Introduction to Finite Volume Methods-I
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Lecture – 13

So, far we have discussed about the modeling part and the discretization part, derivative parts and now we will move towards for the finite volume formulation.

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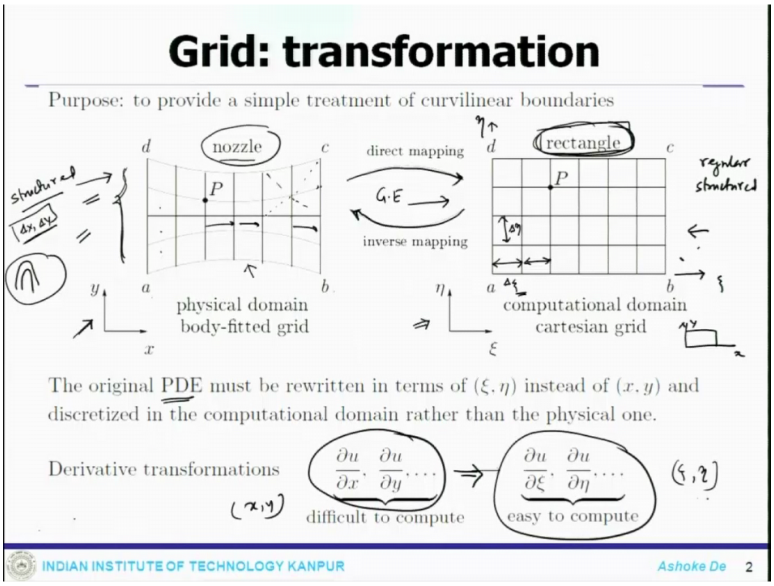

So, before moving ahead to that we will start with the one more important topic that I would like to touch upon is the grid transformation. So, what it happens that that involves also some sort of a derivatives and then how the derivatives are transformed and this would be very very I would say healthy or for doing some complicated geometry calculations. I mean it is not necessarily that always you would be able to that as we have discussed, you can always use the structure grid. I mean sometimes you use unstructured grid these are options, but one more option is that still you can be on the structured system, but then you need to do the transformation.

So, what it essentially does? That the whole purpose is that if you look at this geometry, this is a geometry of a nozzle and this is your physical geometry or the realistic geometry and if theoretically if you need to generate the grid around this nozzle then you need to have this sort of a I mean body fitted structured grid. Now what happens that if this happened then the volume of these each element they are not short of uniform. What we mean to say that once you look at this grid the grids spacing like delta x delta y they are no more uniform.

So, far what we have discussed for the Taylor series discretization that we stick to this kind of this thing, but not necessarily you need to have a uniform spacing. You can always device the higher order scheme or the any particular order derivative using a non-uniform grid spacing and that is another important task that one can achieve through the Taylor series expansion.

Now coming back to this nozzle problem, if you look at this particular problem and this is a kind of grid that you can have. Now this grid if you need to have, so, they look some sort of a I mean organized, but they are not very structured in that pressure. So, what happens is that one approach is that you can I mean transform this grid to I mean regular system. So, this is your regular structured system ok.

When you are here actually with the body fitted system, then you are in the curvilinear system. So, one can always argue and do that I can generate some unstructured grid. So, like this I can have some triangles like that and then have it so, that is one option. So, one can always, but this is one of the technique which is very handy for complicated geometry where people can still because depending on the grid you need to have your solver or your code to be devised whether it is a structure solver because all your discretization should be on the structured mesh or it could be un structure. As we move along with the course we can see how that makes the difference, but as of now let us say you have this kind of a body fitted system which is in the curvilinear system.

Now, I want to map this to a rectangular system. So, this is your global coordinate system and x and y and these are the end points a b c d and once you transform them like a direct mapping, so, from here to here if you transform them what happens; you transform them to a regular system. Now from my x y reference frame, now I have move to xi eta reference frame and the important thing is that now this a b this line or this plane actually this is the if you look at this plane this is a curve plane.

