Computational Science in Engineering
Prof. Ashokee De
Department of Aerospace Engineering
Indian Institute of Engineering – Kanpur

Lecture – 39
Numerical Analysis

So, let us continue the discussion on this ODE. So, we have looked at the stability of the single step method, now we are going to look at the stability of the multi-step method.

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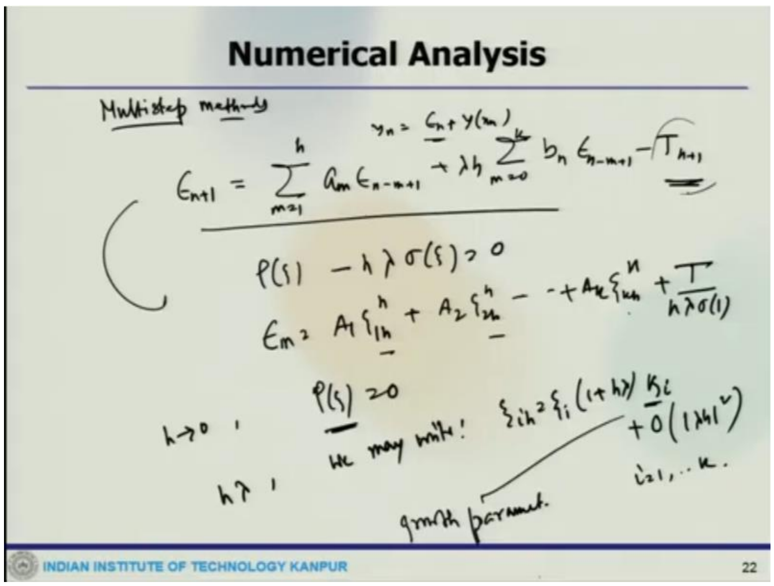

And before moving to something else, so in the multi-step method similarly, we have the expression for the multi-step system where with the error if we replace that, so, this is what we get and this is the local truncation error which is independent of epsilon and again. So, if we assume this is constant and equal to T then the characteristics equation of this which will become like

$$\rho(\xi) - h\lambda\sigma(\xi) = 0$$

And the general solution would be

$$\epsilon_m = A_1\xi_{1h}^n + A_2\xi_{2h}^n + \cdots + A_k\xi_{kh}^n + \frac{T}{h\lambda\sigma(1)}$$

So, A are the constants to be determined from the initial error and all these $\xi_{1h}, \xi_{2h}, \ldots \xi_{kh}$ are the distinct root of the characteristic’s equation. For h tends to 0, the root of the equations $\rho(\xi) = 0$. So, the equation is called a reduced characteristics equation.

And for sufficiently $h\lambda$, we may write

$$\xi_{ih} = \xi_i(1 + h\lambda)k_i + O(|\lambda h|^2)$$

where i goes from 1 to K. So, k_i called the growth parameters. So, this is the growth parameter, one can say this is the growth parameter.

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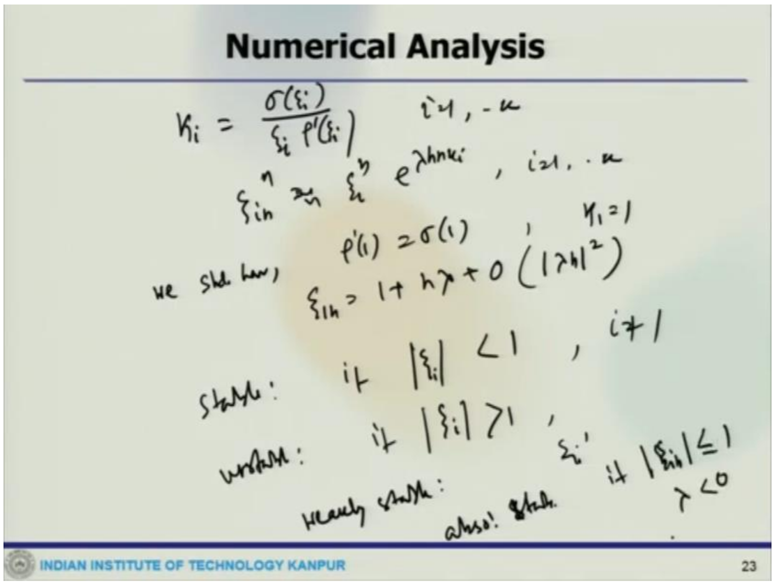

So, what it does then we can neglect the last term, so that we can say this would be

$$k_i = \frac{\sigma(\xi_i)}{\xi_i \rho'(\xi_i)}$$

where i goes 1 to K and then we can write that $\xi_{ih}^n \approx \xi_i^n$, which could be approximated as $e^{\lambda h_n k_i}$, where i goes from 1 to K and for a consistent method we should have $\rho'(1) = \sigma(1)$ where $k_1 = 1$.

