Introduction to Aircraft Control System

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Lecture – 37

Small Perturbation Theory

Here we are going to start how we can use the small perturbation theory to the equation of motion of the aircraft and how we can come up with the linear model of the non-linear system because the aircraft equation of motion are highly complex and non-linear. And to design the linear controls, we should have the system into linear form. Before we apply to the aircraft equation of motion, first let's study what is small perturbation theory and how we can use it to the aircraft system step by step. So here why we have to consider for aircraft. So let me highlight the aircraft motion is governed by non-linear differential equation due to the non-linear nature of aerodynamic forces and movements.

As well as the kinetic, let me write kinematic and dynamic relationship involved in flight. The dynamic and kinematic equation we already found in the last lecture. These non-linear equations while accurate are generally complex and difficult to solve analytically for understanding the aircraft behavior in response to control input. So basically the systems which are we derived for the aircraft equation of motion are highly complex and non-linear. And to understand how the system behaves over time in linear regime it is quite difficult. So we need to come up some linear form using the perturbation theory. Now let's look what is the small perturbation theory. The essence of small perturbation theory leads to approximate the non-linear behavior of an aircraft around a steady equilibrium flight condition. The way we have done for the pendulum system in the beginning of the classical control part. So there how you linearize the non-linear equation of motion of the pendulum system using the small perturbation in the equation of motion of the pendulum. The same way you will be positioning here also flight condition. Often generally we use often straight and level flight using linear equation. This is achieved introducing small perturbation.

Since it's our deviation from the equilibrium point it changes our deviation from the equilibrium point to the aircraft state variables such as velocity, angle of attack, pitch angle etc. And linearly relating these perturbations to the forces and moments of the aircrafts acting on the aircraft. So this is basically how we will be doing in the small perturbation theory with a non-linear system in the aircraft equation of motion.

And let's look what are the different states we have to consider for finding the linearized model of the equation of motion. This is the step we will be finding the linear model of the non-linear aircraft equation. The first step is we need to define the steady state flight condition. Under this first you have to identify a steady state flight condition around which the aircraft responds to be analyzed. The same way we have done for the pendulum system. So in pendulum system we had two equilibrium point, one is upside down, another is above. So here we have one equilibrium point which is $(\pi, 0), (0,0)$. So we analyze the pendulum system about this equilibrium point into the small perturbations. So for small deviation from the equilibrium point, how the system going to behave? So this thing we have done during the study of non-linear equation of motion of the pendulum system. We have discussed this part. So same way you can analyze here. This involves setting the time derivative of the steady variable to zero. It's implying no acceleration. This is the first part we have to do for finding the linearized model of the equation of motion of the aircraft. And then we'll introduce about the equilibrium point of the system. So here we shall introduce small perturbation with the steady variable. The same process will be following for the longitudinal motion and lateral motion.

We'll be doing while we'll be deriving the linear model, these are the steps we'll be following. Perturbation to the steady variables and aerodynamic forces and moment. For example, if you consider v is the steady state velocity of the aircraft. Let's assume δv is added. It may be due to external disturbance or the wind disturbance. So v has been changed by the small part of δv. So we can write $v + \delta v$. So this is how we, part of the steady variable of the system. Then third step is linearize aerodynamic coefficient. So here we'll expand the aerodynamic coefficient. Basically here, expand the aerodynamic forces and moments using Taylor series around the reference point or the equilibrium point and retain only the linear terms. So here we'll neglect the higher order terms, assuming the perturbation. If the perturbation are very small, we can ignore the higher order terms. So I can write higher order terms are neglected assuming the perturbations are small. So from this step, we'll find the linear equations, we'll retain only the linear terms in the perturbations. We'll neglect the coupling term for higher order terms. Then we'll come to the equation of motion in linear regime. Derive the equation. So here we'll substitute perturbed variables and linearize aerodynamic expressions into the original non-linear equation of motion.

So this result a set of linear differential equations, that, this equation will derive in first order ODE basically. So just the degree of the, order of the equation will be one, first derivative only, equations that describe the dynamics of the aircraft in terms of linear differential equation in terms of, we can write the part of, this linear differential equation will be in terms of part of, from the equilibrium state. So assume, suppose we had velocity of the aircraft was for example, V and due to perturbations, this velocity has been changed $V + \delta V$. The perturbed equation we'll be writing in terms of δV. So our main motivation is we need to design the control such that δV is going to zero. So this is the main motivation. So the linear model we'll be finding in terms of part of variables. Final step is simplify the equation of motion in the state space form. See once we have the linear equation of motion of the aircraft system, we can write in state space form.

Here we are going to work on two equation of, two set of equation of motion, which is longitudinal motion, another is the lateral directional motion. So for both the equation of motion, we'll come up with the state space form and we'll be designing the control system. And also how it, we'll be looking how we can find the single input and single output equation from the state space model, those part also we'll be looking at. Okay, the final step is simplify into state space form. So here the linearized and arranged equations are further simplified and arranged into our state space form. So how we can come up with the state space form, those part already have explained. The pendulum system we assume that the second order system and how we can come up with the two ODE form, or the differential equation form. So and also we wrote that system into state space form and which makes the easier to analyze the system. So making it easier to analyze the system using control theory technique. Since in this course we are going to dealing with the linear control theory, so the system needs to be written into linear form.

