Introduction to Aircraft Control System

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Week – 02

Lecture – 09

Introduction to Laplace Transform and Transfer Function

In this lecture, we'll be discussing the concept of convolutional integral from the concept of convolutional sum. Then we'll apply Laplace transform on this convolutional integral and how we'll find the linear relationship between input and output. From this linear relationship, we'll find the transfer function concept, which will relate between input and output. Also, we'll be discussing why we have to assume zero initial conditions for finding the transfer function, because if you assume non-zero initial conditions, the principle of superposition cannot be applied. Then we'll conclude the lecture with some important properties of Laplace transform, which we'll be using in the later lectures. In this lecture, we'll be finding the transfer function for our system.

Here we'll be using the concept of convolutional integral and how we can find the transfer function from this concept. If you remember, we have done this part already. If you remember, if you have LTI system, and if you apply some input $x(n)$ and the output, the impulse response, we can write

$$y(n) = \sum_{k=-\infty}^{\infty} u(k)h(n-k)$$

So this is how we came up with this expression.

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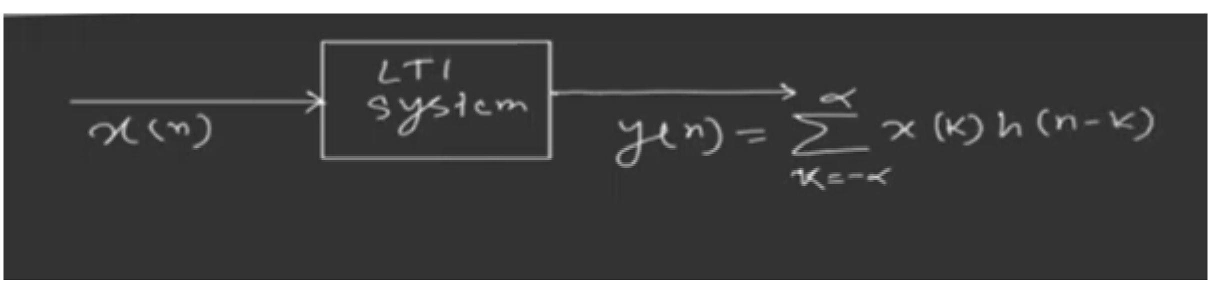

Now we'll be using this convolutional sum. How can you use this concept to find the transfer function? Now from this, we can write in the series form

$$y(t) = \sum_{k=-\infty}^{\infty} u(k)h(t-k)$$

This is also the same thing, basically the same expression we are writing in, just which is instead of n which is the time. Now the summation of the above equation number 1, is replaced by the following integral. So we can write in terms of integral, instead of summation

$$y(t) = \sum_{-\infty}^{\infty} u(\tau)h(t-\tau)$$

This expression also can be written from this convolutional sum. And let me define this equation number 2. Now this is actually called convolution sum and this is called convolution integral because it is in the summation form and this is in integral form equation 2. Now we shall use this concept to find the convolutional Laplace domain. Now it is in the time domain, now if you write, if you want to find the transfer function from this approach is quite difficult.

So now we will be considering some indirect approach using the Laplace transform. And for any function $y(t)$, the Laplace transform, we can write $Y(s) = \int_{-\infty}^{\infty} y(t)\, e^{-st} dt$ So this is basically we can write Laplace transform of $y(t)$. And applying the convolution theorem here in place of $y(t)$, let me write equation number 3, if you substitute equation 2 in equation 3, in place of $y(t)$, we can write

$$Y(s) = \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} h(\tau)u(t-\tau)d\tau \right] e^{-st} dt$$

So here, since we are talking about the LTI system, and it means the linear time invariant system, instead of this, we can write this expression, we just change the time. So it is the same expression, the same value will come up, and it does not change if you shift the time. So that is possible. Now if you change the integration order, we can write

$$Y(S) = \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} u(t-\tau)e^{-st} dt \right] h(\tau)d\tau$$

Let me write this is equation number 4, this is equation number 5. Now if you change the variables of the inner integral by defining $t - \tau = \eta$, we get the following expression.

$$Y(S) = \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} u(\eta)e^{-s(\eta+\tau)} d\eta \right] h(\tau)d\tau$$

Just we are changing the inner integral and we are replacing the time with this parameter η . And we can further write from this expression

$$= \left[\int_{-\infty}^{\infty} u(\eta) e^{-s\eta} d\eta\right] \int_{-\infty}^{\infty} h(\tau) e^{-s\tau} d\tau$$

And from this we can write, this is basically U(S), this is the Laplace transform of U(t) and from the second part we can write H(S).

