Introduction to Aircraft Control System

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Lecture – 38

Linearization of Longitudinal Aircraft Dynamics

In this lecture, we will be starting how we can find the linearized equation of motion of the aircraft. Before we proceed, let me rewrite the trim flight condition of the longitudinal equation because it will be required while deriving the state space model. So, the trim flight condition for the longitudinal motion, we had

$$X_0 - mg\sin\theta_0 = m\dot{u}_0$$

$$Z_0 + mg\cos\theta_0 = m\dot{w}_0 \ldots\ldots Eq(1)$$

$$m_0 = 0$$

Now, we will start to find the what type equation of motion of the original equation, the force equation in the longitudinal motion force along the x axis will start. Let us begin with considering forces along in the x axis or x direction, we can write

$$X - mg\sin\theta = m(\dot{u} + qw - rv) \ldots\ldots Eq(2)$$

We know from previous lecture

$$u = u_0 + \Delta u, v = v_0 + \Delta v, w = w_0 + \Delta w$$

$$p = p_0 + \Delta p, q = q_0 + \Delta q, r = r_0 + \Delta r$$

$$X = X_0 + \Delta X, Y = Y_0 + \Delta Y, Z = Z_0 + \Delta Z$$

$$m = m_0 + \Delta m, n = n_0 + \Delta n, l = l_0 + \Delta l$$

$$\theta = \theta_0 + \Delta\theta, \phi = \phi_0 + \Delta\phi, \psi = \psi_0 + \Delta\psi$$

So, this is basically the first term indicates the reference model and the second term perturbed variable. And if you substitute these terms in the Eq.(2), we get

$$X_0 + \Delta X - mg\sin(\theta_0 + \Delta\theta) = m\left[\frac{d}{dt}(u_0 + \Delta u) + (q_0 + \Delta q) + (w_0 + \Delta w) - (r_0 + \Delta r)(v_0 + \Delta v)\right] \dots \dots Eq(3)$$

And if you ignore the product of perturbed variables and if you assume the trim flight condition, okay, let me write during trim flight we know that $v_0 = p_0 = q_0 = r_0 = \phi_0 = \psi_0 = 0$

So, we are neglecting the product of perturbed variables because the magnitude of these terms are very small. So, after introducing this condition in Eq.(3) and solving, we can write

$$X_0 + \Delta X - mg\sin\theta_0\cos\Delta\theta - mg\cos\theta_0\sin\Delta\theta = m\dot{u}_0 + m\Delta u + m\Delta q w_0 \dots \dots Eq(4)$$

Further, if we assume perturbations are small

$$\cos\Delta\theta \approx 1,\ \sin\Delta\theta \approx 1$$

$$X_0 + \Delta X - mg\sin\theta_0 - mg\Delta\theta\cos\theta_0 = m\dot{u}_0 + m\Delta u + m\Delta q w_0 \dots \dots Eq(5)$$

Using Eq.(1), Eq.(5) can be written as

$$\Delta X - mg\Delta\theta\cos\theta_0 = m\Delta\dot{u} + m\Delta q w_0 \dots \dots Eq(6)$$

Okay we'll proceed with another assumption. If we align the body x axis along the direction of airplanes velocity vector in that case we can write $w_0 = 0$ (stability axis)

$$\Delta X - mg\Delta\theta\cos\theta_0 = m\Delta\dot{u} \dots \dots Eq(7)$$

Here the force ΔX is the change in aerodynamic and propulsive force. Because already we have discussed X actually is the total force which is coming from the aerodynamic and propulsive forces and ΔX is the change in the aerodynamic propulsive force in the body x axis or x direction we can write and this ΔX can be written in Taylor series form in terms of perturbation variables. So, let us write

$$\Delta X = f(\Delta u, \Delta w, \Delta q, \Delta\delta_e, \Delta\delta_t)$$

$$= \frac{\partial X}{\partial u}\Delta u + \frac{\partial X}{\partial w}\Delta w + \frac{\partial X}{\partial q}\Delta q + \frac{\partial X}{\partial \delta_e}\Delta\delta_e + \frac{\partial X}{\partial \delta_t}\Delta\delta_t \dots \dots Eq(8)$$

Where $\Delta\delta_e$ and $\Delta\delta_t$ are the perturbation variables in elevator input and fraction of maximum thrust. These are basically elevator deflection. So, this causes to deflect the body from the reference point to different point and if it is going to control the ΔX then we can say this is the favorable control input to the system which is going to control ΔX

or which can control indirectly actually we are controlling $\Delta\dot{u}$. So, how we can make time rate change of Δu to zero with the application of control input and perturbation under the maximum thrust. So, we'll be looking while designing the control algorithm. Let me write

$$\frac{\partial X}{\partial u}, \frac{\partial X}{\partial w}, \frac{\partial X}{\partial q}, \frac{\partial X}{\partial \delta_e}, \frac{\partial X}{\partial \delta_t}$$

represent the stability derivatives. This can be found from the wind tunnel testing basically which are evaluated at the steady state value. Basically these are the matrices if you look and generally if you remember while studying of the state space representation we considered the Taylor series expansion of the system of the pendulum system the value of this expression at the equilibrium point or the trim values this is what you have done for the pendulum system. Now the aerodynamic forces and moments can be expressed as a function of all motion variables. However, generally we're gonna use almost all variables are involved to generate this force and moments but we will be using which contributed most so we can write however in these equations only the terms that are usually significant have been considered or have been retained. Another assumption, is the effect of, piece rate q on the X which is basically $X_0 + \Delta X$ x is negligible, because they are in the same direction, we can say that. I mean, we are assuming the rate along the y axis to be zero, we are assuming, and that's why the effect of the force along x direction will be negligible.

And therefore, Eq.(7) reduces to

$$\frac{\partial X}{\partial u}\Delta u + \frac{\partial X}{\partial w}\Delta w + \frac{\partial X}{\partial \delta_e}\Delta\delta_e + \frac{\partial X}{\partial \delta_t}\Delta\delta_t - mg\Delta\theta\cos\theta_0 = m\Delta\dot{u}$$

$$\left(\frac{d}{dt} - \frac{1}{m}\frac{\partial X}{\partial u}\right)\Delta u - \frac{1}{m}\frac{\partial X}{\partial w}\Delta w + g\Delta\theta\cos\theta_0 = \frac{1}{m}\frac{\partial X}{\partial \delta_e}\Delta\delta_e + \frac{1}{m}\frac{\partial X}{\partial \delta_t}\Delta\delta_t$$

And if you denote $X_u = \frac{1}{m}\frac{\partial X}{\partial u}, X_w = \frac{1}{m}\frac{\partial X}{\partial w}$ and so on are the aerodynamic derivatives divided by the airplanes mass

$$\left(\frac{d}{dt} - X_u\right)\Delta u - X_w\Delta w + g\Delta\theta\cos\theta_0 = X_{\delta e}\Delta\delta_e + X_{\delta t}\Delta\delta_t \dots\dots Eq(10)$$

So, this is basically the linear model of the system because these terms are constant, they are perturbed variable. Eq.(10) is actually the linear equation of the forced equation in the X direction. The stability derivatives we find from the wind tunnel testing. So, now we will be moving to the next part of the forced equation along z axis and how we can find the linear form of the force along z axis. So, that part we will continue from the next lecture. Thank you.