So, now, as per this so, for this would be stable if $|\xi_i| < 1$ for i not equals to 1 and it would be unstable straight away if $|\xi_i| > 1$ for some i if there is a multiple root of $\rho(\xi_i) = 0$. Now, this would be weakly stable for ξ_i are simple and if more than one of these roots have modulus unity and it would be absolutely stable if $|\xi_{ih}| \leq 1$, $\lambda<0$.

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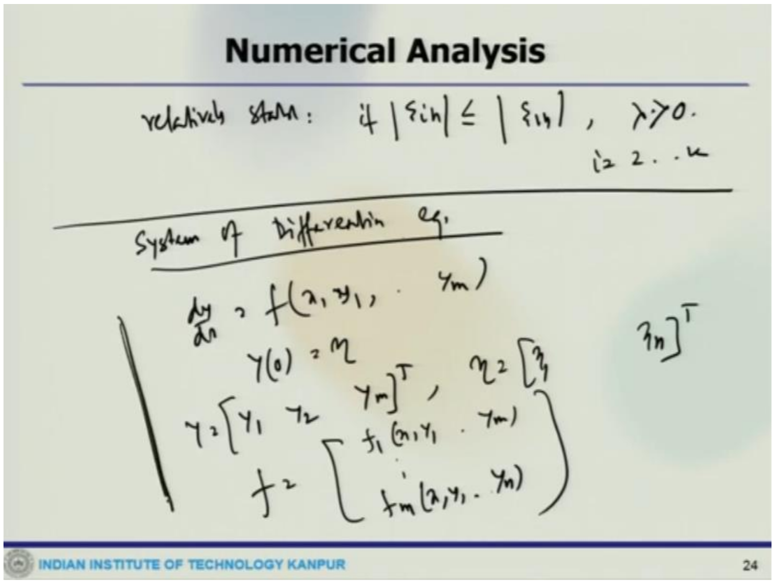

And relatively stable for if $|\xi_{ih}| \leq |\xi_{1h}|$, for λ>0, but i goes from 2 to K. Now it stable in the interval also is absolutely is this. So, this is how the multi-step system would behave. Now, we look at quickly the system of differential equations. So, this is like you have

$$\frac{dy}{dx} = f(x, y_1, \dots, y_m)$$

and $y(0) = \eta$ where this is a system of equations and where

$$y = [y_1 \;\; y_2 \qquad y_m]^T$$
$$\eta = [\eta_1 \;\; \eta_2 \qquad \eta_m]^T$$

Another single step and multi-step method developed already that we have discussed or developed, they can be directly written for the system also the system of equation.

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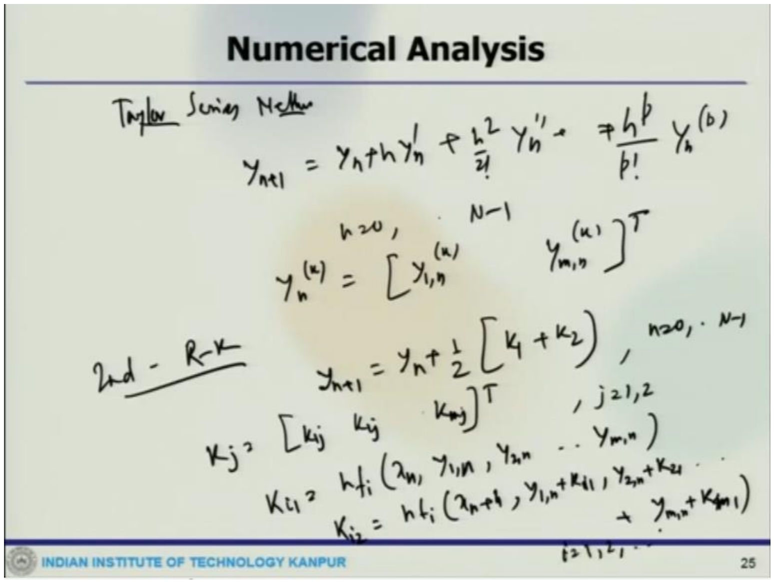

So, now, we can write the Taylor series method. So, which if we write that we can write that

$$y_{n+1} = y_n + hy'_n + \frac{h^2}{2!}y''_n + \cdots \frac{h^p}{p!}y^{(p)}{}_n$$

where n goes from 0 to (N – 1) and

$$y_n{}^{(k)} = \left[y_{1,n}{}^{(k)} \qquad y_{m,n}{}^{(k)}\right]^T$$

Now the second order RK method or RK 2, so, which we can write

$$y_{n+1} = y_n + \frac{1}{2}[K_1 + K_2]$$

n goes from 0 to (N – 1) and

$$K_j = \left[K_{1j} \quad K_{2j} \qquad K_{mj}\right]^T$$

j goes from 1, 2. And

$$K_{i1} = hf_i\left(x_n, y_{1,n}, y_{2,n}, \dots y_{m,n}\right)$$

And

$$K_{i2} = hf_i\left(x_{n+h}, y_{1,n} + K_{11}, y_{2,n} + K_{21}, \dots y_{m,n} + K_{m1}\right)$$

where i goes to 1, 2 to m.

