Introduction to Aircraft Control System

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Week – 02

Lecture – 07

Linearization of Nonlinear System

In this lecture, we'll be discussing the standard method of linearization around the reference point through that Taylor series expansion. And then we'll apply this concept for the pendulum system and how we can get two different models of linear state equations of the pendulum system. Then the linear state equation, what we will be finding from this lecture, we can validate to the previous lecture what are the findings we had, then we'll conclude the lecture. Now, we'll be using the concept of linearization. How can we use this concept to convert the nonlinear system into linear form? This is called the linearization technique. So, let's consider a scalar function, first we'll assume a scalar function then we'll extend to the vector function, a scalar function $f(z)$ which is differentiable, so the function behaves something like this, this is my $f(z)$ function, this is z is the independent variable and let's assume the function something like this.

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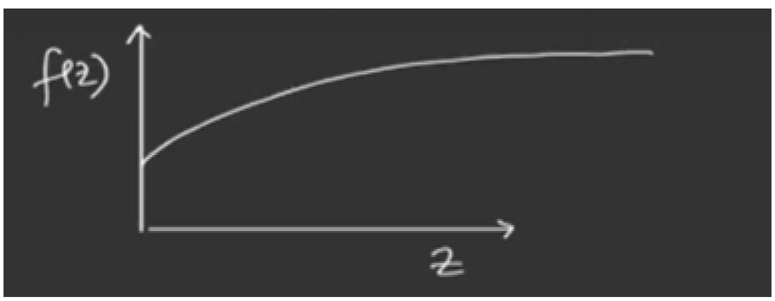

Now, it is found to be equilibrium point on some reference point somewhere here, for example, this is my some reference point or you can say that equilibrium point and around this point will approximate this nonlinear function something like this which is basically first order approximation, we can say at $z = \bar{z}$. Now, for values of z close to $\bar{z}$ we can replace $f(z)$ with its approximation using the Taylor series, around $\bar{z}$ we can write $f(z) \equiv f(\bar{z}) + \left(\frac{df}{dz}\right)_{z=\bar{z}} (z - \bar{z})$, so this is very standard expression if you apply the Taylor series expansion function $f(z)$. Here we are not considering the higher order terms because the magnitude of those higher order terms are quite less compared to the first order term, so we are neglecting those terms. So, this is very basically first order approximation around z.

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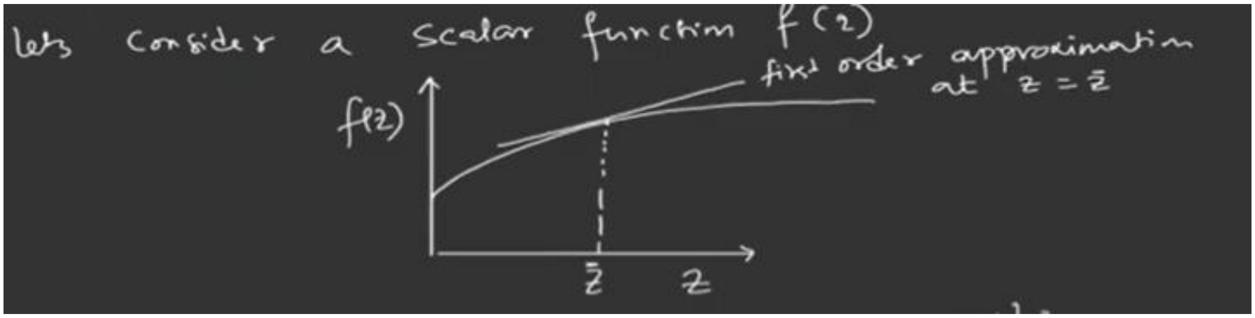

So, here basically the main concept of this technique is that how the system is going to evolve around this equilibrium point or some reference point $\bar{z}$. Now, this is basically a scalar case if you extend this concept to our higher order functions, so if you consider a vector valued function like this $f^n\colon IR^n x\ IR^m \rightarrow IR^n$ and if you also assume the expression $\dot{z} = f(z\,,u\,)$ for example, so this is more complex cases where we are having some disturbance or any external perturbations or some controls on the system, then we can write in general form in this expression, where z is basically a vector valued function or z is basically vector and u can be also vector. So, now in that case we can approximate this function, let me write this equation number one, we can approximate this equation number one, $f(z\,,u\,)$ at point $(\underline{z}\,,\underline{u}\ )$. So, this is basically some equilibrium point is $(\underline{z}\,,\underline{u}\ )$, some fixed point we can say is in the previous case, in the previous case we had only $f(z)$, now we have $f(z\,,u\,)$, so there are two different states and they can be two different equilibrium points. So, in this case we can extend the above concept, we can approximate $f(x\,,u\,) \equiv f(\bar{x},\bar{u}) + \left[\frac{df}{dz}\right]_{z=\bar{z}} (z - \bar{z}) + \left[\frac{df}{du}\right]_{z-\bar{z}} (u - \bar{u})$.

