Introduction to Aircraft Control System

Prof. Dipak Kumar Giri

Department of Aerospace Engineering

IIT Kanpur

Week – 02

Lecture – 10

Transfer Function with Example

Now, this lecture is continued from the last lecture. In this lecture, we'll be taking the example of a mechanical system and how we can find the linear relationship between the input and output. Then we'll be considering the example of a gear-free system and we'll find the relationship between the rotational angle of the shaft and applied torque. Then we'll shift our attention to the block diagram concept, which is very important for the aircraft system because in aircraft there are multiple subsystems connected in series or parallel and how the overall mass function can be derived between the different inputs and output. Then we'll conclude the lecture. In this lecture, we'll be finding the function of a system. As you have already mentioned, we'll be dealing with the positive values of time. So we'll replace 0^+ by 0 in all relevant applications of Laplace transform

Let's consider we have a linear system in ordinary differential equation form,

$$a_n \frac{d^n y(t)}{dt^n} + a_{n-1} \frac{d^{n-1} y(t)}{dt^{n-1}} + \ldots + a_1 \frac{dy(t)}{dt} + a_0 y(t) = b_m \frac{d^m u(t)}{dt^m} + b_{m-1} \frac{d^{m-1} u(t)}{dt^{m-1}} + \ldots + b_1 \frac{du(t)}{dt} + b_0 u(t)$$

So this is we can assume, this is the equation or in a differential equation. For simplicity, we assume the initial conditions for the input $u(t)$ and its time derivatives and the output $y(t)$ and its derivatives are zeros. So if you have these conditions in the Laplace domain, we can take the Laplace transform of the equation, let me write this equation number one.

In Laplace domain, if we apply Laplace classification one, we have

$$S^n a_n Y(S) + S^{n-1} a_{a-1} Y(S) + \ldots + S a_1 Y(S) + a_0 Y(S) = S^m b_m U(S) + S^{m-1} b_{m-1} U(S) + \ldots + S b_1 V(S) + b_0 U(S)$$

And if you take y as common from the first equation from the left hand equation and from the right hand equation, we can write

$$(S^n a_n + S^{n-1} a_{n-1} + \ldots + S a_1 + a_0) Y(S) = (S^m b_m + S^{m-1} b_{m-1} + \ldots + S b_1 + b_0) U(S)$$

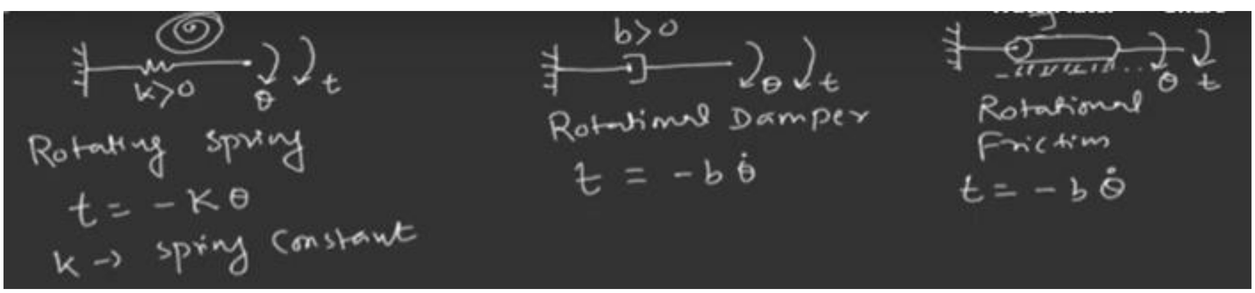
And from this we can write,

$$G(S) = \frac{Y(S)}{U(S)} = \frac{S^m b_m + S^{m-1} b_{m-1} + \ldots + S b_1 + b_0}{S^n a_n + S^{n-1} a_{n-1} + \ldots + S a_1 + a_0}$$

This is basically a transfer function as we have done before, basically the relation between the Laplace transform of output divided by Laplace transform of input, Laplace transform of Y(S) divided by Laplace transform of U(S). So this is how we define the transfer function of a linear system. Now we will take an example of how we can come up with the same concept for a non-SUM system. Let me go through some basics. Let us go to rotational mechanics for systems.

