Introduction to Aircraft Control System

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Week – 09

Lecture – 42

Linearization of Lateral/Directional Dynamics (Cont.)

In the last lecture, we have discussed how we can come up with the linearized model of the equation of motion of the force equation basically in the y direction. So, we had the force equation

$$Y + mg\cos\theta\sin\phi = m[\dot{v} + ru - pw]$$

And through the small perturbation concept, we came up with the linear model, and we worked on two more equations, which are moment equation in the lateral direction motion which are

$$\Delta\dot{v} = Y_v\Delta v + Y_p\Delta p + (Y_r - u_0)\Delta r + \left(Y_\phi + \mathrm{g}\cos\theta_0\right)\Delta\phi + Y_{\delta r}\Delta\delta_r \dots\dots Eq(1)$$

$$l = \dot{p}I_{xx} - \dot{r}I_{xz} + qr\left[I_{zz} - I_{yy}\right] - pqI_{xz} \dots\dots Eq(2)$$

$$n = -\dot{p}I_{xz} + \dot{r}I_{zz} + pq\left[I_{yy} - I_{xx}\right] + qrI_{xz} \dots\dots Eq(2)$$

Eq.(2) represents moment equations which will be considered in this lecture and find the linear model of this equation. And now let us start applying the perturbation variable in this equation. The variables in Eq.(2) involve the reference values and perturbed values.

$$n = n_0 + \Delta n, l = l_0 + \Delta l, p = p_0 + \Delta p, q = q_0 + \Delta q, r = r_0 + \Delta r$$

After applying the small disturbance theory to Eq. (2), the following perturbed equations are obtained.

$$\Delta l = I_x\Delta\dot{p} - I_{xz}\Delta\dot{r} \dots\dots Eq(3)$$

$$\Delta n = -I_{xz}\Delta\dot{p} + I_z\Delta\dot{r} \dots\dots Eq(4)$$

I am not showing the full steps the same way you can proceed what you have done previously. You can ignore the small the product of perturbed variables and the same process we can follow what you have done in the previous lectures. We need to see what are the moments, what are the variables are involved to generate this Δl and Δn and if you can find that error series of those variables I mean those variables which are involved to find this Δl and Δn then we can come up with the full equation model non-linear equation. So, now as mentioned the perturbation Δl and Δn happen to be variations of $v, p, r, \delta_a, \delta_r$. So, these are the variables actually involved to generate this perturbation in moment Δl and Δn. This is basically the control torque to the moment equation. We can write the expression of Δl and Δn in terms of Taylor series as

$$\Delta l = \frac{\partial l}{\partial v}\Delta v + \frac{\partial l}{\partial p}\Delta p + \frac{\partial l}{\partial r}\Delta r + \frac{\partial l}{\partial \delta a}\Delta \delta_a + \frac{\partial l}{\partial \delta r}\Delta \delta_r \dots \dots Eq(5)$$

$$\Delta n = \frac{\partial n}{\partial v}\Delta v + \frac{\partial n}{\partial p}\Delta p + \frac{\partial n}{\partial r}\Delta r + \frac{\partial n}{\partial \delta a}\Delta \delta_a + \frac{\partial n}{\partial \delta r}\Delta \delta_r \dots \dots Eq(6)$$

Using Eq.(5), Eq.(3) yields

$$\Delta \dot{p} = \frac{I_{xz}}{I_x}\Delta \dot{r} + L_v \Delta v + L_p \Delta p + L_r \Delta r + L_{\delta a}\Delta \delta_a + L_{\delta r}\Delta \delta_r \dots \dots . Eq(7)$$

Where $L_v = \frac{1}{I_x}\frac{\partial l}{\partial v}$ and so on are the aerodynamic derivatives. Similarly using Eq.(6), Eq.(4) yields to be

$$\Delta \dot{r} = \frac{I_{xz}}{I_z}\Delta \dot{p} + N_v \Delta v + N_p \Delta p + N_r \Delta r + N_{\delta a}\Delta \delta_a + N_{\delta r}\Delta \delta_r \dots \dots Eq(8)$$

Where $N_v = \frac{1}{I_z}\frac{\partial n}{\partial v}$ and so on are the aerodynamic derivatives

It can be noticed that the rolling and yaw moment equations are coupled with each other. In order to yield the first order differential equation for $\Delta \dot{p}$ and $\Delta \dot{r}$ we need to do the following mathematical manipulations, substituting Eq.(8) in Eq.(7) yields

$$\Delta \dot{p}\left[1 - \frac{I_{xz}^2}{I_x I_z}\right] = \frac{I_{xz}}{I_x}\left[N_v \Delta v + N_p \Delta p + N_r \Delta r + N_{\delta a}\Delta \delta_a + N_{\delta r}\Delta \delta_r\right] + L_v \Delta v + L_p \Delta p + L_r \Delta r + L_{\delta a}\Delta \delta_a + L_{\delta r}\Delta \delta_r \dots \dots Eq(9)$$

Consider $\eta = \left[1 - \frac{I_{xz}^2}{I_x I_z}\right]$

$$\Delta \dot{p} = \left[L_v^* + \frac{I_{xz}}{I_x}N_v^*\right]\Delta v + \left[L_p^* + \frac{I_{xz}}{I_x}N_p^*\right]\Delta p + \left[L_r^* + \frac{I_{xz}}{I_x}N_r^*\right]\Delta r + \left[L_{\delta a}^* + \frac{I_{xz}}{I_x}N_{\delta a}^*\right]\Delta \delta_a + \left[L_{\delta r}^* + \frac{I_{xz}}{I_x}N_{\delta r}^*\right]\Delta \delta_r \dots \dots Eq(10)$$

Where $L_v^* = \frac{L_v}{\eta}$ and so on

Similarly for $\Delta\dot{r}$ equation we get the following expression. The students are suggested to derive on their own.

$$\Delta\dot{r} = \left[N_v^* + \frac{I_{xz}}{I_x}L_v^*\right]\Delta v + \left[N_p^* + \frac{I_{xz}}{I_x}L_p^*\right]\Delta p + \left[N_r^* + \frac{I_{xz}}{I_x}L_r^*\right]\Delta r + \left[N_{\delta a}^* + \frac{I_{xz}}{I_x}L_{\delta a}^*\right]\Delta\delta_a + \left[N_{\delta r}^* + \frac{I_{xz}}{I_x}L_{\delta r}^*\right]\Delta\delta_r \dots\dots Eq(11)$$

Where $N_v^* = \frac{N_v}{\eta}$ and so on

So if you notice above two equations are linear equations because the coefficient of the variables are basically assumed to be constant for the particular flight regime which will be derived from the wind tunnel testing so these are basically constant and the system are basically linear LTI system. Let's stop it here we'll continue from the next lecture where we'll be finding the state space model of the lateral directional motion of the aircraft and how we can connect the control algorithm to the equation of motion which will be defined in linear form. Thank you.