Introduction to Aircraft Control System

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Week – 07

Lecture – 31

Control System Design in Frequency Domain

Here we are going to start how can we design the control system in frequency domain. Here first we would like to find the relation between the time-driven specification what we have done already and the frequency domain specification. For the given time domain specification we can come up with the frequency domain specification which are basically the gain margin and the phase margin of the system. Then we will study how we can mitigate the slow varying disturbance signal and fast varying measurement noise in the system using the controller in frequency domain. What should be the controller structure or the control algorithm should be there to mitigate the disturbance in noise in the approach of control system. Then we will study how we can find the phase margin and gain margin from the given frequency response.

Basically in the frequency response we plot the gain of the system with respect to frequency and phase of the system with respect to frequency. For the given gain cross over frequency and phase cross over frequency you can come up with the gain margin and phase margin of the system. Then we will study how one can design the PID control in frequency domain for the aircraft attitude control problem. Then we will conclude this part. So in the next few lecture we will be involving to discuss this part and how we can design the controller in frequency domain. Then we will wind up this part. In this lecture we will be discussing the control system design in frequency domain. If you notice in the previous lectures we have discussed the control system design of a system where the system model is known but in frequency domain analysis it is not necessary the system analytical model of the system needs to be known. And also the control system we have designed so far it is difficult to comment on the robustness of the system in the presence of uncertainty and measurement noise.

But this problem can be dealt with frequency domain analysis. So let me go through some advantages of the frequency domain and domain analysis control system design. So let me give the advantages. The control system design based on frequency domain analysis is robust to the modeling of the system, uncertainty and attenuation of measurement noise. So the control system based on the frequency domain analysis can deal with this problem, the modeling and measurement noise.

Second advantage the closed loop system stability can be analyzed based on the open loop frequency response. Third point is unlike like root-locus method the analytical model of the plant is not necessary for control synthesis. If you notice in the previous lectures to design the PID control the analytical model of the plant needs to be known but in the frequency domain it is not necessary based on the experimental data or frequency response of the system we can come up with the control system design. So you can write on the experimental system stability can be analyzed and also quite this is important because most of the cases the suppliers may not provide the detailed analytical model of the system they only provide the data and from the given data we can come up with the control system analysis. So for aircraft attitude control suppliers may not provide detailed model, analytical model of the different subsystem for designing the control system.

So this kind of problem can be dealt in the frequency domain analysis even the analytical model not available to us we can come up with the control system stability. So these are the few advantages in frequency domain analysis. Now let's go back to our aircraft PID control system, aircraft PID controlled attitude problem and here we'll come up with some relation how we can come up some quality conditions based on the frequency domain measurements as well as time domain analysis and based on these relations we can come up with the control system design. So first let's go with the control system PD control system for the aircraft attitude control problem. So there are two different model we have discussed if you remember one was the modified implementation of PD controlled aircraft attitude problem so the block diagram is given as below

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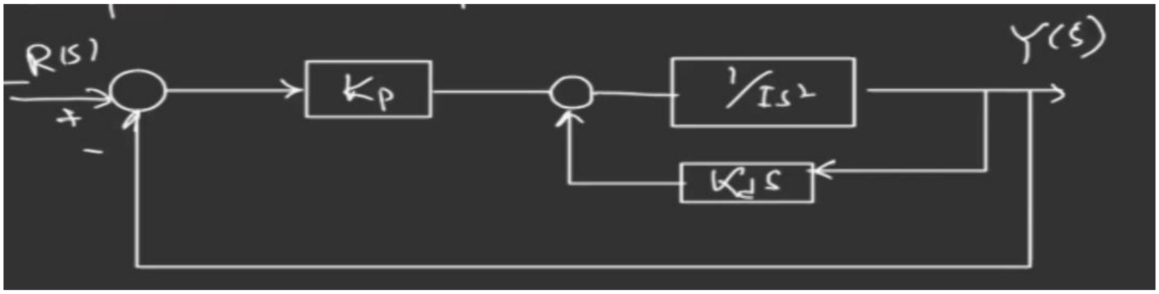

Here we have another summing point and this is our plant which is one upon is squared and in the feedback we had kd s this is our output which is which was generated by Y(s) this is the feedback and we have summing point which basically connect the negative feedback and this is our reference signal R(s) to be tracked and this is the negative feedback and this is this is actually modified implementation of the PD control system of the aircraft attitude problem and we have standard implementation as well if you look the PD control system the standard implementation of aircraft attitude control problem so their block diagram is

