Introduction to Aircraft Control System

Prof. Dipak Kumar Giri

Department of Aerospace Engineering

IIT Kanpur

Week – 02

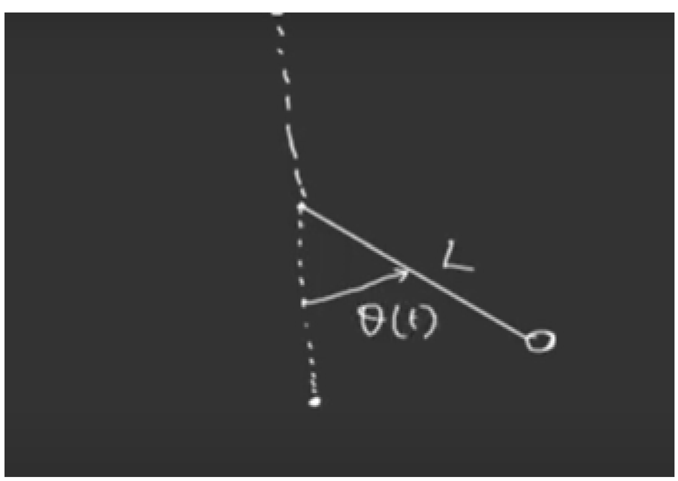
Lecture – 06

Linearization around Equilibrium Point

In this lecture, we'll be discussing the concept of linearization around the equilibrium point. Based on the equilibrium point, we can linearize the nonlinear system and we can get the linear model. In this direction, we'll be taking an example of the pendulum system and how we can linearize the pendulum system at the different equilibrium points. Then also we'll try to connect the concept of dynamic stability and static stability to this particular system and then we'll conclude the lecture. In this lecture, we'll be talking about how we can find the equilibrium point of a nonlinear system.

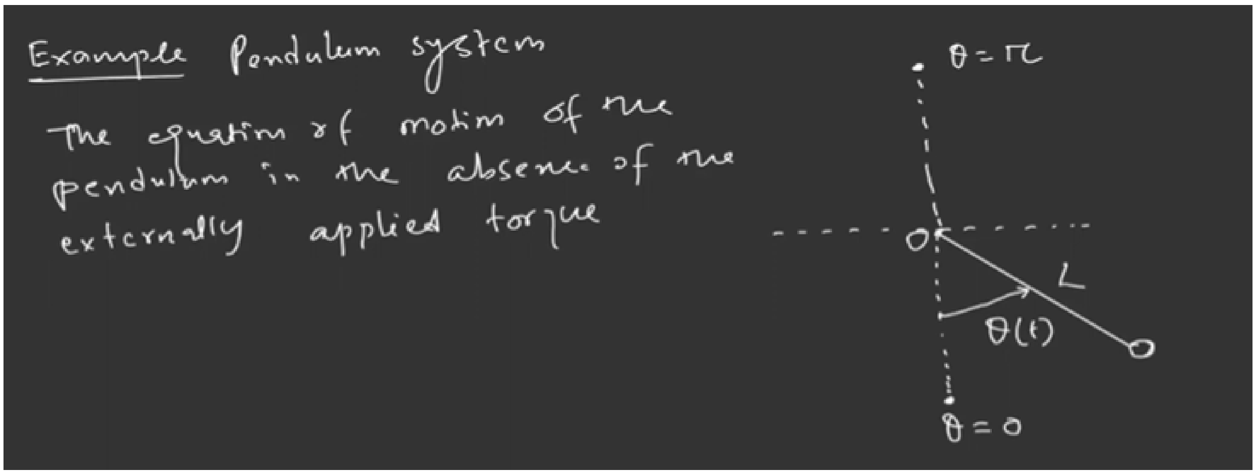
So let's take an example of a pendulum system and let's assume the length of the pendulum is L, suppose this is my starting point or a reference point and the pendulum is the ball which will rotate in 360 degree rotations. The length of this pendulum system is L and this is one point where the pendulum can stay. There's another point. This is also a pendulum that can stay here as it is, but at other point it will disturb and it may fall down to this reference point or this point.

(Refer Slide Time 2:21)



So now the angular position in this situation, let's assume $\theta(t)$. This is my reference point O and in this condition, let's assume $\theta = 0$ and in this condition $\theta = \pi$. This is my reference line I can say. So this is how the pendulum rotates about the reference point O. The equation of motion of the pendulum in the absence of the externally applied torque.

