Introduction to Aircraft Control System

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Week – 08

Lecture – 39

Linearization of Longitudinal Aircraft Dynamics (Cont.)

In the last lecture we have discussed how we can come up with the linear model of the force equation in the x direction of the longitudinal motion of the aircraft. So our equation in the x direction in the longitudinal equation of motion we had

$$X - mg\sin\theta = m[\dot{u} + qw - rv]$$

Using small perturbation theory, we came up with the linear model of equation

$$\left(\frac{d}{dt} - X_u\right)\Delta u - X_w\Delta w + g\Delta\theta\cos\theta_0 = X_{\delta e}\Delta\delta_e + X_{\delta t}\Delta\delta_t \ldots\ldots Eq(1)$$

This is basically the linear model of the force equation in x direction. So this equation we have already found in the last lecture how it so we have gone through the steps to find this expression. Now we will start how we can find the linear model of the force equation in the z direction in the longitudinal motion. We can write

$$Z + mg\cos\theta\cos\phi = m[\dot{w} + pv - qu] \ldots\ldots Eq(2)$$

Now we will proceed the same way what we have done for the x force equation. So we will apply the perturbation of the system. Considering forces in z direction and including the perturbation perturbation variables. Let me write

$$Z = Z_0 + \Delta Z, \theta = \theta_0 + \Delta\theta, \phi = \phi_0 + \Delta\phi, w = w_0 + \Delta w$$

$$p = p_0 + \Delta p, q = q_0 + \Delta q, u = u_0 + \Delta u, v = v_0 + \Delta v$$

Eq.(2) can now be written as

$$Z_0 + \Delta Z + mg\cos(\theta_0 + \Delta\theta)\cos(\phi_0 + \Delta\phi)$$
$$= m\left[\frac{d}{dt}(w_0 + \Delta w) + (p_0 + \Delta p)(v_0 + \Delta v) - (q_0 + \Delta q)(u_0 + \Delta u)\right] \dots \dots Eq(3)$$

So this is the equation which is written in perturbed variable form . Now as we know the for the trim flight condition we have

$$v_0 = p_0 = q_0 = r_0 = \phi_0 = \Delta\phi = 0$$

There is no lateral perturbations in the longitudinal flight. So these are the assumptions so we are considering for finding the linear model of the z force. In Eq.(3) for small value of $\Delta\theta$ the term $\cos(\theta_0 + \Delta\theta)$ yields to be

$$\cos(\theta_0 + \Delta\theta) = \cos\theta_0 - \Delta\theta\sin\theta_0 \dots \dots Eq(4)$$

Hence, we can write

$$Z_0 + \Delta Z + mg\cos\theta_0 - mg\Delta\theta\sin\theta_0 = m\dot{w} + m\Delta\dot{w} - mu_0\Delta q \dots \dots Eq(5)$$

Assume the product of perturbed variable assumed to be zero and also we have introduced trim flight condition in Eq.(3) and under this assumption we got Eq.(5). Also what you have done in the previous lecture that for the trim flight condition for force in z direction yields to be

$$Z_0 + mg\cos\theta_0 = m\dot{w}_0$$

Hence, Eq.(5) can now be written as

$$\Delta Z - mg\Delta\theta\sin\theta_0 = m\Delta\dot{w} - mu_0\Delta q \dots \dots Eq(6)$$

The force term ΔZ can again be expressed by means of the taylor series in terms of the perturbation variable

$$\Delta Z = f(\Delta u, \Delta w, \Delta\dot{w}, \Delta q, \Delta\delta_e, \Delta\delta_t)$$

$$= \frac{\partial Z}{\partial u}\Delta u + \frac{\partial Z}{\partial w}\Delta w + \frac{\partial Z}{\partial \dot{w}}\Delta\dot{w} + \frac{\partial Z}{\partial q}\Delta q + \frac{\partial Z}{\partial \delta_e}\Delta\delta_e + \frac{\partial Z}{\partial \delta_t}\Delta\delta_t \dots \dots Eq(7)$$

On Eq.(7), $\frac{\partial Z}{\partial \dot{w}}$ represents the rate of change of normal velocity. If you combine Eq.(6) and Eq.(7)

$$m\Delta\dot{w} - mu_0\Delta q + mg\Delta\theta\sin\theta_0 - \frac{\partial Z}{\partial u}\Delta u - \frac{\partial Z}{\partial w}\Delta w - \frac{\partial Z}{\partial \dot{w}}\Delta\dot{w} - \frac{\partial Z}{\partial q}\Delta q = \frac{\partial Z}{\partial \delta_e}\Delta\delta_e + \frac{\partial Z}{\partial \delta_t}\Delta\delta_t \dots\dots Eq(8)$$

We'll divide this expression by mass on both sides, we get

$$\Delta\dot{w} - u_0\Delta q + g\Delta\theta\sin\theta_0 - \frac{1}{m}\frac{\partial Z}{\partial u}\Delta u - \frac{1}{m}\frac{\partial Z}{\partial w}\Delta w - \frac{1}{m}\frac{\partial Z}{\partial \dot{w}}\Delta\dot{w} - \frac{1}{m}\frac{\partial Z}{\partial q}\Delta q = \frac{1}{m}\frac{\partial Z}{\partial \delta_e}\Delta\delta_e + \frac{1}{m}\frac{\partial Z}{\partial \delta_t}\Delta\delta_t \dots\dots Eq(9)$$

Denoting $Z_u = \frac{1}{m}\frac{\partial Z}{\partial u}$ and so on are the aerodynamic derivatives divided by the airplanes mass, Eq.(9) can be written as

$$\Delta\dot{w} - u_0\Delta q + g\Delta\theta\sin\theta_0 - Z_u\Delta u - Z_w\Delta w - Z_{\dot{w}}\Delta\dot{w} - Z_q\Delta q = Z_{\delta_e}\Delta\delta_e + Z_{\delta_t}\Delta\delta_t \dots\dots Eq(10)$$

Also we know that $\Delta q \approx \Delta\dot{\theta}$

$$-Z_u\Delta u + (1 - Z_{\dot{w}})\Delta\dot{w} - Z_w\Delta w - \left(Z_q + u_0\right)\frac{d}{dt}\Delta\theta + g\Delta\theta\sin\theta_0 = Z_{\delta_e}\Delta\delta_e + Z_{\delta_t}\Delta\delta_t$$

$$-Z_u\Delta u + \left[(1 - Z_{\dot{w}})\frac{d}{dt} - Z_w\right]\Delta w - \left[\left(Z_q + u_0\right)\frac{d}{dt} - g\sin\theta_0\right]\Delta\theta = Z_{\delta_e}\Delta\delta_e + Z_{\delta_t}\Delta\delta_t \dots\dots Eq(11)$$

So this is the expression for the force equation along the z direction in longitudinal dynamics also yields to be linear form. We started with the non-linear equation of the force equation in the z direction and finally came up with the linear form so this is the non-linear equation so using small perturbations we come up with the linear form so if you notice the above terms basically are stability derivatives and we can find these terms for a given wind tunnel testing for a given flight regime or flight. Now our only equation is left is the moment equation along y-axis how the moment going to affect the longitudinal motion so we'll derive the linear form of the moment equation along y-axis in the next lecture, one more thing so we come up with the linear form of the x and z axis force along x and z so once you have the linear form of the moment along the y-axis then you can combine all the equation and we can come up with the linear state space model. So in the next lecture we'll be discussing how we can come up with the linear state space model of the longitudinal motion of the aircraft. Thank you.