So, this is also a first order approximation, but this is for different multi variables where x and u have two different various state vectors. So, we can write where

$$\left[\frac{\partial f}{\partial z}\right] = \begin{bmatrix} \frac{\partial f_1}{\partial z_1} & \cdots & \frac{\partial f_1}{\partial z_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial z_1} & \cdots & \frac{\partial f_n}{\partial z_n} \end{bmatrix}_{nXn}$$

So, this is called the Jacobian matrix. This is a very powerful part in this case because this matrix will be useful for how we will be finding the eigenvalues or the roots of the system and how we can comment on the stability of the system also.

This is a very simple method you can come up with the conclusion on stability. I mean we'll be talking those things later in the subsequent lectures and here

$$\left[\frac{\partial f}{\partial u}\right] = \begin{bmatrix} \frac{\partial f_1}{\partial u_1} & \cdots & \frac{\partial f_1}{\partial u_m} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial u_1} & \cdots & \frac{\partial f_n}{\partial u_m} \end{bmatrix}_{nXm}$$

The dimension of u vector is m here so that is we'll go till m partial derivative. So, the dimension of these we can write n x n and n x m. So, this is also a Jacobian matrix for the control part but this is called the state matrix ,control matrix but we'll be talking later how we'll connect these things to our systems.

So, this is how we'll be finding the Jacobian matrix for the system. Now, if you apply this concept to the systems $\dot{x} = f(x\,,u)$ and $y = g(x\,,u)$. So, this is same system, here instead of z we are writing x here, so it is structured by same and if you apply this above concept the way we have found the Jacobian matrices for a non-linear system, then the approximations at $(x\,,u) = (\underline{x}\,,\underline{u}\,)$ we can write $f(x\,,u) \equiv f(\bar{x},\bar{u}) + \left[\frac{df}{dx}\right]_{x=\bar{x}\ u=\bar{u}} + \left[\frac{df}{du}\right]_{x=\bar{x}\ u=\bar{u}} (u - \bar{u})$. So, this is the first order approximation of the system and also the output equation,in the previous case we have done for the state equations, also the similar procedure we can apply for this output equation and so here we'll apply the same concept for the output equation as well so the output equation we can write $g(x\,,u) \equiv g(\bar{x}\,,\bar{u}) + \left[\frac{dg}{dx}\right]_{x=\bar{x}\ u=\bar{u}} (x - \bar{x}) + \left[\frac{dg}{du}\right]_{x=\bar{x}\ u=\bar{u}} (u - \bar{u})$ this is a very standard technique in Taylor series expansion, so in Taylor series basically we find the series of a function about some reference point. This is our reference point and how we are going to find the linearized model about this reference point so that is similar. So this is how we find the linearized model of the output equation so here if you define $A = \left[\frac{df}{dx}\right]_{x=\bar{x}\ u=\bar{u}}$ so this is basically, if you notice, a matrix , these are matrices. So we can write a matrix a in define and we can define $B = \left[\frac{df}{du}\right]_{x=\bar{x}\ u=\bar{u}}$ and this is called in control system state matrix and this is called the control matrix. Welcoming these things extensively in the future lectures for the time being please remember this is we call the state matrix for the output equation and this is also the control matrix for the output equation.

So here we define $C = \left[\frac{dg}{dx}\right]_{x=\bar{x}\ u=\bar{u}}$ and $D = \left[\frac{dg}{du}\right]_{x=\bar{x}\ u=\bar{u}}$ So these are basically matrices for the output equation, now we'll find the linear equations from the above concept. From the above concept if you write in terms of matrices if you write from this system and we can write equation as $x \equiv f(\bar{x},\bar{u}) + A(x - \bar{x}) + B(u - \bar{u})$ and similarly for the output equation we can write $y \equiv g\,(\bar{x},\bar{u}) + C(x - \bar{x}) + D(u - \bar{u})$ This is basically what we are getting from the Taylor series expansion.

