Introduction to Aircraft Control System

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Week – 10

Lecture – 48

Period, Time to Half or Double the Amplitude and Number of Cycles (Cont.)

Now, how we can come up with the mathematical derivations for the period, time to half or double amplitude, number of cycles for doubling or halving amplitude in the response. So, let us consider a system have a root

$$\lambda_{1,2} = \eta \pm i\omega$$

So, this is the roots coming from some characteristic equation of the system which is second order system and this is the roots or the eigenvalues of the system. The period for this eigenvalues can be defined because here ω actually is the frequency and η basically is the real part of the roots. The period of oscillation because due to the imaginary part in the roots the system will go through some oscillations and the period of those oscillations is related to imaginary part of root which is basically ω So,

$$P = \frac{2\pi}{\omega}$$

So, this is how we can find the period of the system and rate of growth or decay of oscillations is determined by sign of the real part of the complex root. So, it is nothing, but η here. So, η will tell us how amplitude of the response will be. So, negative real part produces decaying oscillations while positive real part causes motion to grow. This part we have extensively discussed while the example of pendulum system in the beginning of the course. So, if you are having negative real part, the system can go like this and with time it converts to some reference value. But if the positive real part is there so over time the oscillation will increase and it will go like this. So, this is how we can come up with the response with the polarity of the real part in the root.

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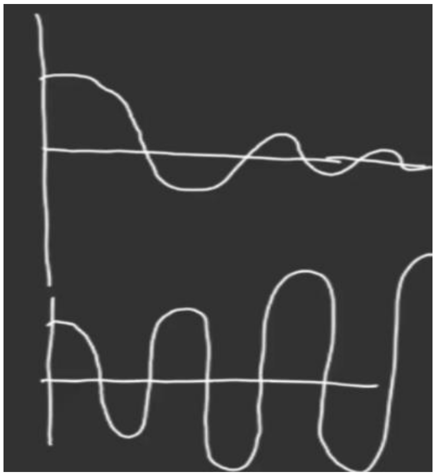

Now, a measure of rate of growth or decay oscillation can be obtained from the time for halving or decaying initial amplitude of disturbance. Why it is initial amplitude disturbance? Because we are looking in the states $(\Delta\theta, \Delta u, \Delta\alpha, \Delta q)$. So, these are basically disturbed variables from the trim point or the steady level flight. Since we are talking about the growth or decay which can be defined by the real part of the roots. So, we can write

$$x(t) = Ae^{\eta t}()$$

Here A is the amplitude of the oscillation and η actually here the real part of the pole of the root. So, here I am not considering the sinusoidal part. So, how we can find the amplitude of the oscillation. So, here this is how we can define the magnitude of the response. So, time to half amplitude we can write

$$Ae^{\eta t_{1/2}} = \frac{A}{2}$$

$$t_{1/2} = \frac{0.693}{|\eta|}$$

So, this is the time to half amplitude this expression is very important expression. Now, let us find time to double amplitude, for example we can write

$$Ae^{\eta t_2} = 2A$$

$$t_2 = \frac{0.693}{|\eta|}$$

Now, to find the number of cycles of half amplitude we can divide the time period by the period we can find the number of cycle in the system. So, et's define the number of cycle for halving amplitude as

$$N_{1/2} = \frac{t_{1/2}}{P}$$

$$N_{1/2} = 0.110 \frac{|\omega|}{|\eta|}$$

So, this is very very important takeaway. If you notice here period to be defined by the frequency in the system so we are having frequency term and from that frequency you can find the time period of the system and again if you know the real part of the root you can find $t_{1/2}$ and t_2 and also number of cycle we can define by the frequency in the system. So, if you consider the longitudinal motion or if you consider the phugoid or short period version, if we come up with some roots of the system because if you consider the longitudinal motion we have $\dot{X} = AX$ so for the whole longitudinal motion the A matrix with dimension 4 x 4 so we are supposed to get 4 roots. We can come up with these relations with the number of cycles, the time to half the amplitude, double the amplitude. If you take the short period motion

$$\begin{bmatrix} \Delta\dot{\alpha} \\ \Delta\dot{q} \end{bmatrix} = A \begin{bmatrix} \Delta\alpha \\ \Delta q \end{bmatrix}$$

