**Wind Energy**

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**Lecture 04: Fluid Mechanics- Integral form of Conservation Equations**

Welcome back. So, we'll continue our discussion on this fundamental software mechanics. So, if we connect back from our last session to where we're talking about the intensive and extensive properties. So, obviously, I mean, just to recall that extensive properties are the one which is very much, I mean, include the mass vary with the amount of the substance. So, that is what the extensive properties are. So obviously, one can think about mass, weight, volume, whereas in the intensive properties do not depend on the amount of the substance.

So now, what we talk about today, we are talking about the system which might change in volume with time. And then from there, we try to kind of derive those equations from the control volume system. Now here, if you look at it, you have a moving system. So, the system is moving from here to here to here, so which may change its volume and same and that is happening.

If you see the situation here you see the situation here and you see the situation here so, the system here it's moving toward a fixed control volume and here the boundary of the system at time instant tree and here the system leaving the control volume at particular time instant so that means i have a fixed control volume here so, it's just like i have a particular fixed control volume and there is a system which is moving it can come and overlap with the control volume then again goes out so this is how this moving system is behaving. So, at time t the system definitely coincides with the fixed control value as we have talked about it because the system is going from this to that. So, what we would try to make a connection that the system total derivative and their partial derivative that means which is your Lagrangian description with that. So, that brings to an important equation in any continuum mechanics I would say. Obviously, here we are talking about fluid mechanics.

So, we would have the... So, here the Reynolds transport theorem says that for any extensive property B and their corresponding intensive property beta, RTT states that the total derivative or the material derivative of that particular system which is a partial derivative their fluxes across the control surface obviously the beta or b of the system because the extensive property is b which is the βdm depend on the time. So, here essentially what it tries to connect is that your lagrangian description with your eulerian description for an arbitrary volume of it so what we can write is that if you look at the simple derivative so here this is my total change of the b, i'll write in terms of limiting variable over ∂t, t + ∂t and t so i'll do little bit of algebra here so i'll get this term segregated okay! and add this one and subtract the same thing so, if you correlate the terms together so what we can do is that this gives me the partial one and this is the derivative due to the control cell so, what happens is that if you try to look at more from pictorially this is at time t, the volume, and this is what happens in the, so this is what the change of B.

$$\frac{dB_{sys}}{dt} = \frac{\partial}{\partial t}\int_{CV} \rho\beta d\mathcal{V} + \int_{CS} \rho\beta(\vec{u}.\vec{n})dS$$

$$\text{where } B_{sys} = \int_{mass} \beta dm = \int_{CV(t)} \rho\beta d\mathcal{V}$$

Material derivative, also written as $\frac{DB_{sys}}{Dt}$

$\vec{n}$ : unit vector at the $CS$ pointing outward from the $CV$

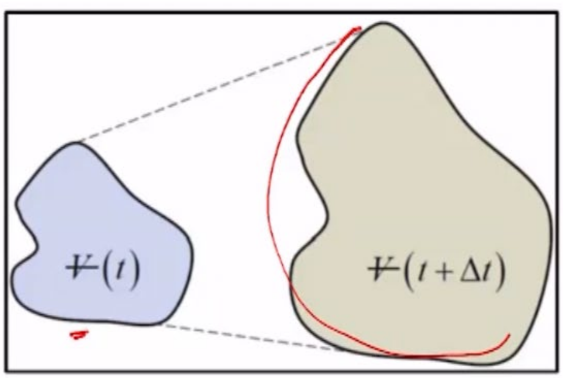
$$\frac{dB_{\text{sys}}}{dt} = \lim_{\Delta t \to 0} \left( \frac{1}{\Delta t} \left[ \int\limits_{\forall(t+\Delta t)} \rho\beta(t+\Delta t)\,d\forall - \int\limits_{\forall(t)} \rho\beta(t)\,d\forall \right] \right)$$

$$= \lim_{\Delta t \to 0} \left( \frac{1}{\Delta t} \left[ \int\limits_{\forall(t)} \rho\beta(t+\Delta t)\,d\forall - \int\limits_{\forall(t)} \rho\beta(t)\,d\forall \right] \right)$$

$$+ \lim_{\Delta t \to 0} \left( \frac{1}{\Delta t} \left[ \int\limits_{\forall(t+\Delta t)} \rho\beta(t+\Delta t)\,d\forall - \int\limits_{\forall(t)} \rho\beta(t+\Delta t)\,d\forall \right] \right)$$

$$= \frac{\partial}{\partial t} \int\limits_{\forall(t)} \rho\beta(t)\,d\forall + \lim_{\Delta t \to 0} \left( \frac{1}{\Delta t} \left[ \int\limits_{\forall(t+\Delta t)-\forall(t)} \rho\beta(t+\Delta t)\,d\forall \right] \right)$$