Now this curve plane has been now mapped to a regular rectangular plane. Similarly, if you see d c; d c has been also move to a regular orthogonal plane. Now along this a b lies in the xi direction, a d lies in the eta direction and importantly these are now uniform spacing and like this would be delta xi, this would be delta eta.

So, what happens? So, it helps you to transform a curvilinear system like this kind of geometry; it could be cylinder, it could be some other u bend kind of stuff anything like you have this kind of nature in a complicated system, you can always convert this system to this rectangular system. Now, physically what you are going to do? You are going to solve your problem in this particular domain; that means, now you have a nicely if you look at this particular system the map system this map system you have a nice rectangular cell and they are uniformly spaced by delta xi and delta eta and the whole system of equations are solved and when you solve this equation then finally, you can always do the inverse mapping and get back the things on the particular elements.

Now one may immediately ask when I do that when I come back to this map system my delta xi delta eta are now uniform, but if you look at the physical problem here the distance and here the distance or here the distance they are not uniform. So, how that helps? This is one of the beauty of this mapping. Since you do not have the regular uniformity here, one can have a solver completely based on this kind of system that is call unstructured solver you can always have that and that would be one of our point of discussions that will as we go along with the course we will discuss that.

But other option is that I can still stick to a structured system in that case since there is a non uniformity in the physical problem I will map that and now once I map it to this rectangular system the map system looks completely uniform and very nicely situated. So, it is quite similar if you look at a Cartesian system like your x y. But now only thing is that. So what happened once you transform that? This can transform the grid points like the nodal points you map it to here and you have some information of xi and eta. So, there will be some correlation, but important thing is that now since we well be solving our equation in this transform plane are the same set of equation can be solved? It is not true.

Now what you need to do? Your partial differential equation or the governing equations they need to also transform to this plane. And how you do that? Now your all these PDE's are essentially they are consist I mean consist of some sort of derivatives whether the first derivative or the second derivative. So, you need to now transform this derivatives which are the core ingredient of your PDE's or the governing equations to this system.

That say that xi eta system. So, this is in my x y system. Now once I transform them I will transform them to xi eta system. So, not only the transformation of grid is required I have to now map my equation system to this particular system through the map system and while doing that essentially I need to transform my derivative to this particular system and once I do that I can solve my problem in this map domain and then do the inverse mapping to get back to the physical problem of interest.

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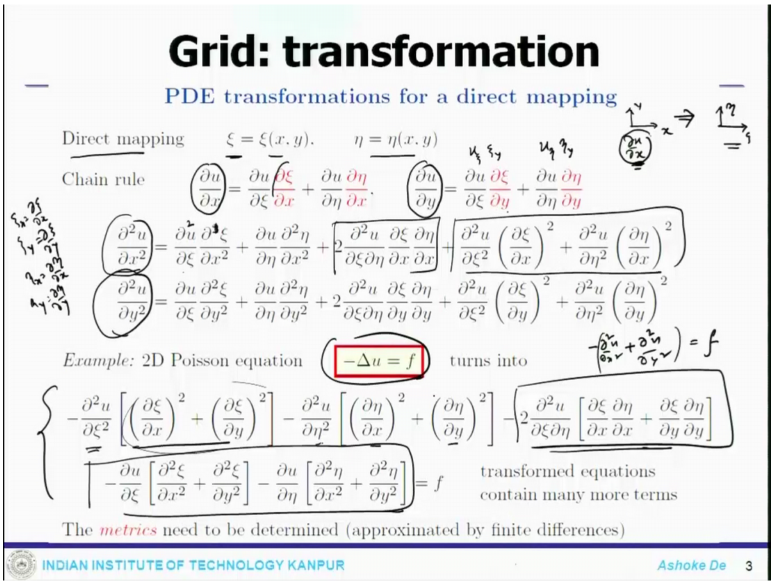

So, how do you do that? Now, so, how you do the direct mapping? So, you have xi and eta. So xi essentially in the xi coordination system; so what you have done? You have a x and y; from there you have map to xi and eta. So, xi is a function of x and y eta is also a function of x and y. So, you use the famous chain rule. So, if I need find out del u by del x.