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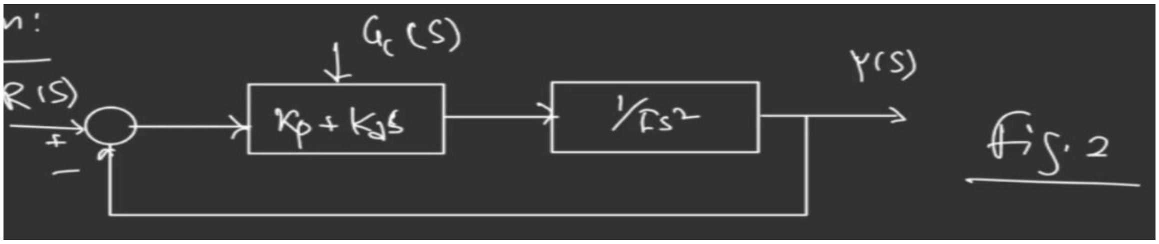

This is the modified implementation of the aircraft attitude control we will take this block diagram for come up with the relation between the frequency domain analysis and time domain analysis so based on the time domain parameters we can come up with the conclusion on frequency domain part and if you are having frequency domain specification we can come up with the time domain specification this is very important relation that we are going to have now. From figure number two the open loop transfer function we can write

$$G_0(s) = G_c(s)G_p(s)H(s)$$

$$= \frac{k_p + k_d s}{Is^2} \ldots \ldots Eq(1)$$

As we have done before in the time domain analysis we had two important relations between the damping ratio and natural frequency and the controller gains if you remember

$$\omega_n^2 = \frac{k_p}{I}, 2\xi\omega_n = \frac{k_d}{I}$$

If you substitute these relations in equation number one, OLTF is

$$G_0(s) = \frac{2\xi\omega_n s + \omega_n^2}{s^2} \ldots \ldots Eq(2)$$

Now this is basically in time domain we can say now we'll come up with the frequency domain analysis now we will have to start how we can come up with the relation between the frequency domain specification and time domain specification so to carry out the formulation in frequency domain we'll substitute in place of s, $j\omega$ the open loop transfer function this is equation number two let me write the open loop transfer function in frequency domain we can write

$$G_0(s)|_{s=j\omega} = -\left(\frac{\omega_n}{\omega}\right)^2 - j2\xi\left(\frac{\omega_n}{\omega}\right) \ldots\ldots Eq(3)$$

Now if you come up with the magnitude of this open loop transfer function you can write this specific gain of this transfer function open loop gain in frequency domain we can write

$$|G(j\omega)| = \sqrt{\left(\frac{\omega_n}{\omega}\right)^4 + 4\xi^2\left(\frac{\omega_n}{\omega}\right)^2} \ldots\ldots Eq(4)$$

Now if you want to come up with the phase angle of this equation number three we can write

$$\angle G_0(j\omega) = \frac{\angle(2\xi\omega_n j\omega + \omega_n^2)}{\Gamma\angle(-\omega^2)} = \tan^{-1}\left(\frac{2\xi\omega}{\omega_n}\right) - 180° \ldots\ldots Eq(5)$$

This is the angle the minus 180 degree coming from the denominator part because there are two poles at origin and for each pole and since there is no a real part only imaginary part if you suppose there is ω^2 so there are two poles at origin for this two poles at origin we can come up minus 90 minus 90 and it is minus 180 degree. Since the frequency ω is positive here the phase angle of $\angle G_0(j\omega)$ lies between

$$-180° < \angle G_0(j\omega) < -90° \ldots\ldots Eq(6)$$

How it is happening because when ω equal to zero that time tan inverse zero equal to zero degree and ω goes to infinity then tan inverse infinity equal to 90 degree. We'll come up with the another important point in frequency domain gain cross over frequency and phase covers possible frequency those are very important part we've come up with the specification in frequency domain but this is I hope it is clear this is the magnitude of the transfer function and the phase angle of the transfer function which is denoted by equation four and five so now let's determine an expression for the phase margin the definition of the phase margin I'll come very soon how we can define for our system but for the time being how we can determine the phase margin for this open loop function find to determine the phase margin we need to we need to calculate the gain cross over frequency for the time being please bear with me I'll explain all this what is gain cross of frequency how to define the phase margin I'll explain very soon so at the gain cross over frequency generally we find the phase margin and gain cross for frequency actually we are defining the gain cross over frequency let's denote ω_g at which the gain of OLTF open loop transfer function is unity. So it means our open loop transfer function we have denoted by g naught the gain of open loop transfer function at the gain cross over frequency

$$\left|G_0(j\omega_g)\right| = 1 \ldots\ldots Eq(7)$$

Using Eq (4) so total gain already we have found of a system this is the gain of the system equation number four and at gain cross over frequency equation number four can be written as

$$\left(\frac{\omega_n}{\omega_g}\right)^4 + 4\xi^2\left(\frac{\omega_n}{\omega_g}\right)^2 = 1 \ldots\ldots Eq(8)$$