So this A matrix was dimensions 2 x 2 so similarly here we can come up with the 2 roots and from those roots we can come up all the relationship similarly we can do for the long period motion. So now in the same way we can do for N_2 as

$$N_2 = 0.110 \frac{|\omega|}{|\eta|}$$

Now let's take an example how we can do all this analysis then I think it will be clear. Find the longitudinal eigenvalues of a general aviation aircraft. This problem I have taken from the Nelson book, you can refer the examples. Compare these results with the answer obtained using phugoid and short period approximation. Given data:

$$X_u = -0.045\ (s^{-1}), X_w = 0.036\ (s^{-1}), X_w = 0$$

$$Z_u = -0.369\ (s^{-1}), Z_w = -0.202\ (s^{-1}), Z_{\dot{w}} = 0, Z_g = 0$$

$$M_u = 0, M_w = -0.05\ (ft^{-1}s^{-1}), M_{\dot{w}} = -0.0051\ (ft\ s^{-1})$$

$$M_g = -2.05\ (s^{-1}), u_0 = 176\ ft/s$$

So these are the data given to us now I'm not writing the equations for the linearized longitudinal motion of the aircraft so here let's go to the solution we had the natural dynamics of the longitudinal motion, we can write

$$\dot{X} = AX + BU$$

but U part we will not assume here so we can write $\dot{X} = AX$. A we had 4 x 4 matrix please go back to the lecture note you can find for it for the linearized longitudinal aircraft motion. Now if you substitute these values in the matrix we get

$$\begin{bmatrix} \Delta\dot{u} \\ \Delta\dot{w} \\ \Delta\dot{q} \\ \Delta\dot{\theta} \end{bmatrix} = \begin{bmatrix} -0.045 & 0.036 & 0 & -32.2 \\ -0.369 & -2.02 & 176 & 0 \\ 0.0019 & -0.0396 & -2.948 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \Delta u \\ \Delta w \\ \Delta q \\ \Delta\theta \end{bmatrix}$$

The Eigenvalues can be determined by

$$|\lambda I - A| = 0$$

The C.E. yields to be

$$\lambda^4 + 5.05\lambda^3 + 13.2\lambda^2 + 0.67\lambda + 0.59 = 0$$

The solution of this C.E. yields Eigne values

$$\lambda_{1,2} = -0.071 \pm i(0.213)$$

$$\lambda_{3,4} = -2.5 \pm i(2.59)$$

In this equation the damping ratio and a natural frequency are less in magnitude compared to other complex conjugate roots, so if the damping ratio and natural frequency are less and that motion basically we are having phugoid. And in another case, the damping ratio and natural frequency are quite large compared to the other complex conjugate and this motion or these roots actually talks about the short period motion. So this is how we can distinguish from these four roots, one pair of roots talks about the phugoid and other roots are the short period motion. So now we can find the time period and the number of cycles for having or doubling the amplitude we can find for both the motions phugoid and short period motion separately. The time period and number of cycles of amplitude are obtained once the eigenvalues are known so once the roots of the system we know we can easily find this number of cycles time period of the response so let's start with the phugoid mode or long period.

$$t_{1/2} = \frac{0.693}{|\eta|} = \frac{0.693}{|-0.0171|} = 40.3\ s$$

Similarly we can find

$$P = 29.5\ s$$

$$N_{1/2} = 1.37\ cycles$$

Now for short period, we can find

$$t_{1/2} = \frac{0.693}{|-2.5|} = 0.28\ s$$

$$P = 2.42\ s$$

$$N_{1/2} = 0.11\ cycles$$

So let's stop it here we'll continue from the next lecture with how we can come up with the same calculations for short period and phugoid approximations. Thank you.