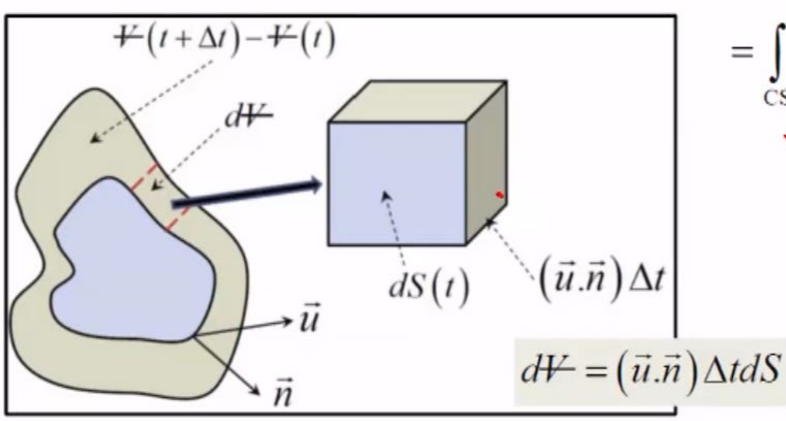
$$= \frac{\partial}{\partial t} \int\limits_{\text{CV}} \rho\beta\,d\forall + \int\limits_{\text{CS}} \rho\beta(\vec{u}.\vec{n})\,dS$$




$$\frac{1}{\Delta t} \left[ \int\limits_{\forall(t+\Delta t)-\forall(t)} \beta\,d\forall \right]$$

$$= \frac{1}{\Delta t} \int\limits_{\text{CS}} \beta(\vec{u}.\vec{n})\,\Delta t\,dS$$

$$= \int\limits_{\text{CS}} \beta(\vec{u}.\vec{n})\,dS$$



This happens within that, and then it becomes like that, so that's the change with respect to the control surface. So, one hand, you try to see the change of that intensive property with respect to time here, and then with respect to the surface. So, the RTT provides you the rate of change of B of the system which is the total derivative then rate of change of B in the Cd and rate of flow of B through the control surface. So, if you have a control volume obviously this can be also be proved for moving or deformable control volumes but this is a very unique theorem and using that all your conservation laws can be defined in field mechanics. I mean needless to mention here this Reynolds transport theorem is applicable to solid mechanics also in continuum assumption under the continuum assumption.

So, that is probably the primary key source for getting the equations for all these solid mechanics. Obviously, it starts with RTT and then with certain assumptions, they bifurcates. If one has to say what is the difference between this and this, if let's say B equivalent to the mass, So you consider a control volume which is just enclosing the fire extinguisher inside this. So here, this is always decreasing if that whole thing is on. But if I look at the total derivative of the system with time, then it's always zero.

That is the difference between. One is that the control volume I can have along this fire extinguisher. In terms of total derivative, this is zero. But whereas if it is open, then it's always decreasing, having a decreased mass. So, that becomes negative.

So, now we can find out the conservation of mass or mass conservation equation in integral form. So, what it says that conservation of mass is that deep the system. So, this is what we get from the RTT. And here V is M and beta is one. If you put it back, this is the DM.

# Conservation of mass: integral form

$$\frac{dB_{sys}}{dt} = \frac{\partial}{\partial t}\int_{CV} \rho\beta d\forall + \int_{CS} \rho\beta(\vec{u}.\vec{n})\, dS \qquad \text{here } B = m \Rightarrow \beta = 1$$

$$\frac{dm_{sys}}{dt} = \frac{\partial}{\partial t}\int_{CV} \rho d\forall + \int_{CS} \rho(\vec{u}.\vec{n})\, dS$$

$$\frac{\partial}{\partial t}\int_{CV} \rho d\forall + \int_{CS} \rho(\vec{u}.\vec{n})\, dS = 0$$

So, the mass, the system is not changing. So, this gives me this equation. So, this is an integral form. I have conservation equation.