(Refer Slide Time 3:46)



Why externally applied torque? Because we can assume that total torque being applied is zero. Let's assume it means there is no disturbance or control torque is being applied to the system. So in this case, the equation of motion can be written as $L\frac{d^2\theta(t)}{dt^2} + g\,sin\,\theta(t) = 0$. So this is very fundamental equation for the pendulum system.Let me write this equation number one.

This equation number one is a nonlinear equation. Due to the $sin\,\theta$ term, the system is nonlinear. The pendulum system, in this case, can be brought to rest at only two positions. $\theta = 0\,, \theta = \pi$.

And let's discuss how this $\theta = 0$ and $\theta = \pi$ is arrived. Let's find out how these points are being derived. These points are actually called equilibrium points. Let's write this equation number one in first order ODE. So ODE is ordinary differential equation. This is also differential equation. This is actually a second order system.

So now we'll write the second order system into two first order systems. So let's denote

$x_1 = \theta(t)$ and $\dot{x}_1 = x_2 = \dot{\theta}(t)$ And from this, we can write this two state variable, one is x_1, one is x_2. We can come up with $\dot{x}_1 = x_2$ and $\dot{x}_2 = -\frac{g}{L} \sin\theta(t)$.

And so why we're converting this first second order system to first order ODEs, because to find the equilibrium point, we generally define the system into first order ODE, because the equilibrium point is assumed to be constant and that rate of change of that value is assumed to be zero. So now from this system, we can write $\dot{x}_1 = f_1(x_1, x_2, t)$ and $\dot{x}_2 = f_2(x_1, x_2, t)$. So two systems we are coming up with are in first order ODE form. From the first equation, $\dot{x}_1 = 0$ and from this we get $x_2 = 0$ because $\dot{x}_1 = x_2$ we are coming up. And from the second equation, $\dot{x}_2 = 0 \Rightarrow -\frac{g}{L} \sin\theta(t) = 0 \Rightarrow \theta(t) = n\pi$ and where $n = 0, 1, 2...$ something like this.

Now for $n = 1$, we have two equilibrium points which are $(0, 0)$ and $(\pi, 0)$. So because here we can come up for $n = 1$, we have $x_1 = \pi$ and for equal to zero also we are having $(0, 0)$. So we are having two equilibrium points. So now what we will do is, we will vary here now $\theta(t)$ between $\theta = 0$ and $\theta = \pi$.

Let us examine the behavior of the system near this equilibrium point, how the system behaves around this equilibrium point. Because if you see the static and dynamic stability we discussed, stability we define how the system behaves around the equilibrium point. So now if you look at equation number one, due to the $\sin(\theta)$, the system becomes non-linear, right? So now we will approximate this $\sin(\theta)$ about the equilibrium point $\theta = 0$. So expand $\sin\theta(t)$ about the equilibrium point $\theta = 0$. With the small perturbations on the system, how the system behaves around $\theta = 0$.

We will check it. So if you extend, the $\sin\theta(t)$ is in Taylor series expansion, we can write $Sin\,\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + ...$, something like we'll be moving. So now if we assume the motion of the pendulum, $\theta = 0$ is considered small angular displacement. let me write equation number two, θ in equation number two consists of small angular displacement, we can write, $Sin(\theta) \approx \theta$, because for small angular displacement, these values are very small in magnitude. So we can write $Sin(\theta) \approx \theta$. So equation number one, in the above condition, equation number one means our nonlinear equation, can be written as $L\frac{d^2\theta(t)}{dt^2} + g\,\theta(t) = 0$.

And this is basically second order ODE and further we can write $\frac{d^2\theta(t)}{dt^2} + \frac{g}{L}\theta(t) = 0$. So this is a simple binary differential equation. And we can come up with a solution because I'm not going to discuss how we can solve it, it's very easy. So because this is a second order system, we'll have two roots and those are basically complex conjugate roots, let me write this equation number three , we can write $\theta(t) = a_1 Sin(\sqrt{\frac{g}{L}}\, t) + a_2 cos\,(\sqrt{\frac{g}{L}}\, t\,)$. So this is the solution of equation number three.

