Introduction to Finite Volume Methods - II
Prof. Ashoke De
Department of Aerospace Engineering
Indian Institute of Technology, Kanpur

Lecture - 05
Linear solvers – V

So, welcome to the lecture of this Finite Volume Method and what we have stopped in the last lecture is on the ILU factorizations. So, this is a one class of factorizations or decomposition that is very handy and can be used for the iterative solvers.

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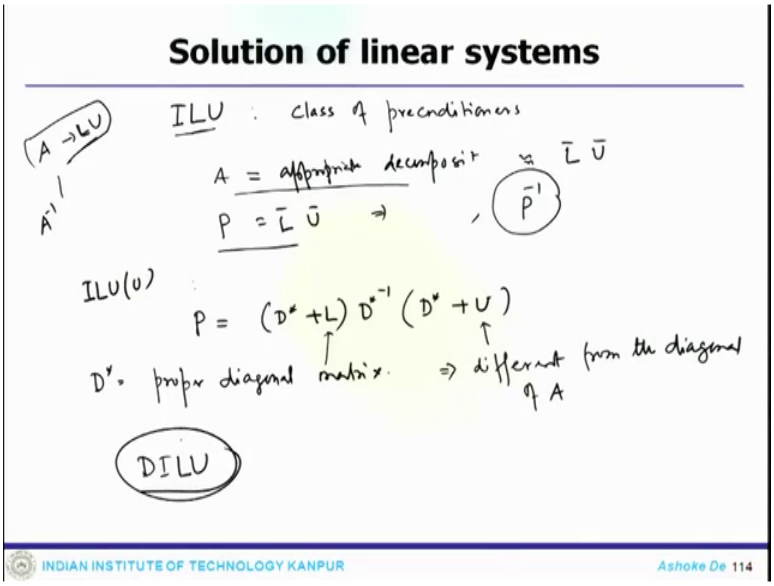

So, what we are talking about that incomplete LU factorizations. So, this is a class of preconditioners. So, essentially this is a class of preconditioners which are populars for the iterative process.

And decomposing a sparse matric into the product of lower and upper triangular system leads to some sequent some sort of substantial feeling which we have already seen while talking about the direct solver.

Now, here when talking about this; so due to the because a preconditioner is only required to be an approximation of A inverse; it is sufficient and also to some extent necessary to look the appropriate decomposition, so, this is appropriate decomposition of A; such that you get L and U. Now you can choose your preconditioner equals to L and

U bar which leads to the efficient evolution of the inverse of the preconditioning matrix P inverse.

So, in turn how good you approximate the decompositioner of A will have an impact on the other hand in the preconditioner or more precisely to find out the inverse of that preconditioner. Because that is one thing which would be required for finding this different. Now for the ILU; 0 the incomplete factorization meanings the non zero elements sparsely of the original matrix such that the preconditioner has exactly the size of the original matrix.

Now, also one can possibly write P as a D star plus L D star inverse multiplied with D star plus upper; where L stand for lower triangular matrix U stands for upper triangular matrix. Now D star here is proper diagonal matrix. So, which is also different from the diagonal of A; so, D star is not exactly the diagonal of A.

So, now the D star matrix is defined and one can find out this D star and which is called the DILU method. So, this will lead to another class of decomposition method.

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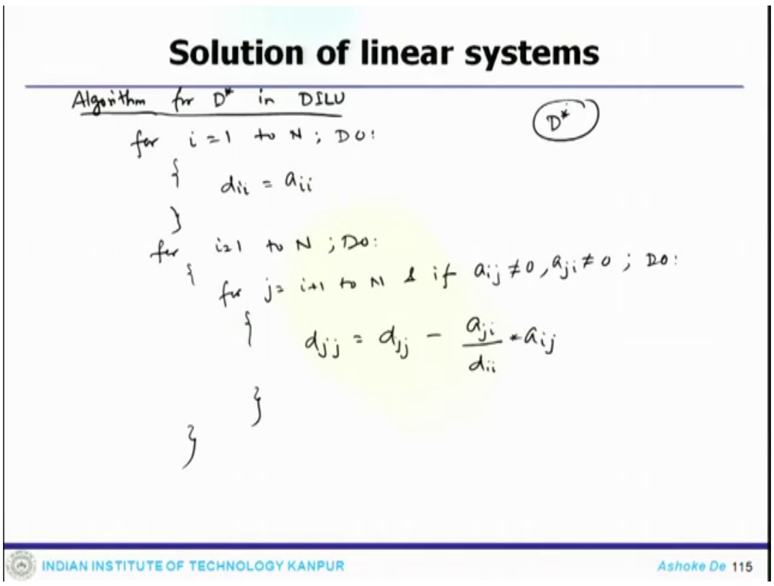

So, the algorithm finds for D star in DILU method. So, this will only find the D star in the DILU. So, what you do? You start going from 1 to N and you do the following where you get d ii equals to a i i and you close that.

