Introduction to Finite Volume Methods-II
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Lecture – 26
Temporal Discretisation - II

So, welcome back to the lecture series of Finite Volume and where will continue our discussion where we left in the last lecture. Now, what happens to the transient diffusion case; now, that is the case where we had a convection case.

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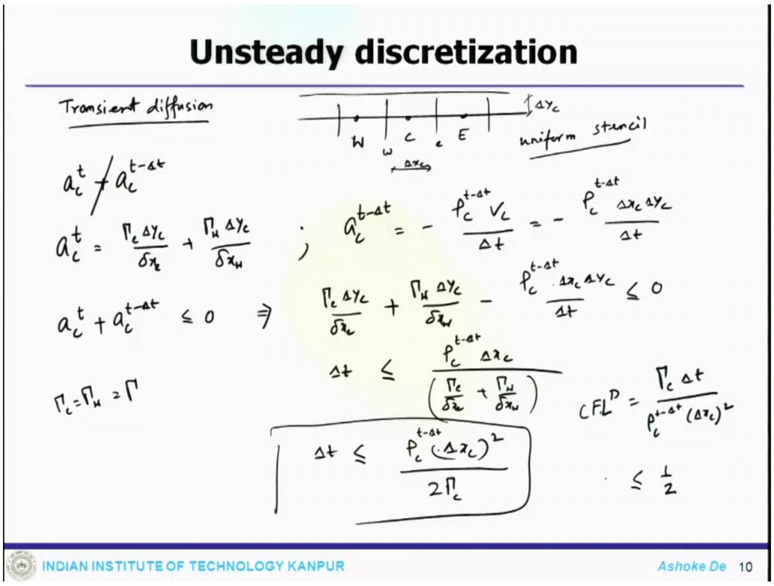

Now, we look at Transient diffusion problem. So, transient diffusion problem for this purpose is again you consider that one-dimensional stencil. So, C, E W; this is small e small w delta x C and if the cell height is there that would delta Y C, so, it is a uniform. Again, it is a uniform stencil. So, once you consider that uniform stencil, now the discretized equation for the transient diffusion term is going to be a C and these are the coefficients in there. So, the a C at present time level is given by gamma e delta Y C by delta x C plus gamma W delta Y C by.

Similarly, at the previous time level, this is minus rho C t minus delta C V C by delta t which is minus rho C t minus delta t del x C del Y C divided by del t. So, the CFL criteria says a C at t plus a C at t minus delta t, they less than 0 which will get you gamma e delta Y C by del x C plus gamma w del Y C by del x w minus rho C t minus delta t del x C del Y C by del t less than 0. So, which will give you an criteria where delta t is less than rho C t minus delta C x C gamma e by del x C plus gamma w by del x w. So, in case you have as we assume the uniform grid and let us say gamma e is same which is gamma; then, this delta t becomes rho C delta x C square divided by 2 gamma C.

So, that is the restriction you get transient diffusion problem. So, the CFL criteria for diffusion problem is gamma C delta C delta C by rho C delta x C square and it has to be less than half for stability. So, you see individually when you look at the convection problem and the diffusion problem, the stability limits are different and when we combine them. So, the complete unsteady convection diffusion system. So, that is the one which you would like to take up next.

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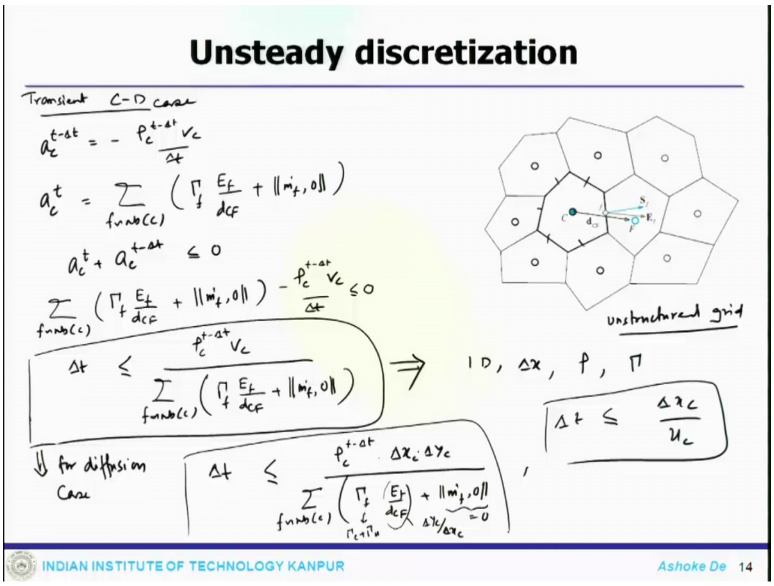

So, transient convection diffusion problem; so now, how one can go about it; let us consider this stencil which is in essentially unstructured grid.