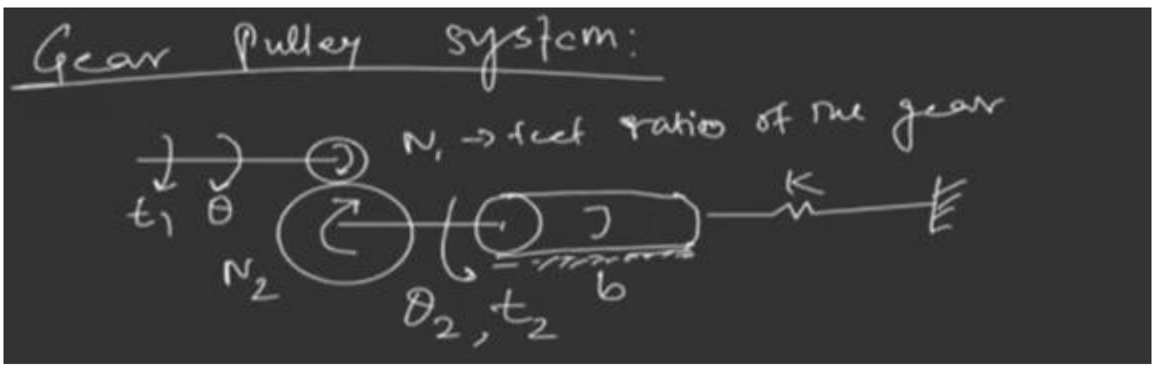
If you have some spring which is connected to some wall, this spring is a rotating spring, and the rotational angle is, let us assume, θ and if the rotation we are generating, we are having some torque t from this spring, the spring is actually rotating like this and the total torque generated from this spring we can write $t = -K\theta$. So here K is the spring constant, and if you have another example, the wall, this is the wall and to this wall a damper is connected ,you can say rotational damper and the rotational angle is θ and the torque, let us assume t again and if we assume the damping coefficient b >0, here k > 0 , spring constant if you assume. So in this case the total torque from this rotational damper we can write $t = -b\dot{\theta}$ and if you have a rotational spring, something is connected like this and not rotational, there is rotational friction. If you assume rotational friction and if you assume J is the some inertia, rotational inertia if you assume J and if you assume the angular motion of θ then total torque from this rotational friction we can write $t = -b\dot{\theta}$. So this is how we can come up with the torque from this kind of system.

(Refer Slide Time 12:12)



Now let's extend to another system, we have some gear-free system, and for this system we'll find the transfer function. If you apply the Laplace transform, if you apply the transfer function, you can write $T(S) = -b\theta(S) \Rightarrow \frac{\theta(S)}{T(S)} = -\frac{1}{b}$ so this is how we can write the transfer function for this system. Let's have this example gear-free system and let me draw the figure. So we have one gear, primary gear and the shaft and the rotational angle is θ and due to this rotation we are also having some torque t_1 and let's also assume the N_1 be the gear ratio for this teeth gear ratio, and we have another bigger gear and N_2 to be the gear ratio for this system and there is a pulley connected like this, this is the rotational angle like this and this is connected to a rotational inertia, the inertia J and again this is connected to one spring, and this is the spring constant K and we are assuming the rotational friction b is here and from this rotation we are getting, θ_2 is the rotation let's assume and we are getting the torque t_2.