Okay. Once I write that, I can now simplify that constant density or mostly non-variable density incompressible flow. So the CV doesn't change essentially, so this is zero. Similarly, for a reflow, where I get this because the CV doesn't change. These are, I mean this is my original mass conservation equation and these are my situations where we try to see whether the flow is. Now, the important thing that we kind of touched upon while talking about what is fluid or for the definition of the fluid and things like that, the Knudsen number in the continuum scale.

# Equation of mass conservation: integral form

$$\frac{\partial}{\partial t}\int_{CV}\rho dV\!\!\!/ + \int_{CS}\rho(\vec{u}.\vec{n})\,dS = 0$$

Constant density (incompressible) flow:

$$\frac{\partial V\!\!\!/_{CV}}{\partial t} + \int_{CS}(\vec{u}.\vec{n})\,dS = 0 \Rightarrow \int_{CS}(\vec{u}.\vec{n})\,dS = 0$$

If CV doesn't change

Similarly for steady flow:

$$\rho\frac{\partial V\!\!\!/_{CV}}{\partial t} + \int_{CS}\rho(\vec{u}.\vec{n})\,dS = 0 \Rightarrow \int_{CS}\rho(\vec{u}.\vec{n})\,dS = 0$$

If CV doesn't change

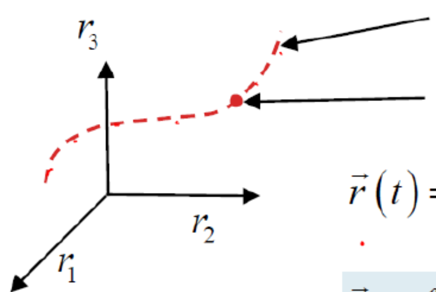
So, continuum hypothesis is applicable in this length scale is very, very high that molecular length scale that is mean free path. So you can see the range, this is what we are dealing with it, which is a very low Knudsen number. and then there is a range which is called the slip flow regime. There is transition regime. This is free molecularism.

So, Knudsen number is defined like an lambda by L. So, here for fluid, this is less than one, very, very much small. So, the continuum hypothesis that we are dealing with all the continuum scale phenomena and all these things, our Knudsen number is absolutely very small. As the Knudsen number increases so in some point of time it could be order of one that means, the lambda is somehow equivalent to the L that means the mean free path and the molecular length scale I mean essentially the mean free path and the actually the characteristics length scale are of the same order if they kind of going towards the transition regime and then goes to the mill I mean once it crosses if it get up then one I mean, obviously, for pain or such that it goes to a pre-molecular job. So that's something one we are not talking about here.

So now, if we try to summarize the things, one, we have a Lagrangian definition, which is particle-based. Another one is a Eulerian definition. Here we model the fluid as a bunch of particles. Maybe many many particles and we track the particles. We track their instantaneous velocity positions and things like that.

So you can derive their local accelerations. Here we talk about it's a many many points in space. So fluid particles arrived at a point and comes and goes out. So, it's more like an individual particles and not another we try to look at the fluid particle when it passes and now, Lagrangian description this is a path line of the particle that means the particle has actually traversed through these points so, this gives me path line of the particle and then the particle position would be a vector so I can find out from the initial position to where it is then it can give me the particle velocity their acceleration. okay! so, in lagrangian description position of a particle is not an independent variation is a function of independent variable time particles are identified based on the initial condition okay! if we look back The Eulerian description.