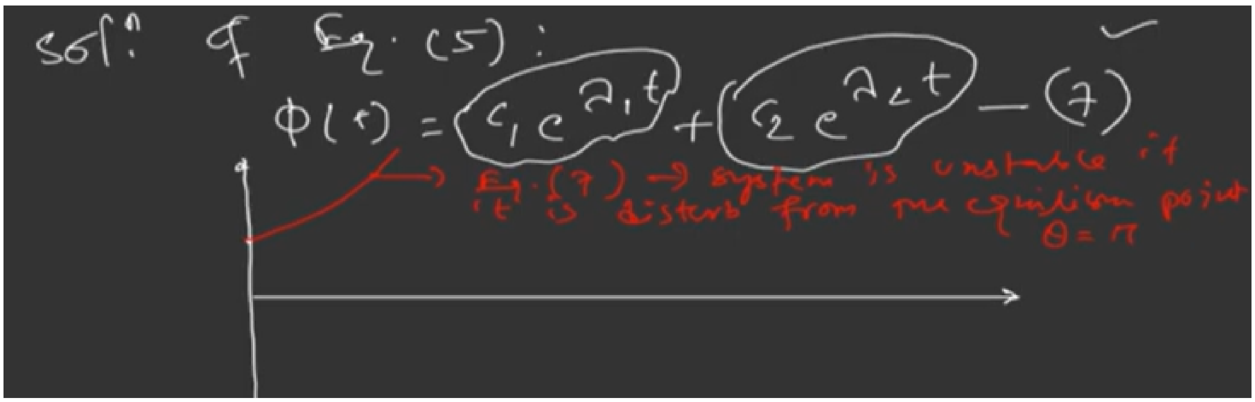
And let me define equation number four. So this is the system how we get from the equilibrium point $\theta = 0$ how the system evolves over time, this is the expression. Now we'll look the expression for $\theta = \pi$ how the system equation yields to be. So similarly, let's look $Sin\,\theta$ about the equilibrium point $\theta = \pi$. Now here, if you displace this $\theta = \pi$, suppose if you displace the body with some perturbation, let me assume (ϕ) , (ϕ) is some angular displacement from the $\theta = \pi$.

So let me write by assuming small angular displacement, denoted by (ϕ) from $\theta = \pi$, we can write $\phi = \pi - \theta$. This is basically the angular displacement ϕ from the equilibrium point π. So θ is basically the angular motion or the pendulum moves in 360 degree, we can write this expression. So now, if you write $Sin\,(\theta)$, we can write $Sin\,(\phi)$, we can write because $Sin\,(\pi) - \phi = Sin(\phi)$, and this is also for small angular displacement of ϕ, we can write ϕ. This is also right again.

So now, what we'll do is , using the above condition, equation one can be written as $L\frac{d^2\phi(t)}{dt^2} - g\phi(t) = 0$ now we are representing the system in terms of ϕ. Now, here ϕ is constant, and from this further we can write $\frac{d^2\phi(t)}{dt^2} - \frac{g}{L}\phi(t) = 0$. Now if we assume the solution of this expression, let me write this is equation number five, let's substitute $\phi(t) = e^{\lambda t}$ and from this we can write $\frac{d\phi}{dt} = \lambda e^{\lambda t}$ and $\frac{d^2\phi}{dt^2} = \lambda^2 e^{\lambda t}$. Now let me write this equation number six, substituting equation six in equation five, else we can write $\lambda^2 e^{\lambda t} - \frac{g}{L}e^{\lambda t} = 0$ and from this we can write $\lambda^2 = \frac{g}{L}$ or $\lambda = \pm\sqrt{\frac{g}{L}}$. The solution for these roots, there are two roots for equation five, the solution of equation five can be written as $\phi(t) = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}$, because there are two roots you can say $\lambda_1 + \sqrt{\frac{g}{L}}$ and $\lambda_2 - \sqrt{\frac{g}{L}}$.

This is the solution for the second equation, equation number five and if you notice carefully here, λ_1 is positive and λ_2 is negative. So if you look carefully in this tweak, this equation number, let me write seven and if you plot the response of the respective system equation number four and equation number seven, we can have, if you find the response, this we can get. If you look equation number seven, this is basically one term is positive, one term is negative, λ_1 and λ_2. Due to my positive λ , in this case due to the exponential part, the system is diverging with time, but if it is even negative, it may be going to zero, but due to the positive term, the response will diverge over time. So if you assume the system is starting from here, initial condition system is starting from here, the system will diverge with time.