So from this we can write $U(S)\ H(S)$. This is a very very important equation from this expression for finding the loss of function. So now let me define it, what is this? So here $U(S)$, I should write Laplace transform of the input time function and $H(S)$ is the Laplace transform of impulse response. So when you are talking about what is the transfer function, we have to write this Laplace transform of the response.

So now by this operation the complicated convolution integral is replaced by a simple multiplication of the transforms. So now we need to interpret this thing that you have done here, how we can do more treatment and how we can come up with a better understanding on transfer function. Now let's have a transfer function $H(S)$ and we are applying some exponential input function $U(t) = e^{st}$ and the output from this transfer function block can write $y(t) = H(S)e^{st}$. So here basically this is a complex number $s = \sigma_1 + j\omega$, again this is the real value, this is the imaginary part. So now we will prove how we are getting this expression through the concept of convolution integral.

So here $y(t) = \int_{-\infty}^{\infty} h(\tau) u(t-\tau) d\tau$ and from this we can write $\int_{-\infty}^{\infty} h(\tau) e^{s(t-\tau)} d\tau$. Further we can write $\int_{-\infty}^{\infty} h(\tau) e^{st} e^{-s\tau} d\tau$ or we can write $\int_{-\infty}^{\infty} h(\tau) e^{-s\tau} d\tau\ e^{st}$. So this is nothing but our $H(S)$ so we can write $H(S)e^{st}$. So this is what we are getting the same expression through this proof. Since we will be dealing with the linear system and also in the next few lectures we will be dealing with the transfer functions and it will be assumed that the initial conditions for finding the transfer function is assumed to be zero.

So why do we have to assume the initial conditions for finding the transfer function in the linear system assumed to be zero. Let's look at why you have to assume. So on the integral it conveys that $-\infty\ to\ \infty$ and which implying that $h(t)$ may have values at any time. So it can take any values from the $-\infty\ to\ \infty$, so the initial condition is not equal but for the causal system if you look $h(t)\ = 0$ for $t < 0$. So if I write in the $h(t)$, basically if you write in the frequency domain you have $\left[H(S) = \int_{-\infty}^{\infty} h(\tau) e^{-s\tau} d\tau\right]$ this is the function we have found here if you notice here.

So now if we assume the causal system, then for $t < 0$ the $h(t)$should be zero. So based on this condition we can write $H(S) = \int_0^{\infty} h(\tau) e^{-s\tau} d\tau$ and correspondingly for causal system $y(t) = \int_0^{\infty} h(\tau) u(t-\tau) d\tau.$ So that's why most of the system will be dealing in the classical contours of the causal system ,we'll be assuming the initial conditions to be zero. So let's look another way, what is the relevance, why you have to assume the initial conditions to be zero. So here if we assume a non-zero initial condition , the superposition principle cannot be applied or validated and in another condition we are assuming that the system starts from an equilibrium point.

If you go back to the lecture's scenario, how the system got disturbed from the equilibrium point, so if they start from the equilibrium point, note that the equilibrium point for the LTI system is always zero. So that's why in the transfer function methods we'll be assuming the initial condition to be zero. So now I'd like to highlight some important properties of Laplace transform and how we can come up with the linearity or this differential or integral and how we can come up with some standard Laplace transform of those functions because we'll be using very repeatedly in our subsequent lectures those important properties of Laplace transform. So first we'll be talking about linearity. If "a" is a constant and $\mathcal{L}f(t) = F(S)$, then Laplace transform of

$$\mathcal{L}\{af(t)\} = a\mathcal{L}f(t) = a\,F(S)$$

Now we'll be taking another example. Second part translation in time. If $\mathcal{L}f(t) = F(S)$ and $a > 0$,a is a positive real number such that if $f(t-a) = 0$ for $0 < t < a$, then we can write $\mathcal{L}f(t-a) = e^{-aS}F(S)$. So these are the functions we'll be using in our subsequent lectures. And third translation in Laplace domain.