Then you move for the another loop where you move from 1 to N and then you do the following. For j goes from i plus 1 to N and if a ij not equals to 0; a ji not equals to 0, then only you do the following. What is that? You calculate the d jj equals to d jj minus a ji by d i i multiplied with a ij. So, that close this then you close the other one. So, this is how you find that D star or the element of the D star.

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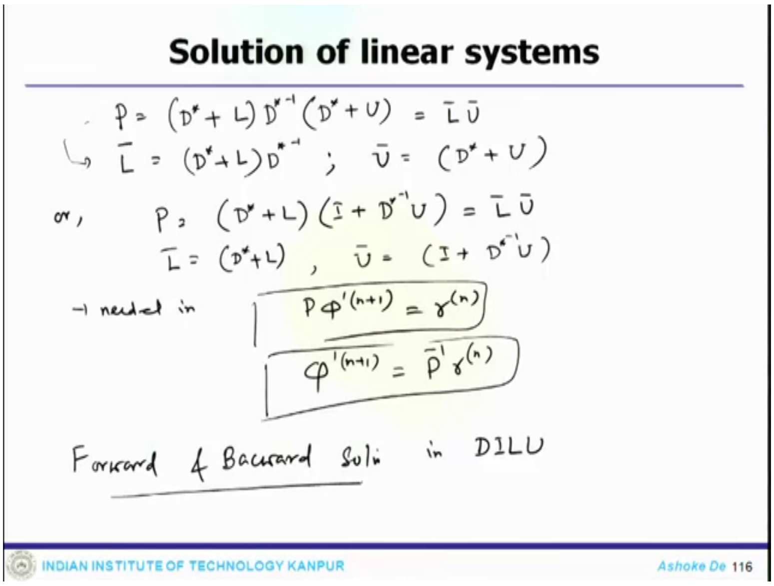

So, if you have in the case of the inverse of the preconditioner; then you can write P equals to D star plus L; D star inverse multiplied with D star plus U which is L bar U bar and your L bar is D star plus L D star inverse and your U bar is D star plus U. So, you just write down the equivalency from this step to this step. Or one can rewrite P equals to D star plus L identity matrix D star inverse U equals to L bar; U bar, where L bar is D star plus L and U bar is I plus D stars inverse U.

Now, where will they be needed? They are needed in this equation where P phi prime n plus 1 is r to the power n. So, that is the correction matrix where the correction field is obtained as P inverse r to the power n and this corrections is used to update the.

Now, once you get this there will be other associated calculations what is the forward and backward solution, backward solution in this DILU method.

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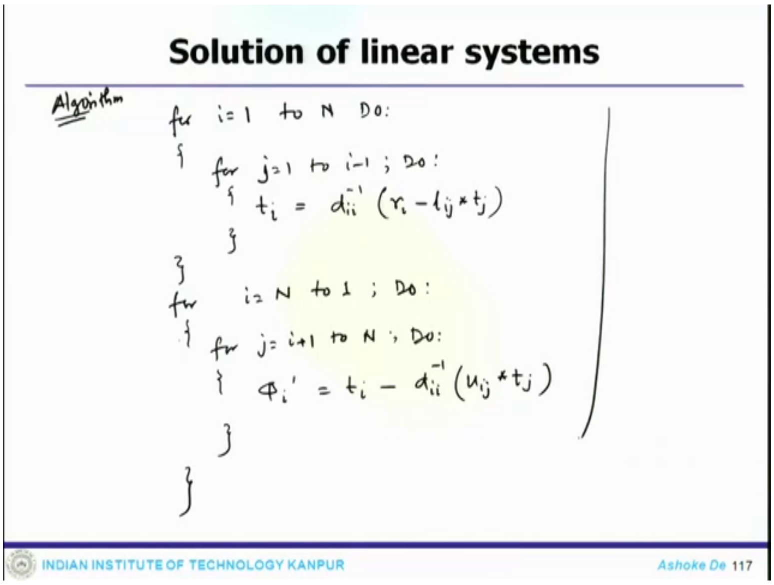

So, what could be the algo for that? Algorithm for that to find out the forward and backward solution in DILU method; so, you go by and loop 1 to N and what you do the following you goes for another loop j 1 to i minus 1 and then you calculate t i; which is d ii inverse r i minus l ij multiplied with t j.

