**Introduction to Aircraft Control System**

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**Week – 05**

**Lecture – 21**

**Initial and Final Value Theorem**

In the last lecture, we discussed how we can define the stability(t)he system based on the location of the poles. And also we have discussed how we can define the type of the system. So in the last lecture, let me rewrite the OLTF equation we had.

$$G_0(s) = \bar{k}\,\frac{(s - z_1)\ldots(s - z_m)}{s^N(s - p_1)\ldots(s - p_n)} \ldots\ldots Eq(1)$$

Where $z_i$ and $p_i$ represents zeros and poles respectively.

Now we define when N equal to 0, this is type 0 system. And when N equal to 1, type 1 system and N equal to 2, then it is type 2 system. And we can relate the reference signal to the closed loop system which can be able to track. So let me define this equation number 1. Now we can open loop transfer function of the reference signal to be tracked. Now let's rewrite the equation number 1 as, the equation 1 as

$$G_0(s) = k\,\frac{(T_{a1}s + 1)(T_{a2}s + 1)\ldots(T_{am}s + 1)}{s^N(T_{b1}s + 1)(T_{b2}s + 1)\ldots(T_{bn}s + 1)}$$

This is basically the same expression but we wrote in different format. If you take an example, example we can write in this above structure how we can write. So for example,

$$G_0(s) = \frac{(s + 2)(s + 3)}{s(s + 5)(s + 8)} = \frac{6}{40}\frac{(T_{a1}s + 1)(T_{a2}s + 1)}{s(T_{b1}s + 1)(T_{b2}s + 1)} \ldots\ldots Eq(2)$$

Where $k = \frac{6}{40}, T_{a1} = \frac{1}{2}, T_{a2} = \frac{1}{3}, T_{b1} = \frac{1}{5}, T_{b2} = \frac{1}{8}$

So this is how we can write the transfer function. Later we will come to, we will be discussing, these are basically time constant but we will discuss later. In the closed loop system while we are talking about the closed loop transfer function, what is the main motivation? Basically we are making the system asymptotically stable where all the poles of the characteristic equation in the closed loop transfer function should lie in the left-hand plane. So let me write assuming that the closed loop system of the aircraft for example is asymptotically stable $(1 + G_0(s))$. Basically if you notice, so what is the transfer function, closed loop transfer function?

$$CLTF = \frac{G_p(s)G_c(s)}{1 + G_o(s)}$$

So the characteristic equation basically(t)he closed loop system is $1 + G_0(s) = 0$ only has roots with negative real parts. So the the roots we are getting from the characteristic equation which are basically poles and the real part of the poles should be negative in sign and then we can say that system is asymptotically stable because system will be decaying to some point reference signal. Now we can examine the steady state error for the different test signals.

As you have done before in the earlier lecture, I mean in the beginning of this course, we discussed two important terms. One is the final value theorem and second is initial value theorem. So now we are going to use final value theorem how we can define the system goes to the steady state of errors. The system goes to the reference signal. What is the error between the basically if you look the response this is my desired response.

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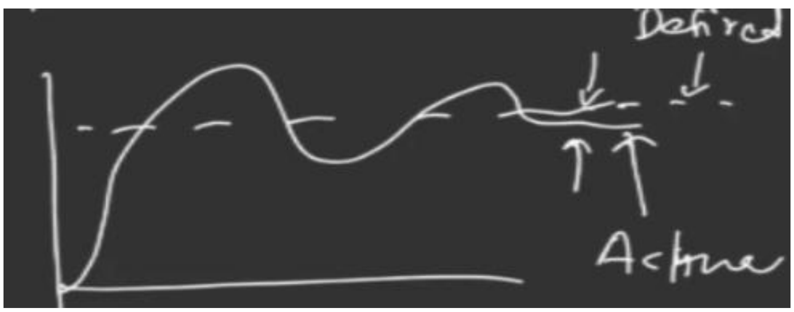