(Refer Slide Time 15:13)



So this is the system and we will find the transfer function for this system. So here I can say t_1 be the input torque and output be the θ_1. Let's assume the angular displacement and we'll find the relation between t_1 and θ_1 for this system. So let's go. From this system you can find the relation, this is the fundamental equation from the mechanics,

$$\frac{t_1}{N_1} = \frac{t_2}{N_2} \Rightarrow t_2 = \frac{N_2}{N_1}\, t_1$$

Let me define this equation number 1 and we have another equation from the angle and T-to-gear ratio. We can write $N_1\theta_1 = N_2\theta_2 \Rightarrow \theta_2 = \frac{N_1}{N_2}\,\theta_1$. Let me define this equation number 2. Now also from the constraint equation or I can say torque balance equation, we can write say $J\ddot{\theta}_2 = t_2 - b\dot{\theta}_2 - K\theta_2$. So this is the second equation, we are writing this equation for the second system here.

So now let me define this equation number 3. From equation 1, 2 and 3 if you substitute in place because what is our end goal we have to find the relation between t_1 and θ_1. So but the system is coming out to be relation between θ_2 and t_2 so we have to convert this equation into θ_1 form. So if you replace θ_2 and t_2 variables in terms of t_1 and θ_1 we can write the equation in terms of t_1 and θ_1. So now like say in place of θ_2 we can write this expression

$$J\frac{N_1}{N_2}\ddot{\theta}_1 + b\frac{N_1}{N_2}\dot{\theta}_1 + K\frac{N_1}{N_2}\theta_1 = \frac{N_2}{N_1}t_1$$

So now our equation has been converted to in terms of t_1 and θ_1 form let me define this equation number. If we apply the Laplace transform we can write equation

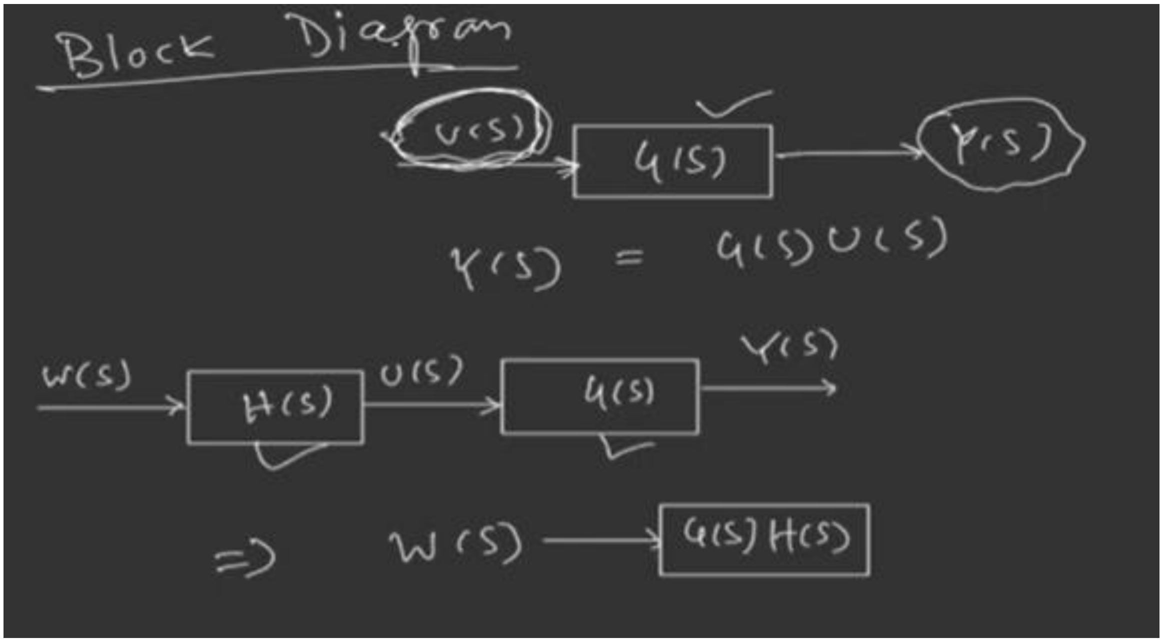
$$\frac{N_1}{N_2}(J\ddot{\theta}_1 + b\dot{\theta}_1 + K\theta_1) = \frac{N_2}{N_1}t_1$$