So, which will actually within this loop and then closes this loop. Outside you move along with another one N to 1 and you do the following; for j equals to i plus 1 to N; you do phi i prime equals to t i minus d ii inverse; U ij into t j then you close the this loop and then final you close this loop.

So, if you look at the complete algorithm of DILU; there is an advantage. Apart from is recursive formulation only it requires the storage of one extra diagonal. Other than that it does not need anything much compared to other version of the ILU factorization.

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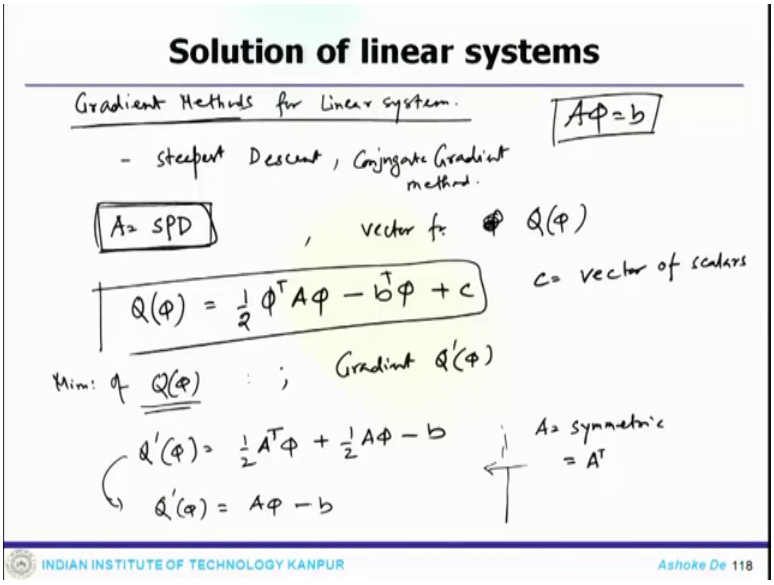

Now, we will move to other class of system which will be helpful to calculate this kind of algebraic system; which are gradient based method or gradient methods for linear system or linear solution of the linear solver.

So, this is another group of iterative process; so decomposition method is one, this is another group of iterative procedure for solving this kind of linear system and the system remains same what we are doing for is the A phi equals to b. So, which requires some sort of an system of equation in the gradient methods; which include some sort of an steepest, descent and the conjugate gradient method; conjugate gradient method.

So, they are initially developed when they are coefficient matrix A is symmetric positive definite; that means, the A was Symmetric Positive Definite; this is another very special case of matrix and while talking about the properties of the matrix; we have discussed about this particular properties of this kind of matrices. So, what you can do the reformulate the minimization problem by using some kind of a vector function like capital Q of phi and which can be written as Q phi equals to half of phi transpose A phi minus b transpose phi plus c.

So, this is how one can reformulate this where c is a victor of scalars and other variables are as per our standard notation. So, now the minimum of this Q phi is obtained when its gradient with respect to phi is 0. So, the minimum one can obtain from this equation when its derivative of the gradient with respect to phi is 0.

So, the gradient; gradient could be defined as Q prime phi and of this vector field Q phi for a given phi. And in the direction greatest increased of Q phi. So, the mathematical formulation lead to that the Q prime phi is half; A transpose phi plus half A phi minus b.

Now, what is the important property? Important property is that A is symmetric. So, as soon as you have a symmetric matrix which means A transpose equals to A; then this equation boils down to. So, using these particular property these equation boils down to Q prime phi equals to A phi minus b.