Now we'll come up with a more compact form. Let's define $\sigma = x - \overline{x}$ and $v = u - \overline{u}$ and $\omega = y - g(\overline{x}, \overline{u})$. So this is my actual values and these are the trim values $\bar{x}\ \bar{u}$ and g is the function at the equilibrium point $\bar{x}\ \bar{u}$ and from this we can come off $\sigma = x - \overline{x}$ since here we are assuming x is constant so we get x dot and if you apply this concept for our system, we can write

$$\dot{\sigma} = (x - \dot{\bar{x}}) = \dot{x}$$

$\dot{\sigma} = f(\overline{x}, \overline{u}) + A\sigma + Bv$ and also you can write $w = C\sigma + Dv$ .This is quite easy you can connect and also it should be noted that at the equilibrium point these values, this function yields to be zero or we can write the approximation is linear when $f(\bar{x}, \bar{u}) = 0$, it's quite obvious because we are finding the equilibrium point making the function $f = 0$ .This is how we are finding the equilibrium point and the solution we are getting from the function when $f = 0$ if the solution from this condition will satisfy this $f = 0$ for those values will come up this condition as well. So this is quite obvious and from this condition we can come out $\dot{\sigma} = A\sigma + Bv$ and $W = C\sigma + Dv$. So this is called linearization of a nonlinear system. $\dot{x} = f(x, u)$ and $y = g(x, u)$ This is quite an important part of how we can come up with the linear form of the nonlinear system. We'll take an example of the same system that you have considered in the previous lecture pendulum system and how we can come up with this form of this structure linear model.

So let's proceed, example we have system $\frac{d^2\theta}{dt^2} + \frac{g}{L} Sin\ \theta = 0$ and after changing the variables we had so for the solution , we have $\dot{x_1} = x_2 = f_1(x_1, x_2)$ and $\dot{x_2} = -\frac{g}{L} Sinx_1 = f_2(x_1, x_2)$ this is we have done before.I can write here x_1 is obviously zero but we can write this is one function and if we apply the principle linearization. We can come up with the linear form but if you notice carefully this system is represented only with the state vector not the control vector, so we do not need to go to this part. It's not required because there is no control part in the system, just we can come up with this structure.

So it means we no need to find the second part because there is no control in the system so we can just come up with the first few terms in the linear model. So now let's start, so here the equilibrium point already you have and the output equation $y = g(x) = x_1(t) = \theta(t)$. So this is the angle we are measuring as the pendulum rotates around the reference point and the jacobian matrix for the system we can write, it's a copy and matrix for the system

$$A = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix}$$

if you notice for this pendulum system we had two equilibrium points,you have found for the system equilibrium point, already you have $\bar{x} = (0, 0), (\pi, 0)$ so it means we are having two jacobian matrix with the respective equilibrium point and if you do the partial derivative for f_1 and f_2 with respect to x_1 and x_2 we get the values for the matrix.

$$A = \begin{bmatrix} 0 & 1 \\ -\frac{g}{L}\cos x_1 & 0 \end{bmatrix}$$

So you can easily find these terms from this f_1 f_2 function. Now we'll find the jacobian matrix for the equilibrium point (0, 0) so let's find A

$$A = \begin{bmatrix} 0 & 1 \\ -\frac{g}{L}\cos x_1 & 0 \end{bmatrix}_{(0,0)} = \begin{bmatrix} 0 & 1 \\ -\frac{g}{L} & 0 \end{bmatrix}$$

at equilibrium point (0,0) .So this is the jacobian matrix for the equilibrium point (0,0). Now if you write in a state equation form

$$\dot{x} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{g}{L} & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Now this is basically a linear time invariant system also we can say linear system and as you discussed earlier that for LTI system the equilibrium point is assumed to be 0 and if you consider here this system, here also we can assume the equilibrium point 0 ,so we can write instead of error equation in that variables we can write this expression ,it is basically that same expression you will be having. So now from this, we can obtain the state equation $\dot{x}_1 = x_2$ and $\dot{x}_2 = -\frac{g}{L}x$. So this is basically the linear system for the pendulum at equilibrium point 0 0. Now we'll look the Jacobian matrix for the different equilibrium point $(\pi ,0)$, so the Jacobian matrix at equilibrium point $(\pi ,0)$ we can write

$$A = \begin{bmatrix} 0 & 1 \\ -\frac{g}{L}\cos x_1 & 0 \end{bmatrix}_{(\pi,0)} = \begin{bmatrix} 0 & 1 \\ \frac{g}{L} & 0 \end{bmatrix}$$

and the corresponding state equations for the pendulum system in linear regime you can write

$$\dot{x} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ \frac{g}{L} & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

and the corresponding state equations if you separate the equations $\dot{x}_1 = x_2$ and $\dot{x}_2 = -\frac{g}{L}x_1$, so this is also linear system of the pendulum at equilibrium point $(\pi ,0)$. Let's stop it here in this lecture. In the next lecture we'll be continuing how we can define the stability based on these obtained state equations and also we'll connect the control part how we can make the system controllable even the natural system is unstable, if you notice here these systems, this equation is basically natural dynamics which we are getting from the original system, if you look the pendulum system we have not assumed any perturbation or control in the system so this is the original dynamics, basically this equation we obtained from the original dynamics so this is natural linear system. So in the next lecture we'll be talking about how to make the system stable using the control. Thank you very much, we'll continue from the next lecture.