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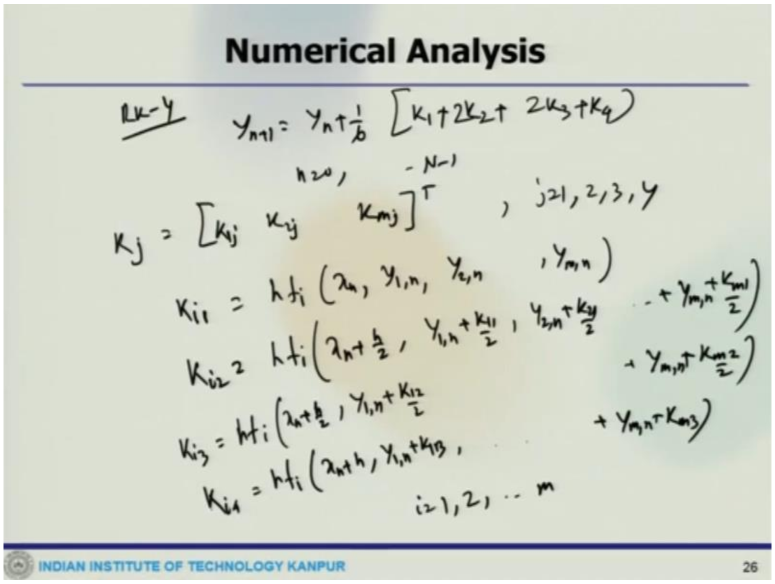

Similarly, one can have RK 4 or fourth order. So, in fourth order similar way you can write

$$y_{n+1} = y_n + \frac{1}{6}[K_1 + 2K_2 + 2K_3 + K_4]$$

n goes from 0 to (N – 1) and

$$K_j = \begin{bmatrix} K_{1j} & K_{2j} & & K_{mj} \end{bmatrix}^T$$

j goes 1, 2, 3, 4 this is fourth order so, goes like that. So,

$$K_{i1} = hf_i(x_n, y_{1,n}, y_{2,n}, \dots y_{m,n})$$

and

$$K_{i2} = hf_i\left(x_n + \frac{h}{2}, y_{1,n} + \frac{K_{11}}{2}, y_{2,n} + \frac{K_{21}}{2}, \dots y_{m,n} + \frac{K_{m1}}{2}\right)$$

$$K_{i3} = hf_i\left(x_n + \frac{h}{2}, y_{1,n} + \frac{K_{12}}{2}, y_{2,n} + \frac{K_{22}}{2}, \dots y_{m,n} + \frac{K_{m2}}{2}\right)$$

And

$$K_{i4} = hf_i(x_n + h, y_{1,n} + K_{13}, y_{2,n} + K_{23}, \dots y_{m,n} + K_{m3})$$

where i is 1 to m. So, this is what you can get for the system of differential equations.

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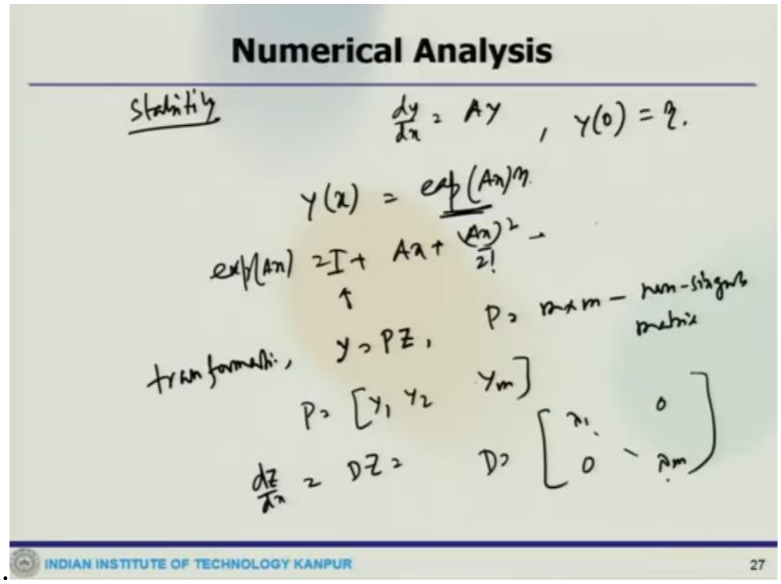

Similarly, whatever we have discussed now also the system of equation the stability is important. Now, the stability of this equation would be here we can write

$$\frac{dy}{dx} = Ay$$

and A is assumed to the constant matrix with distinct values of λ_i. So, we had the initial condition of $y(0) = \eta$. Now we can have the solution of

$$y(x) = \exp\,(Ax)\,\eta$$

$$\exp(Ax) = I + Ax + \frac{(Ax)^2}{2!} + \cdots$$

I is the unit identity matrix or unit matrix.

So, the transformation which we have y = PZ, where P is $m \times m$ non-singular matrix, which is formed by the Eigen vectors corresponding to $\lambda_1, \lambda_2, \lambda_3, \ldots \lambda_m$.