So, del u by del x now there will be two component; one is del u by del xi del xi by del x and second component is del u by del eta del eta by del x. So, del u by del x is sitting here. Now, I am trying to find out actually what is going to happened to del u by del x when I transform the system to the xi eta system. Similarly, del u by del y it is also a function of xi eta. So, del u by del xi and del xi by del y. So, essentially one can say u xi and xi y u eta y so, these are some notation one can use.

So, you transform, now if you see I am trying to find out a derivative del u by del x at my physical domain, but that now consists of some derivative term coming from the xi and eta and if you look at this term this xi x which is nothing, but del xi by del x or xi y which is nothing, but del xi by del y eta x which is del eta by del x and eta y del eta by del y. Now these terms are the derivative of xi and eta with respect to x and that will retain some sort of a information of my coordinate transformation.

So, similarly if you look at the second derivative if you look at the second derivative del u by del x 2. So, that will have del u by del xi del 2 xi by del x 2 ok, similarly del u by del eta del 2 eta by del x 2 ok. So, this is the way you can; so, there is it should be del xi by del x del 2 u by like that. So, similarly you can find out del 2 u by del y 2 and then you end up getting some cross terms here and here.

So, now for example, if you look at the Poisson equation, so, if you try to transform the Poisson equation which is del u equals to f. So, it is del u by del xi 2 del xi u x square plus del xi y square and del u 2 by del eta 2 del eta by x del eta by y square minus this; these are the cross derivative term which turns out to be there and this is the cross derivative term which also and equals to f. So, this is the equation that you get in a transform system.

Now, if you look at quickly in my x y coordinate system, the equation was very simple ok. This was nothing but del 2 u by del x 2 plus del two u by del y 2 equals to f. Once I transform that I get del 2 u by del xi 2, but sitting with a square term del 2 u by del eta 2 some squared some, but some additional term there and this term do appear because of this transformation and so you need to have some sort of a matrix correlation to find out that calculation.

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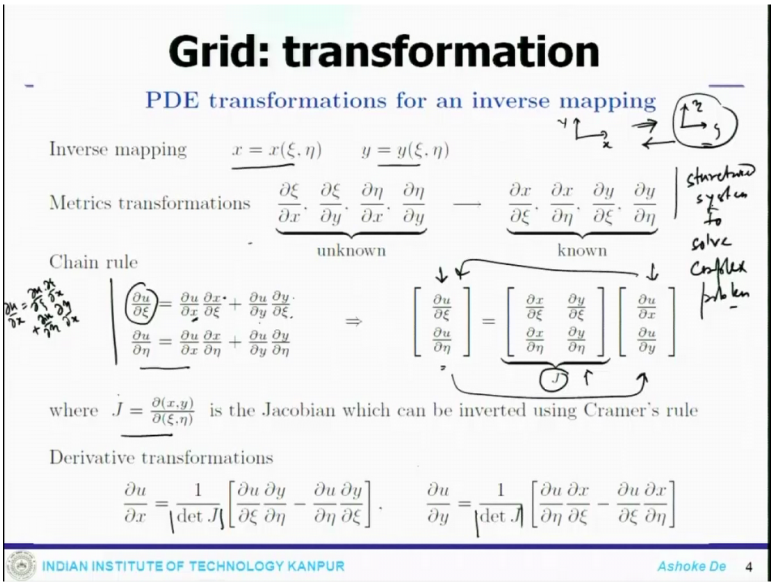

Now, how do you transform that? Now once you transform to the map system you get your solution done and then you do the inverse transformation. How do you do that? Now inverse transformation means I have gone from x and y system to this is the forward transformation xi eta system the inverse transformation will get me back the systems at x y system. So, x is also function of xi and eta y is xi and eta. So, you use your chain rule and find out all these derivatives.