Lets consider $q = \left(\frac{\omega_n}{\omega_g}\right)^2$

$$q^2 + 4\xi^2 q - 1 = 0 \ldots\ldots Eq(9)$$

Solving Eq (9) yields

$$q = -2\xi^2 \pm \sqrt{4\xi^4 - 1} \ldots\ldots Eq(10)$$

Using the positive value of q we find that there is only one gain cross over frequency what is that so let me write

$$q = \left(\frac{\omega_n}{\omega_g}\right)^2 == -2\xi^2 \pm \sqrt{4\xi^4 - 1}$$

$$\omega_g = \frac{\omega_n}{\left(\sqrt{4\xi^4 + 1} - 2\xi^2\right)^{1/2}} \ldots\ldots Eq(11)$$

So this is basically very important value which relates between the natural frequency and gain cross over frequency relates so from this relation we'll come up with the time domain specification relation and we can connect to them that frequency domain specification and time domain specification then how we can design the control system in the given time domain specification how we can come up with the frequency domain control specification we'll discuss now we would like to find the phase margin of the open-loop transfer function if you refer equation number six the phase margin from the from equation six it is clear that the phase margin is positive and given by

$$PM = 180° + \angle G_0(j\omega_g) \ldots\ldots Eq(12)$$

This is very very important to find the phase margins of the system this is the formula we will use now substituting substituting equation 11 and equation 5 into equation 12 yields we can write phase margin equal to

$$PM = 180° + \angle G_0(j\omega_g)$$

$$= \tan^{-1}\left(\frac{2\xi}{\left(\sqrt{4\xi^4+1}-2\xi^2\right)^{1/2}}\right) \dots \dots Eq(13)$$

This equation relates between phase margin and the damping ratio of the system this is very very important equation so here actually for the given damping ratio we can find the phase margin of the system if we consider an underdamped system system where the damping ratio varies between 0 to 1 the relation between damping ratio and phase margin can be plotted as for this damping ratio so we are having the response plot

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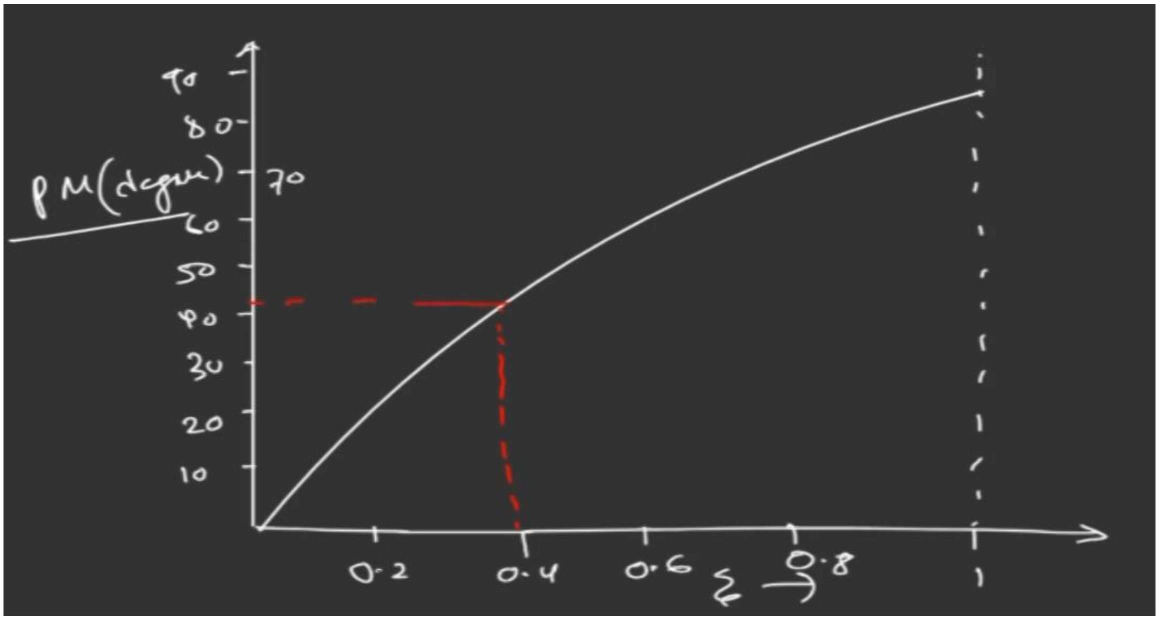