So, when we are in the unstructured grid, then we have a cell connected with neighboring elements which are their faces like this and they are connecting with the. Now, here this is a transient convection diffusion problem or case where your coefficients at the from the matrix these are minus rho C V C by delta t a C at current instance is f NB over C gamma f E f by d CF plus m dot f 0. Now, once we substitute this for the stability criteria which is essentially the criteria of a C t plus a C t minus delta t less than 0, we get summation over F NB C gamma f E f by d CF plus m dot f 0 minus rho C t minus delta t V C by delta t less than 0.

So, which leads to the following condition, the delta t must be rho C t minus delta t V C divided by summation F NB C gamma f E f d CF plus m dot f 0. So, that is the criteria one can get for both convection diffusion system; now, this equation the general requirement for the stability of explicit scheme. Now, the condition obtained earlier for pure convection or diffusion is in one dimensional system and the those could be the special case of this multidimensional problem. Now, again for the case of 1 dimensional diffusion with uniform grid with from here let us say from here one can derive those 1 dimensional case with uniform grid of delta x, density is constant rho uniform diffusion coefficients gamma; then, one reduces this criteria for the for diffusion case.

Unsteady diffusion case that delta t less than equals to rho C which is t minus delta t. This delta x delta Y C divided by summation of F NB C which is now one can write this is gamma f which will be now E f by d CF plus m dot f 0. Now, this guy is 0; this guy is essentially lead to delta Y C by delta x C and gamma f is gamma e plus gamma w. Now so, if the flow moves from left to right, this delta t criteria becomes del x C by u C that is for convection. So, one case you can find out pure diffusion; other case you can find out pure convection.

So, the stability constant which is quite stringent and very restrictive as it forces the use of extremely small cell size and the grid size in the explicit transient formulation. So, when we look at the implicit transient formulation in the latter half of the lecture, we can see that restrictions actually gets relaxed quite a bit. Now, in system with backward Euler.

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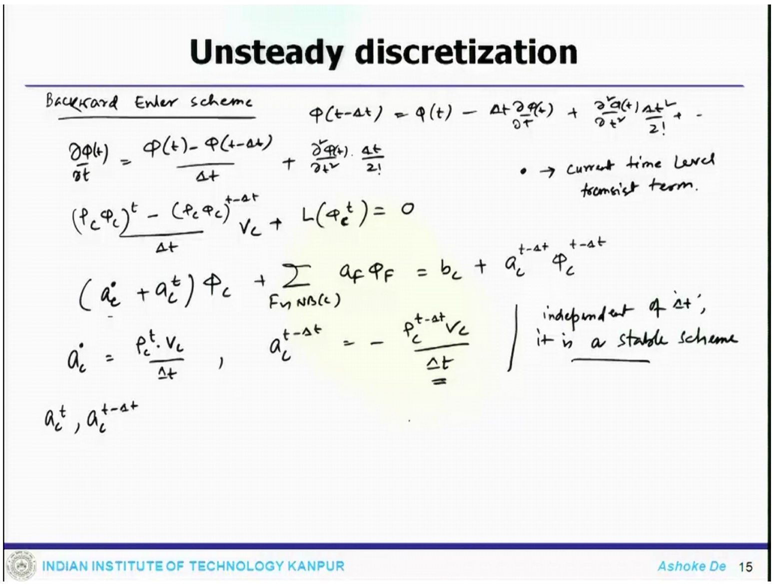

Backward Euler is sort of a implicit scheme. Now again what we can write our any variable t minus delta t is written as t minus del t plus del 2 phi by del 2 delta T square by factorial 2 and so on. So, we can find out the first derivative of these like phi t minus phi t minus delta t divided by delta t which is essentially del 2 phi by del t 2 delta t by 2.

Now, this one can replace in the descritised equations and that it will beings like at t minus rho C t minus delta t by delta t V C plus L phi C at t equals to 0. Now, once we invoke this algebraic relations of the spatial operator and then, complete algebraic transient equation would be this is a C and then a C at the present location, I mean present time step plus phi C. So, this one can define this dot which will represent the transient term at the current time ligand. So, this is the current time level transient term.

Since, we are using same coefficients a C for the space and the time, we will just make it separate like that. So, NB C a F phi F b C t minus delta t phi C t minus delta t; where, the coefficients for the transient term is rho C t V C divided by delta t and a C t minus delta t is minus rho C t minus delta t V C by del t term. So, that is how we can actually get and you can use the field operator to get the nu. Now here, that at the time level t and the time level t minus delta t are also of opposite sign which guarantees that phi C is bounded by this spatial neighbors at the current time step t and C is temporal neighbors from the previous time step t minus delta t.

So, this implies the scheme is stable independent of the time step used. So, this is what the advantage of backward Euler, it is independent of delta t; it is a stable scheme. So, that is the important message. Now, one can actually I mean this is not an ideal scheme as it is also low order and solution obtained with the schemes are of the lower order of accuracy, unless very small time step that is like delta t is used.