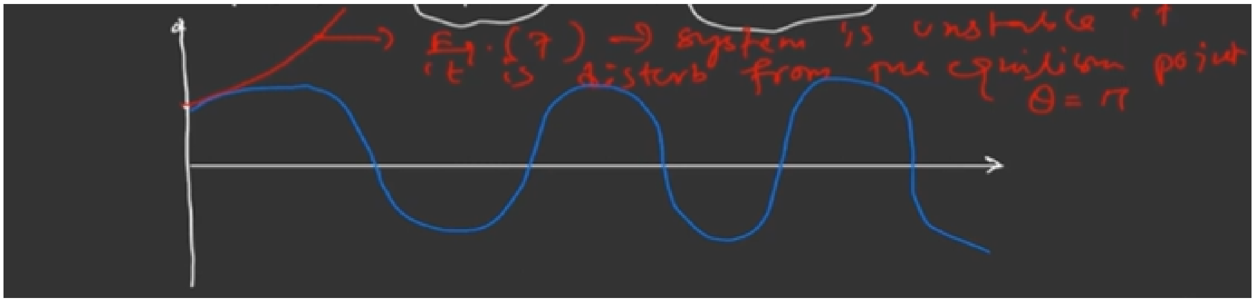
(Refer Slide Time 21:55)



So this is basically equation seven and I can say system is unstable if it is disturbed from the equilibrium point, $\theta = \pi$. So why it is happening? Because if you go back to our system, if you notice with small perturbations on this pendulum, it will fall down. So I can say this equilibrium point is unstable equilibrium point with a small perturbation system will not be at $\theta = \pi$. So this system yields an unstable response.

Now, if you look at equation number seven, and if you start from the same initial conditions, the system will evolve into something like this. So why it is happening? So due to the presence of sine cosine terms, the system will have the sustained oscillation with the same frequency. And here a_1 , a_2 are constant. So I can say that the system is, so equation number five, we can say stable equilibrium point. Quite stable because the system is not diverging with the same magnitude.

(Refer Slide Time 22:37)



The system is oscillating and it will continue like this. And this is how we can define the system stability around the equilibrium point. For equation seven, the response rate, this response system is due to the equilibrium point is unstable, the system response is unstable. And also in the second case, equation number five, the system is stable, it sustains oscillation of the same magnitude. Sometimes it is difficult to analyze the system using the above concept because it is difficult to come up with the linear model of the system, which is highly complex.

For that case, we can use some standard technique, how we can convert the nonlinear system into linear form. Suppose we have a nonlinear system, and we need to convert this nonlinear system into linear form. What you have done in the previous example, if you look at the system, equation number five and equation number three, basically these are linear systems, but to come up with this form is quite tedious. So now we'll be using some simple method, how we can come up with the linear form if you have the equilibrium points of the nonlinear system.

So let's start with the same example. We had $\ddot{\theta} = -\frac{g}{L} Sin\,\theta$. So here it is basically nonlinear equation due to the presence of $Sin\,\theta$. And if you can remember that we come up with some state first order second order systems, where we have that change of variables, $x_1 = \theta$ and $x_2 = \dot{\theta}$ And from these states, we have first order system, $\dot{x}_1 = x_2$ and $\dot{x}_2 = -\frac{g}{L} Sin\,\theta = -\frac{g}{L} Sin\,x_1$.

So here, this is also a nonlinear system. This is the second first order ODE, first order differential equation. And the system is nonlinear, basically, because of $Sin\,x_1$ terms exist in the system. Now, if you define a state vector,vector X, capital $X = \begin{bmatrix} x_1 & x_2 \end{bmatrix}$ basically the states in the system can be written as equation number eight. And let output be $y = x_1$. So here, if you go back to this equation, here $x_1 = \theta$, so θ basically evolves over time, how the system evolves as time proceeds.

So here I can say x_1 is my output for this pendulum system. And in that case, we can write $y = g(x)$, $g(x)$ of some function, which is the function of x. And we can write then the pendulum system in this form $\dot{x} = f(x)$ and $y = g(x)$. So this we call the nonlinear system, the state equation of the nonlinear system. So here, we can write $f(x) = x_2 - \frac{g}{L} Sin\, x_1$ and $g(x) = x_1$.

So this is how we define the nonlinear system in this nonlinear state equation. So now, this is basically for pendulum system, but in general, we can write the state space representation of nonlinear system, we can write $\dot{x} = f(x, u)$ and $y = g(x, u)$. So here, x is the state and u can be any forcing term or control term or any other momentum acting on the system and where the dimensions are $f: 1R^n x\, 1R^m \rightarrow 1R^n$ and $g: 1R^n x\, 1R^m \rightarrow 1R^p$. So this is, we can come up with some general conclusion of the state equation of a nonlinear system.

In the next lecture, we will continue how we can define a nonlinear system in this form and how we can come up with the linear model of that nonlinear system. This is very easy. If you go through this complete process, it is not that difficult and we will be showing the process through an example. So let's stop it here.

We will continue from the next lecture. Thank you.