So we are going to find the dotted line basically desired response and this is basically actual right actual and the dotted line is actually desired one. So using the final value theorem we will be finding how we can find the error between desired value and final value between this how we can find the error between this. So now this steady state error we can define by $e_{ss}$ in frequency domain. So in time domain it will be

$$e_{ss} = \lim_{t\to\infty} e(t) = \lim_{s\to 0} sE(s) = \lim_{s\to 0} s\frac{1}{1 + G_0(s)}R(s) \dots\dots Eq(3)$$

So now for the different test signal how we can find the steady state error in the system. So now the I think the problem is easy now. Let's examine the steady state error for the different types of reference signals. Let's start with the step input this is the most important one we use very frequently in the control system design. So step input we will assume and the Laplace transform of step input we know R(s)=1/s where r(t) equal to one. Applying final value theorem let me define this equation number three. So applying final value theorem,

$$e_{ss} = \frac{1}{1 + \lim_{s \to 0} G_0(s)} \ldots \ldots Eq(4)$$

So if you know the open loop transfer function of the system we can get the steady state error for a reference signal. So here we can define the term $\lim_{s \to 0} G_0(s)$ be defined as $k_{sp}$

$$k_{sp} = \lim_{s \to 0} G_0(s) \leftarrow Static\ position\ error$$

If you substitute this expression in steady state condition or the I mean steady state error condition in equation four we can write

$$e_{ss} = \frac{1}{1 + k_{sp}}$$

Now let's look how the type of the system will influence the error in the system. Now let's consider the type zero system for a type zero system where N equal to zero we can find $k_{sp}$ since $k_{sp}$ is a function of open loop transfer function. Also, if it is type zero system then there is no open loop poles in the system, so N will be 1. So we can write

$$k_{sp} = \lim_{s \to 0} k \frac{(T_{a1}s + 1)(T_{a2}s + 1) \ldots (T_{am}s + 1)}{(T_{b1}s + 1)(T_{b2}s + 1) \ldots (T_{bn}s + 1)} = k$$

This is very interesting conclusion from this part we'll have. Now so what is the takeaway from this part so if the type zero system is there and if you are applying a step input then

$$e_{ss} = \frac{1}{1 + k}$$

So there exist steady state error. This is very interesting so it means that if the system is type zero system and if you are applying step input as a reference signal there will be error and this is the magnitude of the error in the system. Now let's look for type one system or let's assume higher order system N is greater than one so we can write

$$k_{sp} = \lim_{s \to 0} k \frac{(T_{a1}s + 1)(T_{a2}s + 1) \ldots (T_{am}s + 1)}{s^N (T_{b1}s + 1)(T_{b2}s + 1) \ldots (T_{bn}s + 1)} = \infty$$

Now if you put s equal to zero in this case they are in the limit so this part going to be infinite so let me write that therefore for type zero system the steady state to a step input cannot be neglected completely because we have seen this is the error for type zero system but if it is type one and higher order system we can track step input perfectly. It is very interesting and the steady state error we can write

$$e_{ss} = \frac{1}{1+\infty} = 0$$

So it is completely tracked. So by track means this is going to zero means y(t) goes to r(t) difference signal so what is the takeaway from this part so if the system is type one or a higher order system if you if you are having reference signal is step then it can be perfectly we can match this y(t) to r(t) just steady state error can be reduced to zero this is the takeaway from this part so this is the part for the step function now let's look at the ramp input ($R(s) = {}^{1}/_{s^2}, r(t) = t$) if you are having ramp input in the system what will happen ramp input applying the final value theorem we can write

$$e_{ss} = \lim_{s\to 0} \frac{1}{1 + sG_0(s)} = \frac{1}{\lim_{s\to 0} sG_0(s)} = \frac{1}{k_{sv}}$$