So, which is also very restrictive condition because then your calculation would be marching at a slower space compared to; if you add up to a large time step for computational efficiency result would be very erroneous. So, there could be 1 more scheme which can be discussed which is called Crank Nicolson Scheme.

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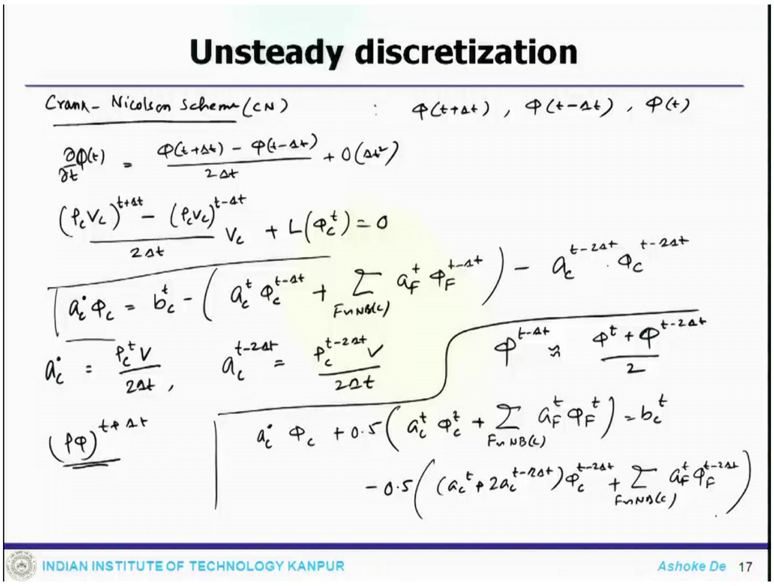

So, the Crank Nicolson Scheme, Crank Nicolson Scheme what it uses it uses the information of t plus delta t; uses the information of t minus delta t and also at the level of t. So, it uses all these information. Now, using Taylor series expansion one can write del phi by del t is phi t plus delta t minus phi t minus delta t divided by 2 delta t. This would be second order accurate now the since the order of accuracy and now if you substitute in the say equation this will give you rho V C t plus delta t minus rho C V C t minus delta t divided by 2 delta t V C plus L phi C equals to 0.

Then, once we invoke the spatial operator. So, we can write that complete system like a C phi C b C minus a C t phi C t minus delta t plus F a F t phi F t minus delta t a C t minus 2 delta t phi C t minus 2 delta t. So, it uses the information at the current time level previous time level and 1 level earlier, where a C this is rho C by 2 delta t a C t minus 2 delta t is rho C t minus 2 delta t V divided by 2 delta t. So, this scheme what is uses that at the t plus rho phi at t plus delta t which can be performed using the old values.

However, the old values are readed from the 2 levels before this one and accordingly it will modify the system. Now one more thing which is required that analysis of the stability of this particular scheme Crank Nicolson Scheme. So, this can be performed after slightly modifying the equation. In that case what modification is done is that which says phi at t minus delta t is approximated as phi t plus phi t minus 2 delta t divided by 2. Then, the complete algebraic equation this one brings down to a C phi C point 5 a C t phi C t plus summation of F NB C a phi F all at the tth level equals to b C t minus 0.5, where we write a C t plus 2 a C t minus 2 delta t phi C t minus 2 delta t plus NB C a F t phi F t minus 2 delta t.

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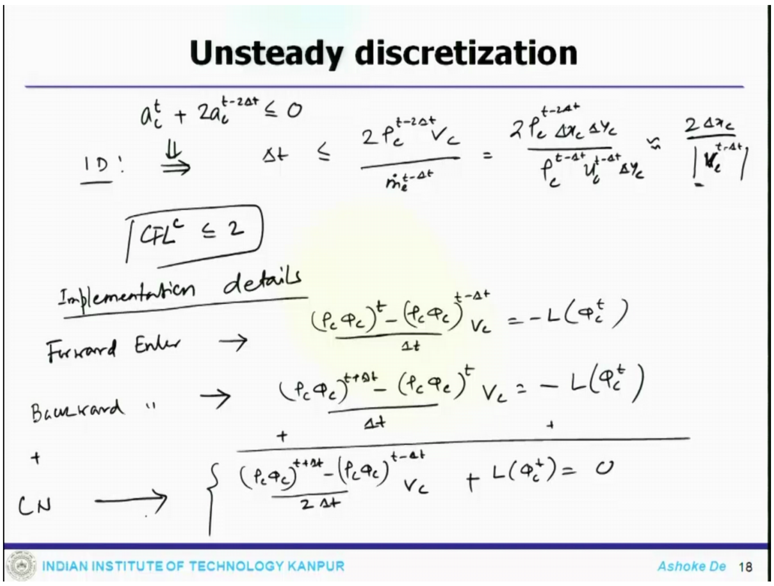

So, that is how you can get and the which becomes a C t plus 2 a C t minus 2 delta t less than equals to 0. Again if you go back to one dimensional system, then this equation provides you the criteria that delta t less than equals to 2 rho C t minus 2 delta V C by m dot e t minus delta t which is 2 rho C t minus delta t del x C del Y C divided by rho C t minus delta t u C t minus delta t delta Y C which is nothing but 2 delta x C by magnitude of t minus delta t. So, which has been assumed that the advection term is descritised using the upwind scheme.