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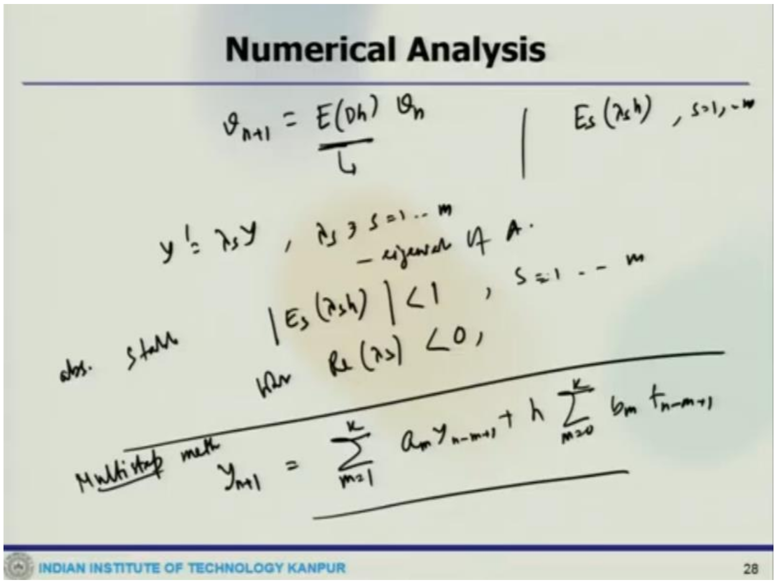

So, if we apply Taylor series what do we get that

$$v_{n+1} = E(Dh)v_n$$

So, this in the approximation to exponential Dh and the matrix $E(Dh)$ is also a diagonal matrix each of its diagonal elements is like $E_s(\lambda_s h)$. So, where s goes from 1 to m is an approximation to the diagonal element of exponential $(\lambda_s h)$. Now, we can have important result that stability analysis for the Taylor series method.

So, the scalar equation which we have now is $y' = \lambda_s y$ and λ_s goes from s = 1 to m of the these are the Eigen values of A. So, this would be absolutely stable the system $|E_s(\lambda_s h)|$, which will be less than 1 for s = 1 to m, where real of lambda s is always less than 0. So, the multi-step method now, basically we can write in general the multi-step method like

$$y_{n+1} = \sum_{m=1}^{k} a_m y_{n-m+1} + h \sum_{m=0}^{k} b_m f_{n-m+1}$$

and a_m and b_m have the same values as in the case of this earlier discussion. So, also the stability method applies to this kind of system.

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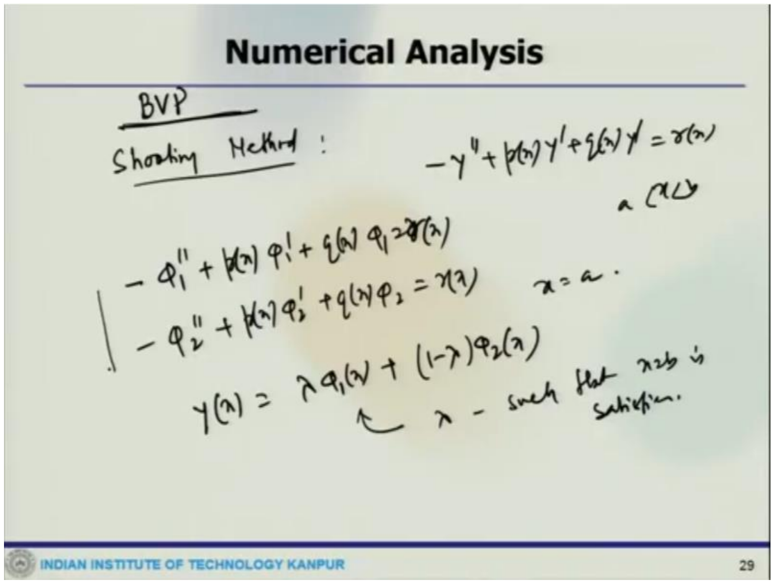

Now, we will talk about some boundary value problem and one of the very important methods to handle the boundary value problem is called a shooting method. So, let us say if you have an non-homogeneous equation like

$$-y'' + p(x)y' + q(x)y = r(x)$$

which is defined in this and this is subjected to some boundary conditions, then this boundary value problem can be solved using the shooting method.

So, this boundary value problem it can be also solved as a 2 non-homogeneous initial value problem like

$$-\phi_1{}'' + p(x)\phi_1{}' + q(x)\phi_1 = r(x)$$

and

$$-\phi_2{}'' + p(x)\phi_2{}' + q(x)\phi_2 = r(x)$$

So, then for the switchable initial conditions at x = a. Now, we write the general solution of the boundary value problem in the form like

$$y(x) = \lambda\phi_1(x) + (1 - \lambda)\phi_2(x)$$

So, and determine lambda so, that the boundary condition and the other end that is so, we can determine lambda such that x = b is satisfied. So, we solved the initial value problem given here.