## Lagrangian description



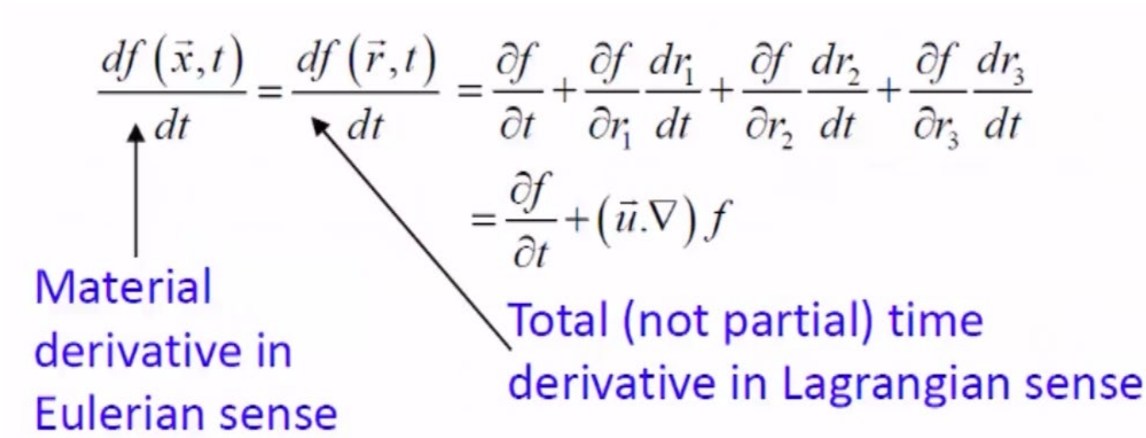

Particle position: $\vec{r} \equiv (r_1, r_2, r_3)$

$$\vec{r}(t) = \vec{r}_0 + \int_{t_0}^{t} \vec{u}\,dt; \text{ initial condition : } \vec{r}(t = t_0) = \vec{r}_0$$

$$\vec{r} = f(\vec{r}_0, t_0, t) \qquad \vec{u} = \frac{d\vec{r}}{dt}, \vec{a} = \frac{d^2\vec{r}}{dt^2}$$

in general, any property $f = f(\vec{r}, t)$

So, there is a point in space. This is obviously in continuum scale. The point is always occupied by some particles. So we can find out the coordinate. Any property is defined space and time.




$$\frac{df(\vec{x},t)}{dt} = \frac{df(\vec{r},t)}{dt} = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial r_1}\frac{dr_1}{dt} + \frac{\partial f}{\partial r_2}\frac{dr_2}{dt} + \frac{\partial f}{\partial r_3}\frac{dr_3}{dt} = \frac{\partial f}{\partial t} + (\vec{u}.\nabla) f$$

The above property is the property of the occupiers and the Eulerian description point coordinate and times are independent. So there is a difference. But whereas you can combine these two. together. We have a path line of the particle.

We have the particle position. We have the point coordinate. So, we can say that this is the material derivative of that at a particular time. Then this could be partial derivative of time and then respect to these vectors. So, which essentially gives me back the equation that we have seen.

Okay! So, that means the Lagrangian and Eulerian you can combine them together and gets the integral form of conservation equation. okay! so, when we look at the Reynolds transport theorem again going back to the system so here we are now looking at the momentum conservation. For momentum conservation what we look at it we say that µb is µ and β is u then if i put back in that RTT so this is my RTT and then we define b and β then i get this so momentum conservation principle says that dmu by dt is the So integral formulation of momentum conservation equation. So F equals to this is the partial derivative. This is the convective derivative.

## Reynolds Transport Theorem:

$$\frac{dB_{\text{sys}}}{dt} = \frac{\partial}{\partial t}\int_{\text{CV}} \rho\beta\, d\forall + \int_{\text{CS}} \rho\beta\left(\vec{u}.\vec{n}\right) dS$$

For Momentum conservation: $B_{\text{sys}} = m\vec{u} \Rightarrow \beta = \vec{u}$

$$\frac{d\left(m\vec{u}\right)}{dt} = \frac{\partial}{\partial t}\int_{\text{CV}} \rho\vec{u}\, d\forall + \int_{\text{CS}} \rho\vec{u}\left(\vec{u}.\vec{n}\right) dS$$

Momentum conservation principle: $\frac{d\left(m\vec{u}\right)}{dt} = \vec{F}$

Momentum conservation (intgral formulation) $\vec{F} = \frac{\partial}{\partial t}\int_{\text{CV}} \rho\vec{u}\, d\forall + \int_{\text{CS}} \rho\vec{u}\left(\vec{u}.\vec{n}\right) dS$

$$\vec{F} = \vec{F}_S + \vec{F}_B$$

$\vec{F}_S$ : Surface force, all forces acting at the control surface

$\vec{F}_B$ : Body forces (gravity, electromagnetic)

So, I have two forces to be balanced out. One is the surface force. So, all forces which are acting on the control surface. So, if I have a control volume like this.