Now if we apply Laplace transform to this equation , we can come up with the equation $\frac{N_1}{N_2}(JS^2\,\theta(S) + b\,S\theta_1(S) + K\,\theta_1(S) = \frac{N_2}{N_1}T_1(S)$ Now we can come up with the θ, θ is the output and T_1 is the input here ,so I can find the relation between T_1 and θ_1 is $\frac{\theta_1(S)}{T_1(S)} = \frac{\left(\frac{N_2}{N_1}\right)^2}{JS^2+bs+K}$. So this is basically our transfer function so we can write

$$(S) = \frac{\theta_1(S)}{T_1(S)}$$

This transfer function basically relates between the input and output and this is a very important example and we'll be following this way for the other system as well for the aircraft system and let's look some concept of transfer function, how we can find the transfer function in different structure the different multiple system in the loop, how we can find the transfer function for the overall system. So this is the part of the block diagram. So a motivational example of how we can find the transfer function for a system and let's go with the block diagram part. What you have learned till now from this part and the previous lecture so if your system $G(S)$ and if you assume we have input to the system by $U(S)$ and if you assume the output $Y(S)$ so we can find the relation $Y(S) = G(S)\,U(S)$. One thing we need to say that $U(S)$ can be output of another system because in this case $U(S)$ is the input but $U(S)$ can be output of another system here $U(S)$ here $Y(S)$ is the output from this system and similarly $U(S)$ can be output of another system . Let's assume we have another system $H(S)$ and the input to this $H(S)$ is $W(S)$ and from this block let's assume we have $U(S)$ as the output from $H(S)$ and this $U(S)$ is going to be input for this block $G(S)$ and the output from $G(S)$ we can write $Y(S)$,so from this we can write $W(S)$ the input to the entire system we can write $G(S)$ and $H(S)$, we can multiply this block transfer function which relates between the $W(S)$ and $U(S)$, this is a transfer function which relates between the $Y(S)$ and $U(S)$.

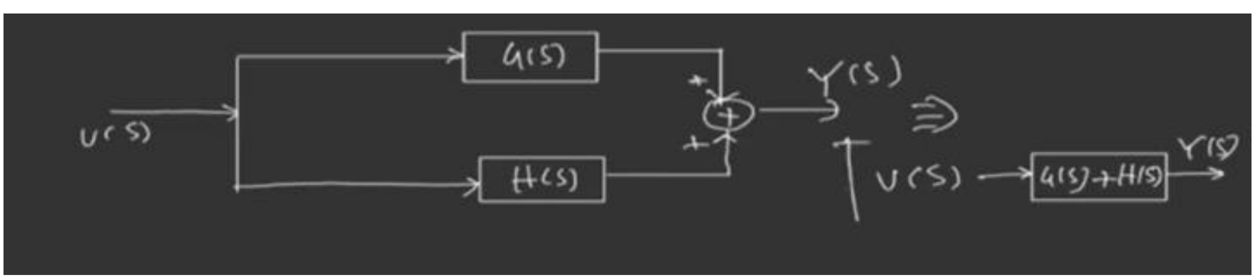
(Refer Slide Time 24:50)



So if you have something like this kind of structure we can multiply the block transfer function like this and the total response from the system will be same as $Y(S)$. So this kind of connection we call the transfer function are connected in series form.Now let's look another structure suppose we have system $G(S)$ and we have another system $H(S)$ and both this block or both the transfer function are relaxed by $U(S)$. So this U is going to both the transfer function and the output of the individual transfer function going through some summing point. If you assume this summing point , I can write + + and the output of this block is let's denote $Y(S)$, if you want to write the relation between between $Y(S)$ and $U(S)$ we can write in this way