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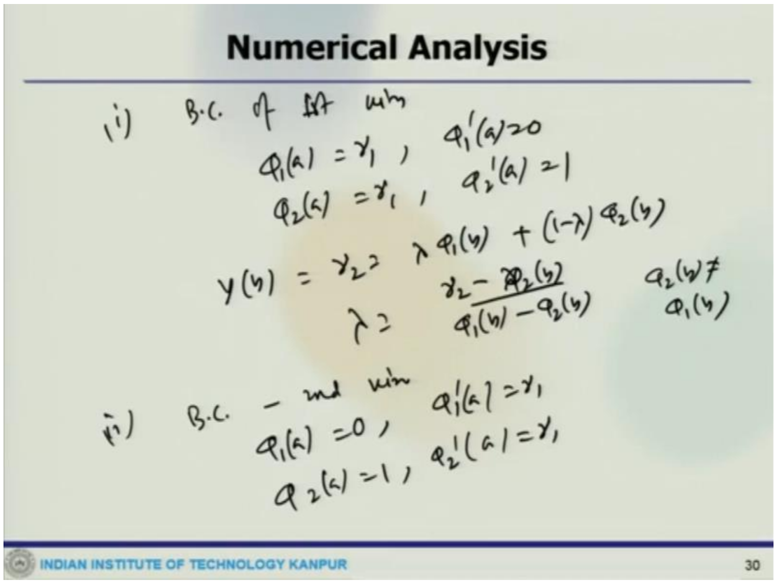

So, now, we first applied a boundary condition of first kind. So, what is that $\phi_1(a) = \gamma_1$, $\phi_1{}'(a) = \gamma_1$, $\phi_2(a) = 1$, $\phi_2{}'(a) = \gamma_1$.

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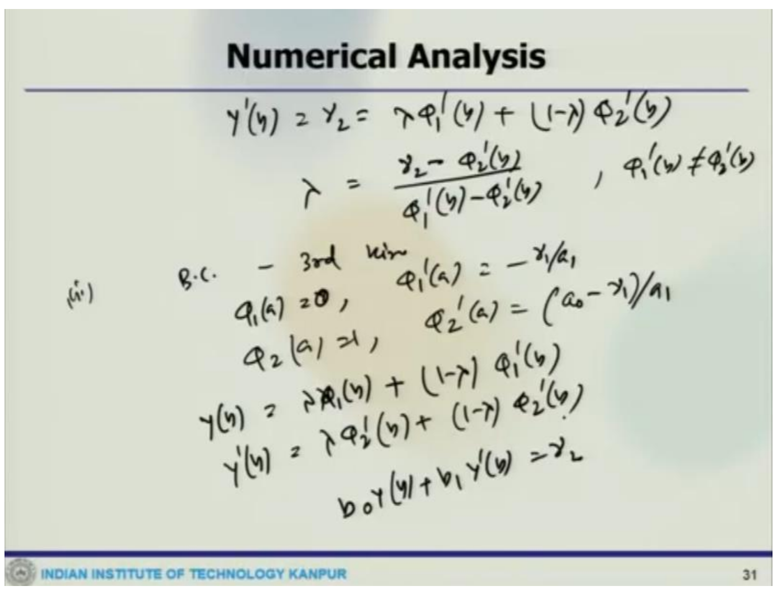

So, we can write them like write the system like

$$y'(b) = \gamma_2 = \lambda\phi_1'(b) + (1-\lambda)\phi_2'(b)$$

from where

$$\lambda = \frac{\gamma_2 - \phi_2'(b)}{\phi_1'(b) - \phi_2'(b)}$$

for $\phi_1'(b) \neq \phi_2'(b)$. And we apply the boundary condition of third kind. So, what we get $\phi_1(a) = 0$, $\phi_1'(a) = -\frac{\gamma_1}{a_1}$, $\phi_2(a) = 1$, $\phi_2'^{(a)} = \frac{(a_0-\gamma_1)}{a_1}$. What do we get

$$y(b) = \lambda(b) + (1-\lambda)\, \phi_1'(b)$$

And

$$y'(b) = \lambda\phi_2'(b) + (1-\lambda)\, \phi_2'(b)$$

So, if we substitute this second the condition like

$$b_0 y(b) + b_1 y'^{(b)} = \gamma_2$$

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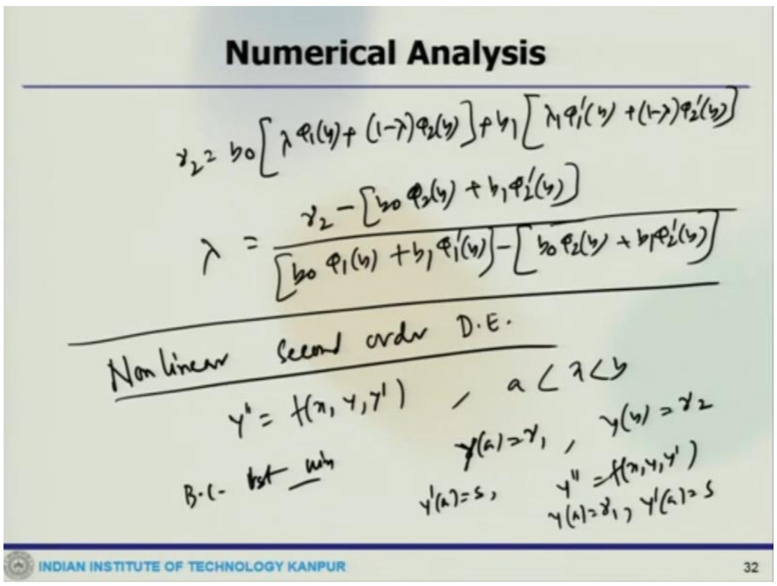

Then what we get is that γ_2. So, which gives the λ as shown on the screen. So, this is one of the ways that one can solve in essentially the initial boundary value problem by shooting method.