So, this is a control surface. So, the forces that is acting on this. and the body forces so this is the one so this is the body forces which could be gravity and the surface forces are usually come from pressures here interaction with the solid objects and this okay! so now if i write them together so what i have is that surface force plus body force is this Lagrangian derivative and the convective derivative. So, obviously in incompressible flow, we assume density is not very much. We can write this. And in a non-deformable control volume and steady flow, we write this.

$$\vec{F}_S + \vec{F}_B = \frac{\partial}{\partial t}\int_{CV} \rho \vec{u}\, dV + \int_{CS} \rho \vec{u}\left(\vec{u}.\vec{n}\right) dS$$

Incompressible flow: $$\vec{F}_S + \vec{F}_B = \rho \frac{\partial}{\partial t}\int_{CV} \vec{u}\, dV + \rho\int_{CS} \vec{u}\left(\vec{u}.\vec{n}\right) dS$$

In a non-deformable CV

0

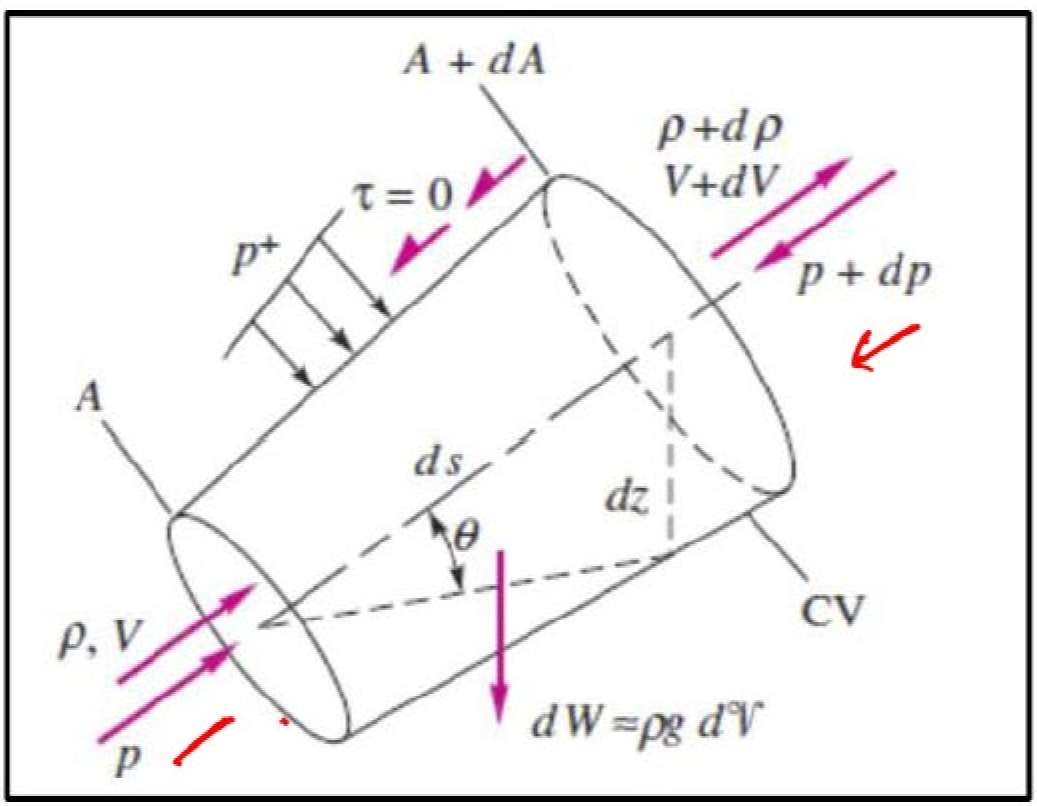
Steady flow: $$\vec{F}_S + \vec{F}_B = \int_{CS} \rho \vec{u}\left(\vec{u}.\vec{n}\right) dS$$

Steady, incompressible flow: $$\vec{F}_S + \vec{F}_B = \rho\int_{CS} \vec{u}\left(\vec{u}.\vec{n}\right) dS$$

So, this term goes to zero. Steady incompressible flow, this again simplifies to this. So, these are the different situations where you can simplify stuff like that. So, pressure force always at the control surface which is towards the control volume is compressive. That means if we have some and pressure is acting like that then this is behaving like a compressive force whereas if pressure is used like that then it is a tensile force. There are some thumb rules of choice for control volume.