Where $k_{sv}$ is the static velocity error constant. Now we'll look at the type 0 system first for a type 0 system where N equal to zero and $k_{sv}$ we can find

$$k_{sv} = \lim_{s\to 0} k \frac{s(T_{a1}s+1)(T_{a2}s+1)\ldots(T_{am}s+1)}{(T_{b1}s+1)(T_{b2}s+1)\ldots(T_{bn}s+1)} = 0$$

If you put s equal to zero this expression goes to zero and so what is the takeaway so here your

$$e_{ss} = \frac{1}{0} = \infty$$

So if the system is type 0 system and if you are applying a ramp input there will be infinite error in the steady state so here we can't apply. Now let's look at the type 1 system where N equal to one

$$k_{sv} = \lim_{s\to 0} k \frac{s(T_{a1}s+1)(T_{a2}s+1)\ldots(T_{am}s+1)}{s(T_{b1}s+1)(T_{b2}s+1)\ldots(T_{bn}s+1)} = k$$

So what is the takeaway from this, so steady state error we can write

$$e_{ss} = \frac{1}{k_{sv}} = \frac{1}{k}$$

If you are applying ramp input there will be steady state error, this is very important conclusion from this. Now let's look the type 2 or higher order system for type 2 or higher order N greater than equal to 2 so let me write

$$k_{sv} = \lim_{s\to 0} k \frac{s(T_{a1}s+1)(T_{a2}s+1)\ldots(T_{am}s+1)}{s^N(T_{b1}s+1)(T_{b2}s+1)\ldots(T_{bn}s+1)} = \infty$$

Here one s will cancel out so I can write in place of $s^N$ I can write $s^{N-1}$ . $S$o now since it is higher order N greater than two so if you put s equal to zero the value of $k_{sv}$ goes to infinite so in this case this is very important takeaway therefore the error steady state error is goes to one upon $k_{sv}$ one upon infinity is it's going to zero

$$e_{ss} = \frac{1}{k_{sv}} = \frac{1}{\infty} = 0$$

So what is the takeaway from this part if the system is type 2 or higher order system and if you are applying ramp input as a reference signal there will be no steady state error in the steady state response so this is how we can choose the different type of signal to be tested in control system this is very very important takeaway from this lecture for now we'll be taking an example then we'll conclude in this lecture example.

Let us assume our pd controlled aircraft attitude system we know

$$G_p(s) = \frac{1}{Is^2 + k_d s} = \frac{1}{k_d s\left(\frac{1}{k_d}s + 1\right)}$$

This is the plant transfer function in the inner loop and the controller transfer function for this particular case the controller transfer function if you remember transfer function we had $G_p(s) = k_p$ and the open loop transfer function we can write

$$G_0(s) = G_p(s)G_c(s) = \frac{k_p}{k_d s\left(\frac{1}{k_d}s + 1\right)}$$

From this the type of the system $G_0(s)$ is one and now let's look how the system going to respond input first let's try with the step input R(s)=1/s and first we'll find the static error constant $k_{sp}$ which is basically

$$k_{sp} = \lim_{s\to 0} G_0(s) = \lim_{s\to 0} \frac{k_p}{k_d s\left(\frac{1}{k_d}s + 1\right)}$$

$$e_{ss} = \frac{1}{1+k_{sp}} = \frac{1}{1+\infty} = 0$$

From this we can say we can track step command completely now let's look with the ramp input so for the ramp input the static velocity error constant we can write

$$k_{sv} = \lim_{s \to 0} sG_0(s) = \frac{k_p}{k_d}$$

$$e_{ss} = \frac{1}{1 + k_{sv}} = \frac{k_d}{k_p}$$

So this is basically non-zero steady state error for this aircraft attitude control system if you apply ramp input we'll have steady state error in the response but if you apply step input step command the reference signal we can come up with zero steady state in the response. Let's stop it here. In the next lecture we'll be discussing how we can consider the disturbance in the system and how it is going to affect the steady state error.