Now, we can look at some nonlinear second order differential equation. Let us say which are given as like $y'' = f(x, y, y')$ between $a < x < b$. So, these are subjected to some boundary conditions and since the equation is nonlinear, we cannot write the solutions what is been already discussed. So, depending on the boundary condition, we can proceed like some like as follows like if you have the boundary condition of first kind.

Then we say $y(a) = \gamma_1$ and $y(b) = \gamma_2$ and we assume $y'(a) = s$ and solve the initial value problem like $y'' = f(x, y, y')$ with $y(a) = \gamma_1$ and $y'(a) = s$ up to x = b using any numerical method.

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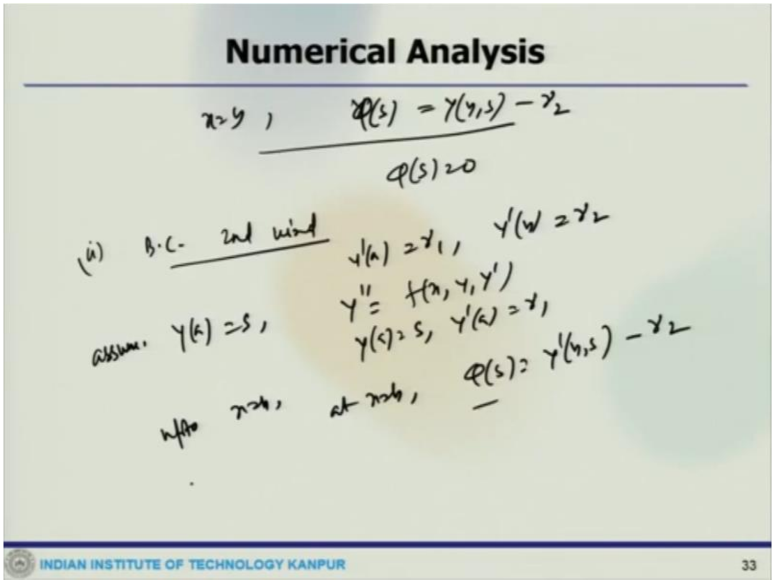

And then the solution of the initial value problem should satisfy the boundary condition at x = b which would be like

$$\phi(s) = y(b, s) - \gamma_2$$

So that we can find s such that $\phi(s) = 0$. Now, secondly, we can have boundary condition of second kind, which is like we have $y'(a) = \gamma_1$ and $y'(b) = \gamma_2$. So, we assume $y(a) = s$ and we solve the initial value problem where $y(a) = s$, $y'(a) = \gamma_1$. So, this would be solved up to x = b and with using the methods that we have already discussed then at x = b it should satisfy that

$$\phi(s) = y'(b, s) - \gamma_2$$

and hence the problem is to find s such that $\phi(s) = 0$.

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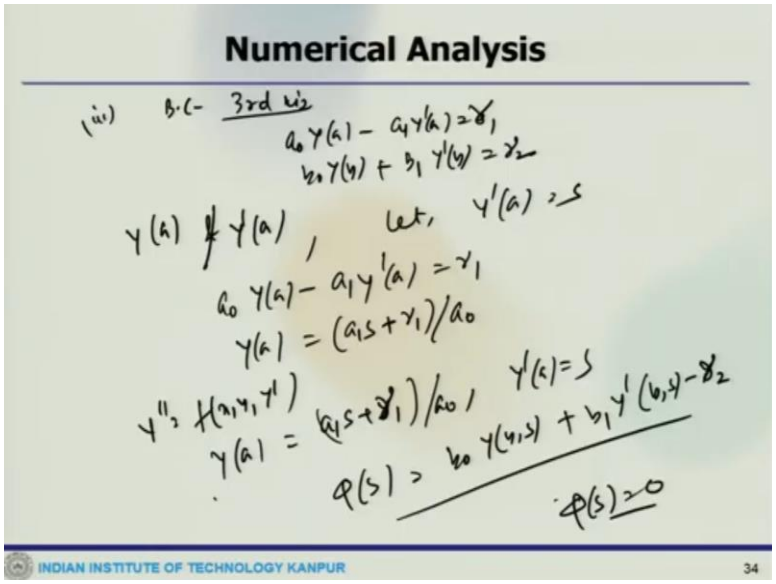

