, we look at some of these non-dimensional of this equation system. So far what we are talking about the equation system that we have obtained either in integral form or differential form. We can look at this. So how do we start the non-dimensionalize? You start with the actual equation. Then you define your reference velocity.

You define your reference scale. And then you define this different, define this quantities. One is the, it's a non-dimensional quantity u star, which is a ratio of the u to u naught, v star, v to u naught, x star, y star. And also you can use this and del u, del v, del x, del y, everything you define in terms of non-dimensional. Now, once I do that, and put it back in the so essentially this and this, these two get me back this then if i multiply both side by l by u naught square finally i get this for the difference in dimensional time with the t star and p star what i get, i get a non-dimensional form of the equation is exact form of the or exactly similar to original dimensional one, dimensional form. But obviously it starts with star and all these things where you define a non-dimensional number called the Reynolds number which is UL by mu, I mean L is the characteristic length.

Nondimensionalization of Governing Equation

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=-\frac{1}{\rho}\frac{\partial p}{\partial x}+\nu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)$$

Reference velocity and length (scales): u_0, L

Define dimensionless quantities:

$$u^* = u/u_0\,, v^* = v/u_0\,, x^* = x/L, y^* = y/L$$

Now use: $u = u_0 u^*, v = u_0 v^*, x = L x^*, = L y^*$

$$\partial u = u_0 \partial u^*, \partial v = u_0 \partial v^*, \partial x = L\partial x^*, \partial y = L\partial y^*$$

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=-\frac{1}{\rho}\frac{\partial p}{\partial x}+\nu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)$$

$$u=u_0u^*, v=u_0v^* \quad \partial u=u_0\partial u^*, \partial v=u_0\partial v^*, \partial x=L\partial x^*, \partial y=L\partial y^*$$

$$\frac{u_0\partial u^*}{\partial t}+u_0u^*\frac{u_0\partial u^*}{L\partial x^*}+u_0v^*\frac{u_0\partial u^*}{L\partial y^*}=-\frac{1}{\rho}\frac{\partial p}{L\partial x^*}+\nu\left(\frac{u_0\partial^2 u^*}{L^2\partial x^{*2}}+\frac{u_0\partial^2 u^*}{L^2\partial y^{*2}}\right)$$

Multiply both sides by: L/u_0^2

$$\frac{L\partial u^*}{u_0\partial t}+u^*\frac{\partial u^*}{\partial x^*}+v^*\frac{\partial u^*}{\partial y^*}=-\frac{1}{\rho u_0^2}\frac{\partial p}{\partial x^*}+\frac{\nu}{u_0L}\left(\frac{\partial^2 u^*}{\partial x^{*2}}+\frac{\partial^2 u^*}{\partial y^{*2}}\right)$$

Further define dimensionless time, pressure:

$$t^*=\frac{t}{L/u_0}, p^*=\frac{p}{\rho u_0^2}$$

$$\frac{\partial u^*}{\partial t^*}+u^*\frac{\partial u^*}{\partial x^*}+v^*\frac{\partial u^*}{\partial y^*}=-\frac{\partial p^*}{\partial x^*}+\frac{1}{\mathrm{Re}}\left(\frac{\partial^2 u^*}{\partial x^{*2}}+\frac{\partial^2 u^*}{\partial y^{*2}}\right)$$

Where Reynolds number $\mathrm{Re}=\frac{u_0L}{\nu}=\frac{\rho u_0 L}{\mu}$

$$\mathrm{Re}=\frac{\rho u_0 L}{\mu}=\frac{\rho u_0^2L^2}{\mu\frac{u_0}{L}L^2}\sim\frac{\text{inertia force}}{\text{viscous force}}$$

$$p^*=\frac{p}{\rho u_0^2}=\frac{pL^2}{\rho u_0^2L^2}\sim\frac{\text{pressure force}}{\text{inertia force}}$$

So Reynolds number is nothing but the ratio of inertia forces to viscous forces and then you can have the p star which is a p by rho u square which is a pressure force to the inertia force because so non-dimensional numbers are kind of indicate the relative dominance of different forces so that's what what we are going to look at it now from this non-dimensional system or non-dimensional equation system how these different forces are going to be dominant at different time or at the different point of time or that under different situations so what we are going to look at it let's say we have a non Navier-Stokes equation so we have inertia here, we have pressure here, we are discussing obviously when one force is small enough the balance to others are sufficient okay it's essentially talks about the balancing act between different forces, whether the inertia, pressure or viscous. Now dimensional numbers are going to be very useful to comparing these forces. For example, if you have a creeping flow, then the small inertia, pressure and viscous forces are of the same order. If you have inviscid flow, then there is a small viscosity effect, then the pressure and inertia is going to be of similar order. Some cases of bounded air flow where you have small pressure viscous and inertia of the same model.

Now, how these things actually come into the picture? One can look at from the right angle-tangle analysis, let's say ABC. This length is A, this is B, this is C. So, C square is A square plus B square and this angle is phi and this is right angle-triangle. What we can say that area of ABC is function of phi and C. i can rearrange i can say phi is function of area of this triangle, so this is dimensionless the phi area is l square this is l so i can say that phi is function of ac by c square and ac by c square is function of phi so rather i can write in this fashion so it's a different way of writing so the area of triangle i can write that c square fp okay where this P is here and similarly area of angle.

So it is ABC ABD would be a square FP and ACD would be B square FP. So the total area of angle this ABC the total area is area of A and area of B. So if I write that you can get to cC square. You can see this dimensionless number can be used to even they can be related and can be used to relate these things in between. So, we can come back and look at the same example where you have a tank and there is a opening here.