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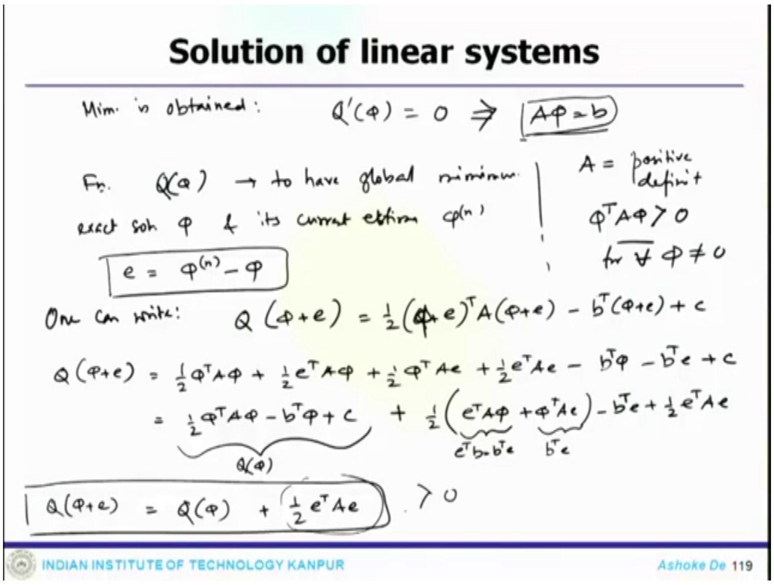

Now, how do I obtain the minimum? So, the minimum is obtained when Q prime phi equals to 0; which leads to A phi equals to b. So, that is what you get back the system. So, therefore, minimising these; Q phi is equivalent to solving the this equation. And the solution of the minimization problem actually gives you the solution of the system of the linear system; so that is the equivalence is one can show.

Now, once you talk about this function; this function Q phi to have a global minimum, now to have global minimum; it is necessary for coefficient matrix A to be positive definite matrix. So; that means, the A is positive definite matrix; that means, it satisfied phi transpose A phi greater than 0 for all phi not equals to 0.

So, that is the property of a positive definite system and these are the properties as I told earlier that while discussing the properties of the linear systems that there going to be used while talking about the linear systems. So, now once this requirement one can establish by considering the relationship of the between the exact solution phi and its current estimate.

So, the current estimate is phi n. So, one can write that error e is phi n minus phi that is the error between the successive iteration. So, that is actually provides the error between two successive iteration and one can write this particular expression what we had the one which we had this equation that; Q phi equals to half phi transpose A phi and b transpose phi plus c.

So, these equation one can write now one can write using this error that Q phi plus e equals to half; Q plus e sorry, phi plus e transpose A; phi plus e minus b transpose phi plus e plus c. So, if you do some sort of an calculation or algebraic manipulation; one can write phi transpose A phi plus half e transpose A phi plus half; phi transpose A e plus half e transpose A small e minus b transpose phi minus b transpose e plus c.

So, I will rearrange the term half phi transport A phi minus b transpose phi plus c; then plus half of e transpose A phi plus phi transpose A e. And then remaining term b transpose e plus half e transpose A e. Now, this term is nothing, but the original function which is Q phi.

And this particular term is e transpose b equals to b transpose e and this guy is b transpose e. So, what one can write after doing this? So, essentially these becomes key function of Q phi plus half e transpose A e; so, that is what you get. Now, property of A is positive definite; so since it is positive definite the second term which is seen here.

So, this is your Q phi plus e; so the second term here will be always positive. So; that means, this term is always positive as the A is positive definite. And this will be always positive except e is 0; in which case the required solution would have been obtained. Further more one can see that A is positive definite or all the eigen values are also positive and the function Q phi has a very unique minimum.

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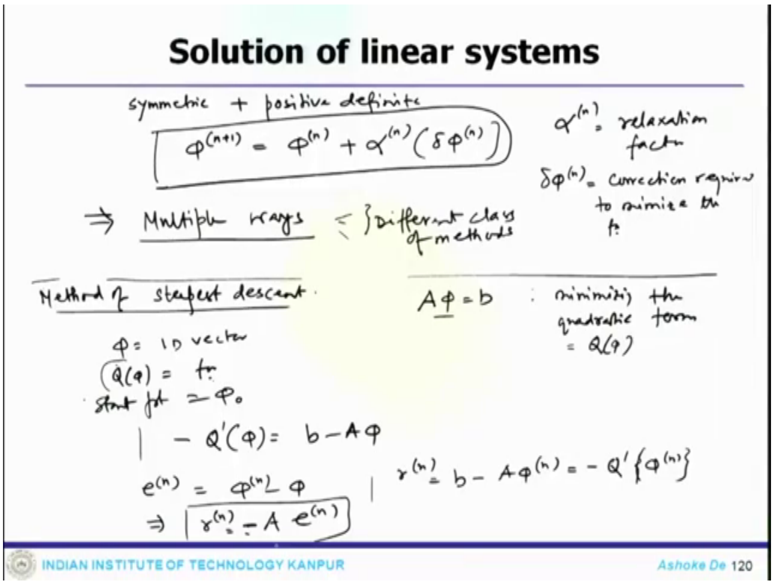