Similarly, if you put boundary condition of third kind, then you get

$$a_0 y(a) - a_1 y'(a) = \gamma_1$$

$$b_0 y(b) + b_1 y'(b) = \gamma_2$$

So, here we assume the value of $y(a)$ or $y'(a)$ let us say we assume $y'(a) = s$ then, we get

$$a_0 y(a) - a_1 y'(a) = \gamma_1$$

where we get

$$y(a) = \frac{(a_1 s + \gamma_1)}{a_0}$$

Now, we solve the initial value problem, which is

$$y'' = f(x, y, y')$$

and

$$y(a) = \frac{(a_1 s + \gamma_1)}{a_0}$$

$$y'(a) = s$$

So, this would be solved up to x = b to be it should satisfy that

$$\phi(s) = b_0 y(b, s) + b_1 y'(b, s) - \gamma_2$$

So, here the problem is to find $\phi(s) = 0$. So, the function $\phi(s)$ is a nonlinear function and we solve for the equation all these different boundary conditions that we have given that the function is given that $\phi(s)$ is solve for this with the different iterative method and find out that so, that we can find the solution to this equation.

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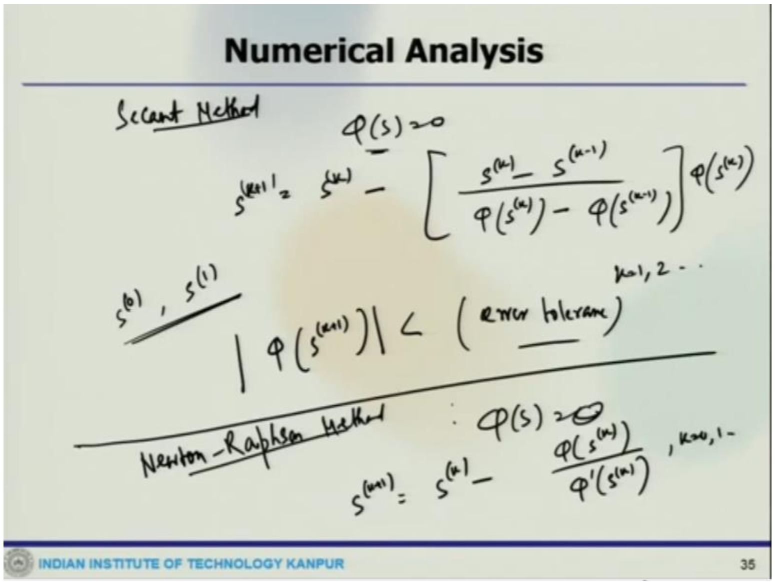

One of quick such method is the secant method where the iteration is done to solve $\phi(s) = 0$. So, we go back to our initial discussion when we are talking about different iterative method to solve a linear equation. These are 2 initial approximations to s to solve the initial value problem which we have already discussed.

And so, that we to guess values of this and keep iterating until we get

$$\left|\phi\left(s^{(k+1)}\right)\right| < (error\ tolerance)$$

that is provided wise. Similarly, one can look the Newton Raphson method for this. So, we are solving here again $\phi(s) = 0$. So, this is to just to solve that nonlinear function $\phi(s) = 0$, how we can solve them. So, here we can write

$$s^{(k+1)} = s^{(k)} - \frac{\phi\left(s^{(k)}\right)}{\phi'(s^{(k)})}$$

where K goes from 0, 1 like that.

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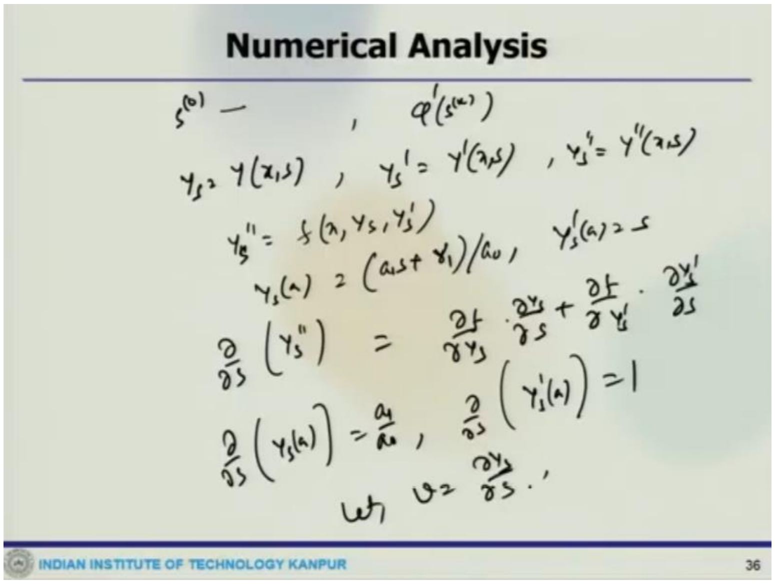

So, this is where $s^{(0)}$ is some initial approximation to S. So, to determine $\phi'\left(s^{(k)}\right)$ we can proceed like let us say

$$y_s = y(x, s)$$

So

$$y_s{}' = y'(x, s)$$
$$y_s{}'' = y''(x, s)$$

So, you can write

$$y_s{}'' = f(x, y_s, y_s{}')$$

So,

$$y_s(a) = \frac{(a_1 s + \gamma_1)}{a_0}$$

So, if we take the partial derivative, what finally we will get.