Surface normal at control surface will be along or opposite to the flow direction. If you have uniform pressure distribution over a closed control surface, so eventually that leads to zero pressure force. Then the reference pressure simplifies the force calculation. so, these are the example of different system where this is a hydraulic turbine versus pelton wheels so, here you need to fix your reference system along the rotating frame of the system so, that you try to find out the down equations or whether get the calculation done for that system Now, what we're going to touch upon is that the look at the momentum equation along a streamline. So, if I have a tank like that and this is connected with the pipe.

So, if there is a stream line coming out so along a steam line we want to write which is known as Bernoulli's equation. So, this is extensively useful for steady frictionless flows. This doesn't require it is not essentially differential not integral formulation obviously, it can allow you to predict the real life situations fairly well. okay! so, how do we find out so you assume a steady incompressible flow along a stream line and then I have some pressure at this and then density, velocity So, I write the mass conservation.




So, VA, which is V + VD. Momentum conservation, the pressure force, body force. So, this phase rho V square A, other phase rho V square D B. So, it simplifies, we get this.

Okay! Now, the pressure force surface, I mean the all surface forces, the pressure force at the surface, which are P A and B plus D B by 2 A P. So, if you do that, that gets a body forces sin theta.

## Mass conservation:

$$VA = (V + dV)(A + dA)$$

## Momentum conservation:

$$F_{ps} + F_{bs} = -\rho V^2 A + \rho (V + dV)^2 (A + dA)$$

$$= -\rho V^2 A + \rho V A (V + dV) = \rho V A dV$$

$$F_{ps} = pA - (p + dp)(A + dA) + \left[\left(p + \frac{dp}{2}\right) A_p\right]_s$$

$$= -\frac{dp}{2} A - \frac{dp}{2}(A + dA)$$

$$= -A dp$$

Momentum conservation: $-Adp - Adz\rho g = \rho VAdV$

$$\rho VdV + dp + \rho gz = 0$$

So, momentum conservation get me this. So, which is essentially this. So, from momentum conservation, if I take the differential of that, so that allows me to get this b square by 2 plus p by rho gz equals to 0, which is constant along a straight line. This is very much well-known equations or known as the Bernoulli's equation, which is applicable to steady incompressible friction and so on. this is very very important to note steady incompressible and frictionless flow obviously Bernoulli's equations if you are i mean interested enough you can look at the textbook Bernoulli's equation can be derived for more general situations such as unsteady compressible viscous flows but yes they might not look so simplified system but it can be done. So now, if I look at that Bernoulli's equations here, one can comment that no real flow is strictly frictionless.

Momentum conservation:

$$\rho VAdV + Adp + A\rho gdz = 0$$

$$\Rightarrow d\left(\frac{V^2}{2} + \frac{p}{\rho} + gz\right) = 0$$

$$\frac{V^2}{2} + \frac{p}{\rho} + gz = B \ \text{(constant along a streamline)}$$

But, the Bernoulli's equation is still used in real flow to have an idea about the pressure. So, what is the consideration here when the flow is from the solid one  you may sometimes be considered as a fixed list, again the flow under the special case known as irrotational flow for example if this is my solid one and i'm talking about flow here, which is irrotational flows when it goes out from the solid world. So obviously in those cases b is constant and all these things so, if i write the boundary equation then it comes with those assumptions like steady fixed and less incompressible but generally fixed and less fixed are very very significant in narrow flow passages, diverging sections, weak region, narrow and separated flow. All these are situations where Bernoulli's equation should not be used because I mean obviously one can use that by I mean, it's a theoretical calculation and one can forcefully use that, but it might give you all kinds of wrong calculations.