So that is the main motivation why we need to construct this perturbation theory. If you are designing nonlinear control, we can go as it is, we can take the equation as it is and we can move on. Nonlinear equation we can take as a reference dynamics, we can design the control. Since we are designing the classical control or modern control for this aircraft system, the system needs to be written into linear form. The state space model consists of matrices that represent the dynamics and control input effect on the aircraft part of state. So if you remember the, after linearizing the system, if you write in state space form, we can write $\dot{x} = Ax + Bu$ and $y = Cx + Du$. So here is the control matrix and this is the output matrix. So this is how we can define the nonlinear equation, higher order equation into state space form. So these steps will be following for finding the linearized model of the equation motion of the aircraft.

We will introduce the perturbation, then we will write the Taylor series expansion and from the Taylor series expansion we will ignore the higher order terms or coupling terms and we can come up with a linear model. And also the linear model we can write in state space form as well. Now we'll move or we'll take our system. The first system we are going to take the longitudinal motion and how we can apply this perturbation theory to come up with a linear model of the longitudinal equation of motion of the aircraft. Before that we need to apply the perturbation to all variables and how we can write the perturb variables of the reference states. So let me write all the variables in the equation of motion are replaced by reference value of the aircraft plus a perturbation or disturbance. As I have written here that if v is the variable for example, due to the perturbations or due to the disturbance the variable is shifted by δv . So we can write $v + \delta v$. So this is how due to the perturbation we need to apply this perturbation to all the states in the equation of motion of the aircraft. So let's look how we can write the state variables plus reference part.

$$u = u_0 + \Delta u, v = v_0 + \Delta v, w = w_0 + \Delta w$$

$$p = p_0 + \Delta p, q = q_0 + \Delta q, r = r_0 + \Delta r$$

$$X = X_0 + \Delta X, Y = Y_0 + \Delta Y, Z = Z_0 + \Delta Z$$

$$m = m_0 + \Delta m, n = n_0 + \Delta n, l = l_0 + \Delta l$$

$$\theta = \theta_0 + \Delta \theta, \phi = \phi_0 + \Delta \phi, \psi = \psi_0 + \Delta \psi$$

So this is the perturbations we apply to the state variable for the reference state of the aircraft. And since we have mentioned the reference model of the aircraft will be assuming level or level flight. So for the level or flight let's look what should be the values, reference values should be. And before that let me consider another assumption to the system. The reference flight condition is assumed to be symmetric and propulsive forces assumed to be remain constant. This implies

$$v_0 = p_0 = q_0 = r_0 = \phi_0 = \psi_0 = 0$$

So for the reference flight condition, we can come up with this. Now we'll assume the longitudinal equation of motion and for this reference values of the state and let's look at the trim condition for the longitudinal motion of the aircraft. Okay so first let me write the equation of motion for the longitudinal motion.

$$X - mg \sin\theta = m[\dot{u} + qw - rv]$$

$$Z + mg \cos\theta \cos\phi = m[\dot{w} + pv - qu] \dots\dots Eq(1)$$

$$m = \dot{q} I_{yy} + pr[I_{xx} - I_{zz}] + I_{xz}[p^2 - r^2]$$

Now for this reference model let's look the what shall be the trim flight condition for the longitudinal motion of the aircraft at trim values

$$v_0 = p_0 = q_0 = r_0 = \phi_0 = 0$$

So this condition actually the rates side velocity of the aircraft at trim condition should be 0 and the flight is level flight. So these are the condition needs to be maintained for the trim condition of the longitudinal flight of the aircraft. Under this condition the equation motion of the longitudinal equation aircraft we can write

$$X_0 - mg\sin\theta_0 = m\dot{u}_0$$

$$Z_0 + mg\cos\theta_0 = m\dot{w}_0 \dots\dots Eq(2)$$

$$m_0 = 0$$

Eq. (2) implies the trim flight for longitudinal aircrafts equation of motion. Now let's look what should the trim flight condition of the lateral direction equation. So let me let us rewrite the equation motion of the lateral direction equations of the aircraft. The lateral direction equation of motion can be written as

$$Y + mg\cos\theta\sin\phi = m[\dot{v} + ru - pw]$$

$$l = \dot{p}I_{xx} - \dot{r}I_{xz} + qr\left[I_{zz} - I_{yy}\right] - pqI_{xz}$$

$$n = -\dot{p}I_{xz} + \dot{r}I_{zz} + pq\left[I_{yy} - I_{xx}\right] + qrI_{xz}$$

Steady state trim flight for lateral directional equations at

$$p_0 = q_0 = r_0 = \phi_0 = 0$$

$$Y_0 = m\dot{v}_0$$

$$l_0 = 0 \dots\dots Eq(3)$$

$$n_0 = 0$$

Eq. (3) implies the trim flight for lateral-directional motion of the aircraft. So these are the model we'll be requiring while finding the state space model which we'll be writing in perturbed variable form. Let's stop it here we'll continue from the next lecture how we can find the linearized model of the longitudinal equation of motion of the aircraft. Thank you.