Now, using the CFL number for the convection defined by these expression, the criteria for the convection would be less than 2. So, larger the CFL limitation is very much helpful in the computing point of view, but the improved accuracy is just more important as it allows for accurate solution which can be achieved without the need to resolved to very small time step. So, specially that the second order derivative is now estimated from the error. So, now, we can see the accuracy security issues now.

Now, implementation if you look at these details. So, the implementation details because end of the day one has to think about from the programming point of view. Now what happens at the forward Euler where you get your rho C phi C at t minus rho C phi C t minus delta t by delta t V C equals to minus L phi C t. Now, Backward Euler you get rho C phi C t plus delta t minus rho C phi C t divided by delta t V C minus L phi C at t. Now if you add this two, Forward plus Backward Euler, what it does? It actually add these term and now, you add these term which will get you rho C phi C t plus delta t. This guy gets cancelled minus rho C phi C t minus delta t by 2 delta t into V C plus L phi C t equals to 0 which is essentially Crank Nicolson Scheme.

So, this points to a simple implementation of the Crank Nicolson Scheme within a implicit scheme framework it is a 2 step procedure. So, in the first step 1 can do the back Euler formulation. So, to use implicitly find the value. So, this and the second step the value at time step t plus delta t can be found.

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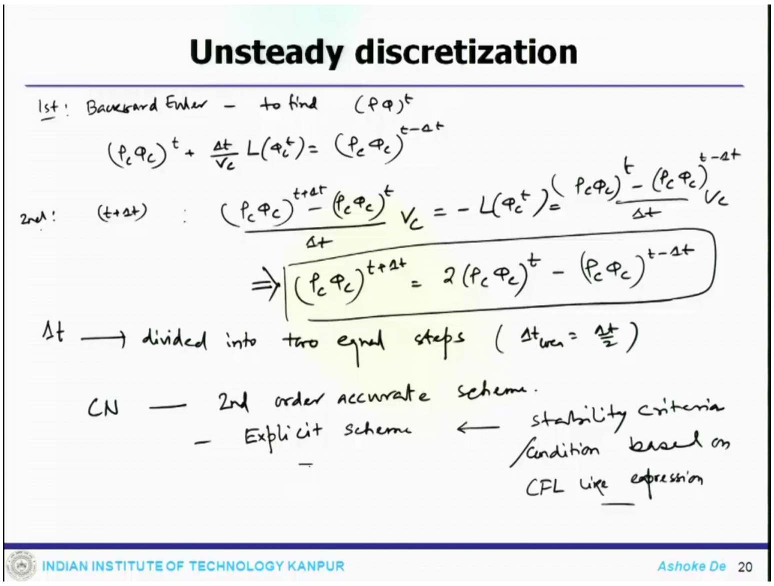

So, at the first step it is the Backward Euler which will get you to find rho phi t from rho C phi C t plus delta t by V C L phi C t equals to t minus delta t. Now the second step find the value at t plus delta t like rho C phi C t plus delta t minus rho C phi C t divided by del t V C minus L phi C t which is which will get you that rho C phi C t minus rho C phi C t minus delta t by delta t V c.

So, that is rho C phi C t plus delta t equals to 2 rho C phi C t minus rho C phi C t minus delta t. So, in these calculation one thing which is assumed as that delta t is divided into 2 equal steps which is delta t local is delta t by 2. Now, one can note here is that the Crank Nicolson Scheme is second order accurate scheme. So, it is an also still it is an explicit scheme, also it is an explicit scheme which has some stability criteria or condition based on CFL like expression. So, this also has the limit of the choosing delta t.

Like the Backward Euler or Forward Euler, they are very much restrictive using our CFL criteria, similarly Crank Nicolson also has certain restriction using CFL like criteria. But one thing which one can I mean note here since you achieve this calculation in 2 equal steps. So, which is essentially provides slightly better stable system compared to Backward Euler or Forward Euler. So, this is a and top of that it is a second order accurate scheme. So, one can note here that the Crank Nicolson is a better option compared to Forward or Backward Euler and it provides higher order accuracy even in the temporal description and also stable and we look at other scheme in the next lecture.

Thank you.