If $\mathcal{L}f(t) = F(S)$ and a is a complex number then we can write $\mathcal{L}\{e^{at}f(t)\} = F(S-a)$. Fourth is real differentiation. If $\mathcal{L}f(t) = F(S)$ and if $\frac{df(t)}{dt}$ is Laplace transformable, then we can write $\mathcal{L}\left\{\frac{df}{dt}\right\} = S\,F(S) - f(0^{+})$. So here small $f(0^{+}) \rightarrow$ the value of $f(t)$ in the $\lim\limits_{t\to 0}$ approaching from the positive side. So this is the first term derivative, if you take the kth derivative so if you have a function $\frac{d^{K}f(t)}{dt^{K}}$, if you want to take the Laplace transform of this function we can write

$$\mathcal{L}\frac{d^{K}f(t)}{dt^{K}} = S^{K}F(S) - S^{K-1}f\left(0^{+}\right) - S^{K-2}\frac{df(0)}{dt} \ldots - \frac{d^{K-1}f(0^{+})}{dt^{K-1}}$$

So this is how we can find the Laplace transform of a derivative differential function. Now we will take an integral and how we can find the Laplace transform real integral function. If $\mathcal{L}f(t) = F(S)$ and the indefinite integral $\int f(t)dt$ is Laplace transformable then we can write

$$\mathcal{L}\left\{\int f(t)dt\right\} = \frac{F(S)}{S} + \frac{1}{S}\int_{-\infty}^{0} f(t)dt$$

but later we will be assuming or since we will be dealing with the positive values of t ,so we will not consider the initial conditions, later we will be explaining how it is coming up. So note that the integral term on the right hand side is zero if $f(t) = 0$ for $t < 0$. So if you have another example, infinite, which is very important for sixth number, initial value theorem, I will find the Laplace transform of the initial value theorem in the time domain.

If $\mathcal{L}f(t) = F(S)$ and $\frac{df}{dt}$ is transformable and $\lim_{t\to\infty} SF(S)$ exists then we can write $f\left(0^{+}\right) = \lim_{S\to\infty} SF(S)$. This we'll be using very frequently while we're talking on control design and steady state behavior of the system if the flight is going to equilibrium point and how we can find whether the exactly going to equilibrium point or not or the desired values or not this thing we'll be using for finding the steady state error in the system. And seven the final value theorem, if $\mathcal{L}f(t) = F(S)$ and $\frac{df}{dt}$ is Laplace transformable and $\lim_{t\to\infty} f(t) = f(\infty)$ exists then we can write $f(\infty) = \lim_{S\to 0} SF(S)$. So this we'll be using for the steady state behavior of the system how we can find the error, suppose this is my desired response that I want to fly my aircraft, now if the system goes like this and if there are some errors at the steady state find this amount of error using this final value theorem.

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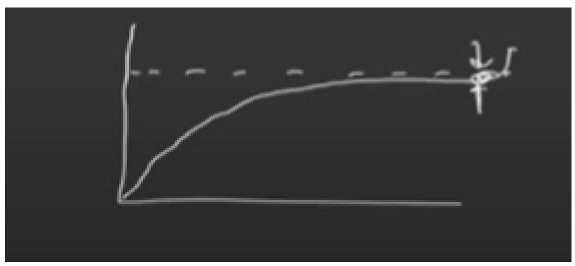

So let's stop it here. We'll be continuing from the next lecture. In the next lecture we'll be talking how we can find the transfer function of our system and also why you have to assume the initial condition to be zero in the transfer function and also we'll be coming up with the concept of pole zero in the system, in the transfer function how you can come the come up with the poles and zeros of the system. So thank you very much, we'll continue for the next lecture.