So, for this symmetric and plus this positive definite property; if you use both these positive definite properties, so the convergence criteria for phi at n plus 1 level would be phi n plus alpha n delta phi n. So, this is what one can write where alpha n actually some sort of a relaxation factor. And delta phi n is the correction which is required to minimise the set function ok.

So, this can be accomplished in a variety of ways and once; so whole idea is that you have a function and iteratively you get some sort of an corrections and that can be obtained in a multiple ways. So, multiple ways one can obtain that and what do that; that actually lead to different different methods. So, that leads to different class of methods. So, being said that we will look at some such class of methods; one is the method of steepest descent; so that is one one.

So, the method of steepest descent; so the linear system that we are looking for is A phi equals to b. So, the method of steepest descent for solving linear system of the equations of this nature is based on the minimising; the quadratic form which is given by the expression of the Q phi. So, it is a minimization of this form Q phi which we have. So, where one can save is 1 dimensional vector of a given scalar phi and Q phi is represent some function ok.

And to finding the minimum of this; this is some sort of an one can think about it is a parabolic function; this para. To find out the minimum of this parabolic function iteratively starting your starting point is starting point is phi 0 is that the you go on doing this until you hit the minimum. So, it is same thing can be extended for multiple dimensions and that case this function Q phi; which can be depicted as a parabola and the solution would be again obtained from the initial solution.

Now, now according to this; the error function what we have obtained is that minus Q prime phi is b minus A phi that is the function. Now the exact solution is going to be phi; so the residual at the every step and the error. So, one can calculate error is phi n minus phi and the residual at every step would be b minus A phi n; which is minus Q prime phi n. So, that means r n is minus A equals to minus A; e power n.

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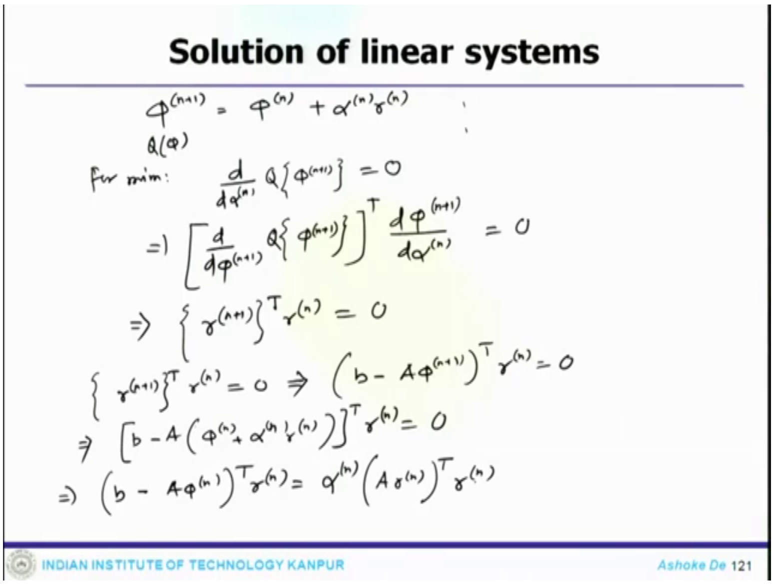

So, now you can move from one iteration to the other iteration and then you can write phi n plus 1 equals to phi n plus alpha n and r n. So, the value of alpha that minimises that Q phi and one can write for minimum condition is that d d alpha n of Q n plus 1; which would be 0.

Now that you can get that d d phi n plus 1 or Q phi n plus 1; transpose d phi n plus 1 divided d alpha n which is 0; so that gets you back r n plus 1 transpose r n equals to 0. So, that indicates that new step should be in the direction normal to the old step. Now, the value alpha n can be calculated from this which would be now r n plus 1 transpose; r n which is 0.