Between point one and two, we try to find out what would be the V2. This is a quasi-steady incompressible frictionless flow and we have derived from Bernoulli's equation V would be root two gh. now the same problem we can try to look at from alternate perspective. So, what we want to look at that v is function of g and h so this is a function of u so i can use made of g and e so it has the dimensional number from here what we can do this is we write in terms of basic length and time scale then the gd and l then we try to find out this solution of m and n and once we get it used so we can find out the function of f2 if i put a constant this is what it is the solution may also be written in terms of pi which is so the idea here is that the function of velocity gravity and h is zero but v g h are all dimensional so this system is equivalent to psi of pi 0 pi is non-dimensional which pi is v by root gh so the way we try to define this different dimensional group of parameters into a non-dimensional group is called the dimensional analysis. So there is a theorem which states how you can do this dimensional analysis or carry out the dimensional analysis for certain physical phenomena which is governed by the f x1 to xn or small xn which are dimensional then the above phenomena can be represented as some of these pi variables which are non-dimensional pi1 to pi1 and the nature of f and psi may be obtained from different experiments.

Now this typical way of doing things is known as Buckingham Pi theorem where you have a set of variables which are dimensional then you group them into some non-dimensional group. So the minimum number of fundamental dimension would be k For example, function of vgh, so the k would be two and n equals to three, so m is one. We can see some example how we can do that. Let's say for this particular example, again coming back, we said that if vg, the function of vgh, which is the dimension parameter zero, and here the minimum variable is two, and the fundamental variables are three so you have only one pi which can be made of vgh so above equation suggests pi equals to vbhc then we can write in terms of mlt system and from there once you solve you get the pi variable is v by root gh so function of v by root gh is zero so that's a non-dimensional grouping of parameter similarly you can kind of take into account the repeating and non-repeating variables. So, selections of the repeating variables, they must be dimensionally independent and together they must include all fundamental dimensions.

So, once we kind of look at this chart, then we can find out the complete system how to analyze it. again this example if we come back so the dependent variables are velocity gravity h and mu so total variables are four minimum dimensional variable so which are essentially our mlt system but we need two so then we have select two repeating variables so let us say the repeating variables are g and h and non-repeating v and mu then we can find out this table but obviously we write down this and find out one pi variable pi 2 again we write that and get the second variable so idea here is that you identify the group of non-dependent variables or the pi variables And then with that, I can write now this dimensional group. I can say that it's a function of this non-dimensional group and this non-dimensional group. V by root g h is another non-dimensional number known as Froude number.

V by g h cube is Froude number by Reynolds number. So one can write function of Froude number or Froude number Reynolds number ratio 0 or Froude number is this i over pi. So this is the advantage of doing dimensional analysis. where instead of solving the equation. So one can definitely solve the Navier-Stokes equation, but without even solving that. using this kind of dimensional analysis where you can group the dimensional variable into non-dimensional numbers and then for example here the Froude number is a function of Froude number and reynolds number.

If it is a frictional flow then it's constant then viscous flow this becomes like that. That's what it tells you what is the advantage of this dimensional analysis. So I had this dimensional group, from where I write this non-dimensional group, that Froude number is V by root gh and Froude number by Reynolds number is this. So I can, I don't need to conduct too many experiments, less number of experiments can be used to get this dimensional system, very expensive, so data reduction becomes easier.

So we can find out single plot to find out. For example, you can find out this variation from the experiments with some points here and there, and then fit the curve. And once you know the curve, then this curve can be used for a particular number. So now, I can extend this concept to a different size and find out that, for example, once I have this particular figure, then I can define a tank size as long as I have this particular figure ready so I can model a tank and find out this size and then their velocity what is going to be at the exit so at a particular point I can find out all these variables. So that is the advantage of the non-dimensional analysis where you can actually use this. So, what is the basic idea behind this model testing because in the present case study we have a Froude number which is function of Froude-Reynolds number.

So, since the non-dimensional number holds good for similar kind of model that means the model and prototype tanks I can write this function and they are going to be same. So, the Froude number for model and similarly certain fluid mechanical phenomena can be governed. So, I can write this pi for model and prototype would be same. So, this is called the complete similarity request. I have kind of a geometric similarity, kinematic similarity and dynamic similarity.

For example, I can have this is called the geometric similarity where length scale matching a model so this is a model and this is a prototype they are geometrically similar if and only if when the body dimensions in all three coordinates have the same linear scale ratio that means whatever quad length is here i am reducing it to 10 times here this is 40 meter to 4 meter this is 1 meter to 0.1 meter so the whole geometric scaling from the model to prototype is done at this particular scale now when we talk about the dynamic similarity this is a forced scale force this is geometric similarity then i can have kinematic similarity when velocity scale matching, so, i have a flow around this bigger cylinder and there is a small cylinder so, model and prototype can be said kinematically similar if the homologous particles lie in the homologous points at every homologous time so essentially this is where you get the kinematic similarity and then you can get the dynamic similarity where you have four scale matching that means the model and prototype are dynamically similar if ratio of any two forces are same on the model and prototype from there So obviously, when you talk about dynamic similarity, dynamic similarity requires geometric kinematic similarities. So one has to check before model testing whether you have geometric similarity plus matching of pi theorem. So one hand again we can reiterate the geometric similarity depends on the proper design, manufacturing, material choice. So proper choice of variable include miscellaneous fluid dynamical effects.

In reality, it is not always possible to attain complete similarity and force to work with partial similarity. It may happen more than three dominant forces also. But ideally, one would like to have these dynamic and geometric similarities. So, that's the advantage of this similarity analysis where without getting into too much of experiments or spending money, once you figure out some of these non-dimensional curves, that can be used for design consideration. So, we'll stop here and continue our discussion on this in subsequent session. Thank you.