So this is the phase margin where the values are in degrees and we can come up with some relation should be something like this so it means for the given damping ratio we can find the phase margin suppose the damping ratio is given 0.4 so we can at 0.4 we can come up with the what should be the value of phase margin we can come we can get those values so this is very very important relation between the phase margin and damping ratio so now we'll see how the damping ratio and maximum overshoot also the phase margin and damping ratio again if you notice the damping ratio maximum overshoot are related by

$$M_p = e^{-\pi\xi/\sqrt{1-\xi^2}} * 100\%$$

In the figure 3 basically it relates between the phase margin and damping ratio and also we can come up with the relation between the phase margin and maximum overshoot so here also the relation between M_p and damping ratio is given by you can come up some relation

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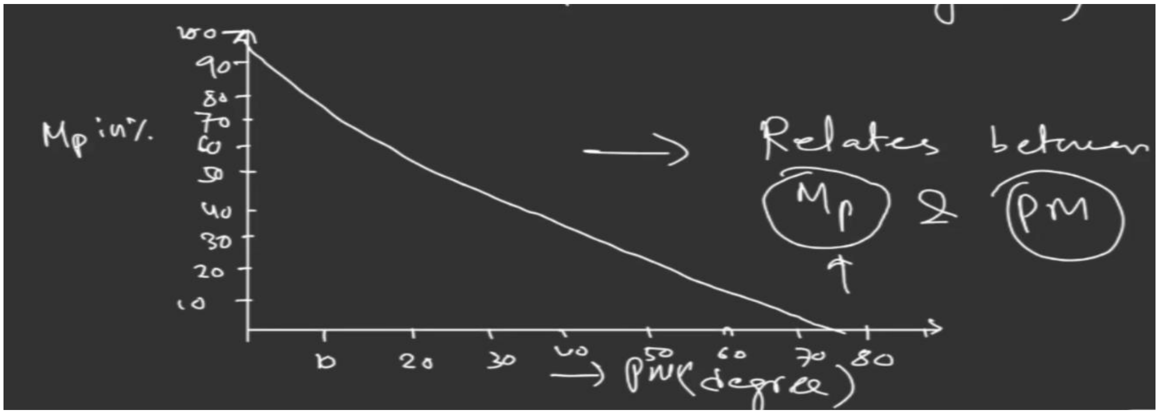

So this is y-axis where we are plotting the maximum overshoot in percentage and this is our phase margin 10 degree and if you plot this and if you look at the y-axis and the relation between phase margin and damping ratio the plots for this can be something like this relates between M_p and gain margin so if the for example in the problem if the phase if the maximum overshoot is given to us then we can from this plot we can come up with the phase margin of the system so this is how we can relate between the time domain specification and frequency domain specification now we have another relation where the rise time and settling time how they are connected let's look with the gain cross over frequency suppose the phase margin has been specified or given to us and also for the given phase margin we can find the damping ratio since phase margin is specified and for the given phase margin from this plot we can find the damping ratio the expression now let's look how the gain cross over frequency and settling time are connected to each other or rise time how they are connected so let's first start with the rise time as we have done as we have done in time domain analysis the expression for rise time we can write

$$t_r = \frac{\pi - \beta}{\omega_n\sqrt{1-\xi^2}}$$

$$\beta = \tan^{-1}\left(\frac{\sqrt{1-\xi^2}}{\xi}\right)$$

Now if you use Eq (11) we can find the relation between the rise time and gain cross over frequency equation in place basically we'll substitute ω_n here because from Eq (11) we can come up with the expression for ω_n and that expression we can substitute here so we can find the relation between the t_r and ω_n so we can write the gain cross over frequency we can have

$$\omega_g = \frac{\pi - \beta}{t_r\sqrt{1-\xi^2}\left(\sqrt{4\xi^2+1} - 2\xi^2\right)^{1/2}}$$

So this relates between gain cross over frequency and rise time. Now we'll look the settling time constraint also from the time domain analysis the expression expression for settling time we can write

$$t_s = -\frac{ln\left(0.02\sqrt{1-\xi^2}\right)}{\xi\omega_n}$$

Again if you substitute the value of ω_n from Eq (11) we can find the relation between the t_s settling time and the gain cross over frequency

$$\omega_g = -\frac{ln\left(0.02\sqrt{1-\xi^2}\right)}{\xi t_s\left(\sqrt{4\xi^4+1} - 2\xi^2\right)^{1/2}}$$

So this is how we can connect the time domain specification with our gain cross over frequency this is how we can find also phase margin gain cross over frequency and if you have phase margin we can come up with the frequency domain control system so we'll come up with those things later so the conclusion of this lecture is we have seen that if the phase margin is given specifications of the closed loop control system namely rise time settling time maybe on the given gain cross over frequency, this is very very important part how we can connect the frequency domain analysis what you are doing in open loop transfer function and we are finding the system specification in time domain specification of the closed loop control system. Let's stop it here we'll continue in the next lecture. Thank you.