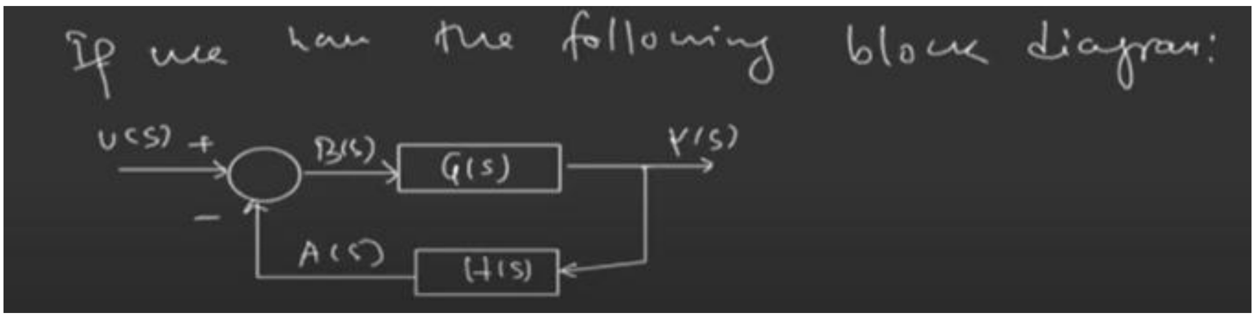
$$u(S) \rightarrow G(S) + H(S) \rightarrow Y(S)$$

(Refer Slide Time 26:49)



So if you have something with this kind of structure ,we can write in this problem. Now let's look another structure as we have done in the beginning of this course: how to find the transfer function for the negative feedback system because if you remember we had example on how to control the speed of the aircraft, so we have used the negative feedback. Now if you have a system something like that, how can we find the transfer function for those kinds of systems? If we have the following block diagram like suppose we have one summing point here and to the summing point we are having a sum input here $U(S)$ and we have in forward loop we have one transfer function $G(S)$ and this is my output from this block $Y(S)$ and in feedback if you assume one transfer function $H(S)$, basically $H(S)$ in the practical life maybe some kind of systems, it may be the sensor we use in the aircraft and this is my negative feedback loop and it's negatively going and this is the positive and this negative and we have $B(S)$ which is nothing but error actually.

(Refer Slide Time 28:57)



Let's also assume the output from $H(S)$ let's assume $A(S)$, so I think the block is now complete and from this block we can write $A(S) = H(S)\,Y(S)$ and $B(S) = U(S) - A(S)$ and also in the forward loop we can write $Y(S) = G(S)\,B(S) = G(S)[U(S) - A(S)] = G(S)U(S) - G(S)\,A(S)Y(S)$,because $Y(S)$ I can write $G(S)$ multiplied by $B(S)$. So now from here $G(S)$ in place of $B(S)$ we can write this expression , and from this we can write $Y(S) + G(S)H(S)Y(S) = G(S)\,U(S)$ and from this we can find the relation between the $Y(S)$ and $U(S)$, because in transfer function we generally find the relation between the input and output, this is also one transfer function but if you come up with the complete one this can be a transfer function also, let me do it, then I'll explain so I can find the relation between the $Y(S)$ and $U(S)$. So from this I can take common $Y(S)\,[I + G(S)\,H(S)] = G(S)\,U(S)$ and from this further I can write $\frac{Y(S)}{U(S)} = \frac{G(S)}{I+G(S)\,H(S)}$ so this is the transfer function of the negative feedback system. Now you can write the same way if instead of negative if you have positive sign ,what will happen. This concept will be extended for the aircraft system in the next lecture. How can we come up with the transfer function of a rotational motion of our aircraft and after finding the rotational motion of the aircraft transfer function will be moving forward how we can design the control autopilot for the same. So let's stop it here. We'll continue from the next lecture.