So; that means, one can write b minus A phi n 1 transpose r n 0; so that leads to b minus A phi n plus alpha n r n transpose; r n equals to 0. Then b minus A phi n which will transpose r n 0 equals to alpha n A r n transpose r n.

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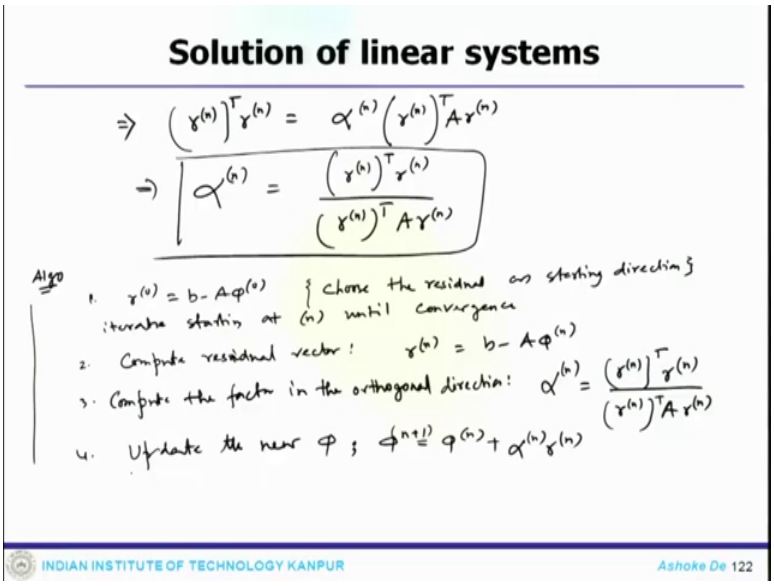

Now, you expand this one further you get r n transpose r n is alpha n; r n transpose A r n; which will get you back the alpha n equals to r n transpose r n divided by r n transpose A; r n. So, that is how you get this factor.

So, the algorithm could be looking like something which one can summarise as like; first you calculate; so that will take you to the algo. So, first you calculate r 0; which is b minus A phi 0; so that means you are choosing the residual as starting direction. And you do now iterate starting at n until convergence.

Now, you compute the residual vector; residual vector which is like r n equals to b minus A phi n then you compute the factor in the orthogonal direction. So, that if you do; alpha n equals to r n transpose r n divided by r n transpose A r n. And then update the new phi as phi n plus 1 equals to phi n plus alpha n r n; so, this is how the algorithm work.

Now, you can see that there you can one of them while doing this performance, there are some matrix and vectors which are calculated.

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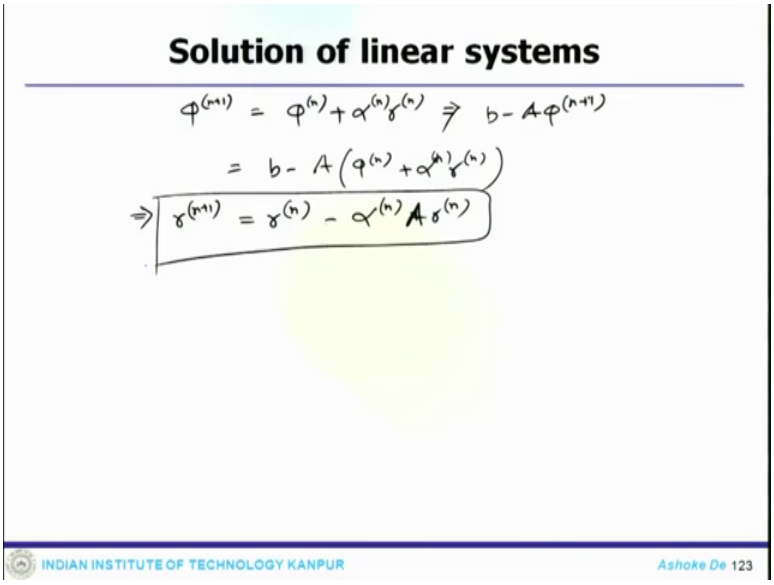

So, one of them can be eliminated by multiplying both sides by negative of A and b. To get phi n plus 1 equals to phi n plus alpha n r n which is b minus A phi n plus 1 and b minus A phi n plus alpha n r n. So, that's get you r n plus 1 equals to r n minus alpha n A r n; so that is what you get. So, we will stop here today and we will take from here in the follow up lectures.

Thank you.