So, here we can see the flow past an airfoil. Stick lines are attached to the solid surface. Here the stick lines are slightly separated. So, this is the zone which is called separated. So, here it is a zero angle of attack.

Here you change the angle of attack. Here there is a weight behind the moving baseball. Here one can clearly figure out this particular case you can apply Bernoulli's equation. But when you are just little away from the solid surface in case 2 and 3, we do have far away from the solid surfaces. So, these locations you can still use, but not here.

This is where you cannot use Bernoulli's equation. This is where you cannot use Bernoulli's equation. So, someone or one has to be careful while using those Bernoulli's equations. We assume steady slow quasi incompressible frictionless flow. So, here is a tank and through which there is a tap connected to that or outlet through which this is coming out. So, you can apply Bernoulli's equation and find out the velocity here.

So, we can apply the mass conservation. So, here and here, then we can use the mass conservation. Obviously it's an incompressible, so density doesn't change. If you can constantly get this, then you apply the Bernoulli's equation between 0.

1 and 0.2, you get the V2. This is a simple example of Bernoulli's equations where we can find out these things easily. Here is another example. Here you are supposed to find out the effects at the flange where there is a water flow of 1.5 meter cube per minute. Again, the assumption is very important, steady, incompressible, frictionless flow.

So the P2 would be atmospheric. So now across this, I'll write the mass conservation. So 0.1, you have A1, U1, here A2, U2, which is the cube. if i use boundaries between one and two i'll get because other terms like there is no elevation because they are at the same plane with the velocity difference i have the pressure differences and the momentum balance if i write then effects plus here they are having the pressure force and then connect to the change in momentum.

We can find out the effects from here. So, one can see that you can easily apply two equations to this kind of situation. Now, we move to the another conservation equation which is the energy conservation equation which write dE by dt Q dot minus W. So Reynolds transport theorem says this, where E is the specific odd unit energy or energy per unit mass and rate of the work would be shaft work, shear pressure, others. If I combine then Q dot minus W is rate of change of energy and the convective transport of that. If you have this kind of thing, here is the surface normal velocity, then I can find the elemental pressure here.

I can find the displacement rate, then I can have W dot pressure at this control surface. If I put it back, then my energy equation becomes like this. So, what it looks like, that I'll have Q dot here, others, and then changing. if you have a steady flow this goes to zero then you get this energy this e has internal energy, kinetic energy and potential energy and i can write in terms of enthalpy as well. So, there's a different way one can write it down and if you write that then you can again put it back in the main equation so you can get so enthalpy is essentially my internal energy and p by rho so, p by rho plus internal energy so again steady non-dependent volume i can write change in h plus u square by 2 g z, this so if there is no palm turbine they are involved in the sap work is not there so if there is no losses then this is a steady flow energy equation that one can derive so this can be used for steady flow energy session in a system so which one can write for isothermal adiabatic one inlet one exist system so this is my internal energy which is constant then adiabatic mass flow rate then i write this equation system like this so i can write that m dot p by rho z square minus losses.

So, these are the system one can apply so if you extend that then you can connect this is the pressure head this is the velocity head this is elevation head this is the shaft work this is the head loss so from here you can find out so find out the power so it can be easily applied to a system where your pumps and all these things are involved now you can look at this particular example where there is a kind of a dam through which this there is a turn involved and you can apply this equation and find the loss so from head loss you can find out how much power is going to be used. So, this is an example so if you put in let's say how much available friction and that's if you the total power you can find out. Similarly in this particular case you can find out how much power is going to be so that energy equation with the head loss one can apply and then find out so connectivity between bernoullis equation and the energy equation is that bernoulli equation is kind of applied along a stream line in a frictionless flow where energy equation, velocity, pressure are over control surface. They look similar for adiabatic, isobary, isothermal frictionless flow with no work interaction but in some cases they may be different. Okay! So, if we try to recap what we have been talking about, that we have talked about this RTT, then assume the control volume doesn't change over time, that being there, then now.

So, from RTT, you get your mass conservation and then you get momentum conservations and all this. So using RTT, you can actually find out all this conservation equations like mass conservation, momentum, energy conservation equation. Obviously, one can, these are all written in integral form. Now, one can also write in the differential form which is shown here and that we will discuss in further sessions.