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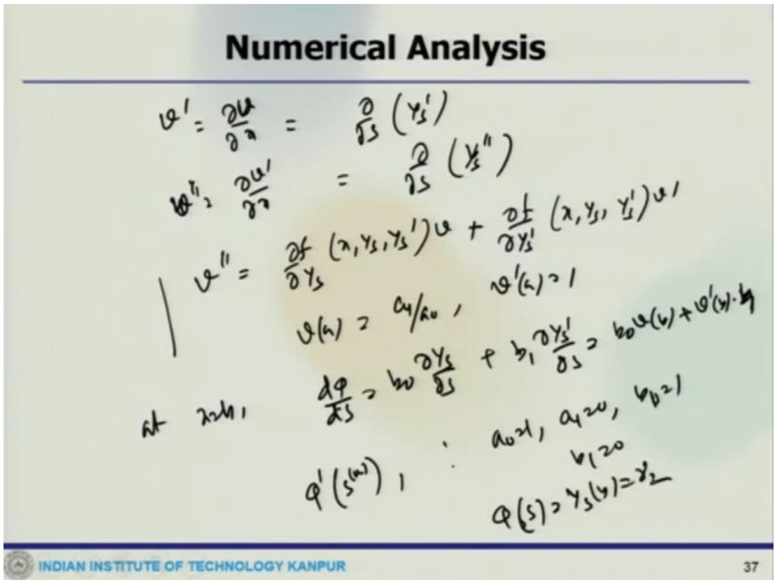

Let

$$v = \frac{\partial y_s}{\partial x}$$

by then what we get is that

$$v' = \frac{\partial v}{\partial x} = \frac{\partial}{\partial s}(y_s{}')$$

$$v'' = \frac{\partial v'}{\partial x} = \frac{\partial}{\partial s}(y_s{}'')$$

So, the differential equation which is given is the first variation equation and it can be solved step by step and can be solved together also as a single system.

So, now at x = b what you have

$$\frac{\partial \phi}{\partial s} = b_0 \frac{\partial y_s}{\partial s} + b_1 \frac{\partial y_s{}'}{\partial s} = b_0 v(b) + b_1 v'(b)$$

We have the values of $\phi'\left(s^{(k)}\right)$ and which can be used. So, if the boundary condition of the first contour is given then we have $a_0 = 1$, $a_1 = 0$ and $b_0 = 1$ $b_1 = 0$. So, we get

$$\phi(s) = y_s(b) = \gamma_1$$

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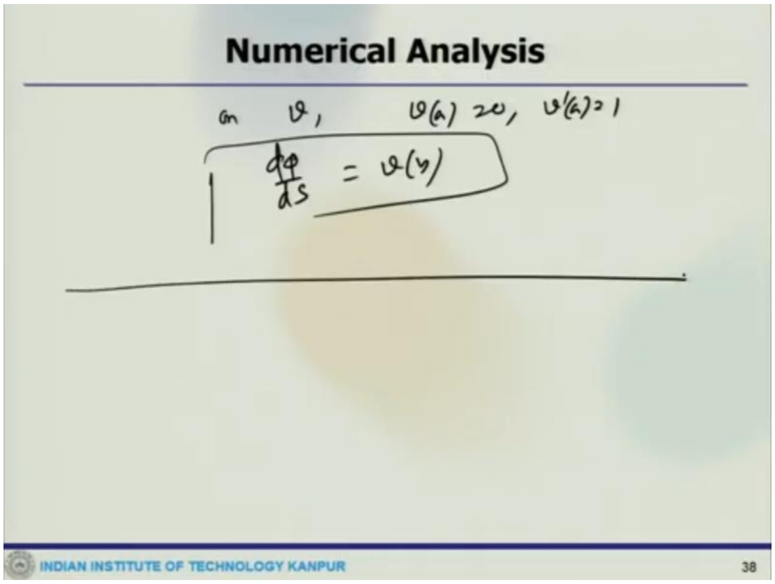

And initial conditions on v becomes $v(a) = 0$ and $v'(a) = 1$. So, we have

$$\frac{d\phi}{ds} = v(b)$$

So, this is what we finally have. So, you can see how the nonlinear system also can be solved with the different kind of iterative procedure. So, that pretty much talks about to like different ordinary kind of different ordinary differential equation that you have and that you can solve. So, we started with the simple methods then we looked at the multi-step method.

And then we looked at the convergence and different stability criteria. Also, we have looked at different variants of methods like first order method or second order method such that so, and depending on and most of the time we talked about or restricted the discussion on generic system, so that for a particular value you can pick and you can have a particular order of method. So, that pretty much talks about the ODE and we will stop it here and just quickly go over some of the issues of the partial differential equation in next session.