So, for example, if I want to find out del u by del xi. So, that will be del u by del x del x by del xi del u by del y del y by del xi; similarly del u by del eta it just the reverse calculation, previously what we try to find out is that like del u by del x which was del u by del xi del xi by del x plus del u del u by del eta del eta by del x. Now in the reverse case you are trying to find out the derivative at this coordinate system xi eta system. So, del u by del xi that will retain del u by del x and del x by del xi del u by del y del y by del xi, similarly del u by del eta.

Now if you put them in a matrix del u by del xi and del u by del eta here you get this; this is called the jacobian of transformation and del u x by del y. So, jacobian is nothing but the determinant of this particular matrix and if you use that you can always have if you look at this is the derivative in the transform system and this is the derivative in my physical system and you have a nice correlation through this jacobian, always you can go back from this side to this side and this side to this side. One case it is multiplied with the jacobian, other case it is the inverse of the jacobian. So, for example, if you look at del u by del x, so the jacobian comes here and you get this term and del u by del y the jacobian comes here and you get this term.

So, it is a very nice correlation. So, you can always have the second derivative and other derivatives and one of the I mean beauty of this particular system is that you can have your code still based on your structured mesh, but you can get an input from non-conformal grid; that means, non-orthogonal non-uniform system and you can transform them to xi eta system and then transform your governing equation in that solve it get back to.

So, that allows you to still stick to the xi eta system I mean structured system, but the typically the structured system to solve complicated or complex problem ok. In very often we say that that the structure solver is not good and as we keep moving along the lectures we will see they are not always good, but this is one approach which can be adopted for the structured system and one can still stick to that.

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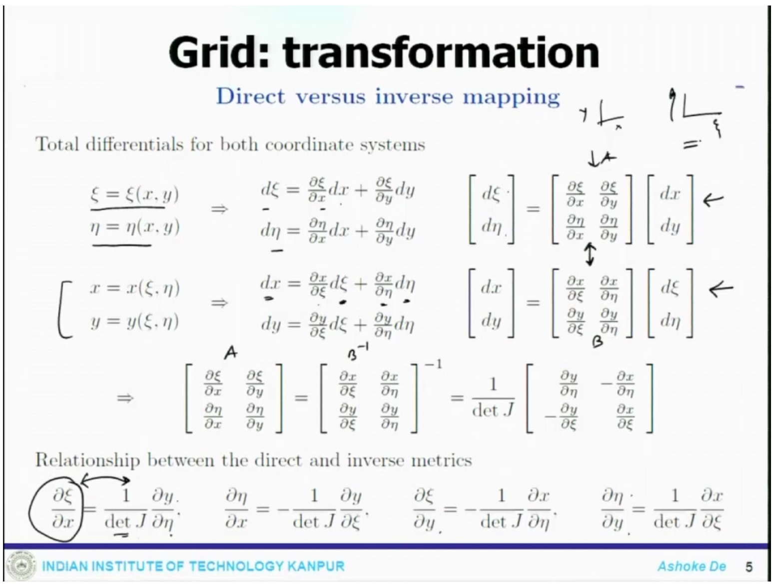

Now, so, finally, if you put down everything together what we have discussed so far. So, we have xi is a function of x y. So, d xi is if I look at using the chain rule d xi is del xi by del x dx eta is x y function of x y, so d eta I can write. So, d xi and d eta with del xi by del x and del eta that is one system inverse mapping dx is del dx del x by del xi into del xi del x by del eta del eta and this is the inverse one.

But if you look at the similarity of these two matrices one is inverse of another. So, essentially if I write down the first matrix A and this is the B, then A is B inverse or essentially one I can find out this. So, if you look at that the relationship between that now you can find out del xi by del x equals to 1 by determinant of del y by del eta. So, you get so, similarly you get for del eta by del x you get for del xi by del y you get for del eta by del y.

Now if you look at that now you get a nice correlation between all the derivative, derivative both in xy plane and also xi eta plane. So, and one can switch from one to another, it just a matter of factor of jacobian. So, this jacobian is called the jacobian of transformation for grid mapping or map system and this is very handy that one can use it and move his complex geometry, the grid to the xi eta rectangular system solve the problem and go back. So, this will be very useful those who want to still have a structured solver for his calculation.

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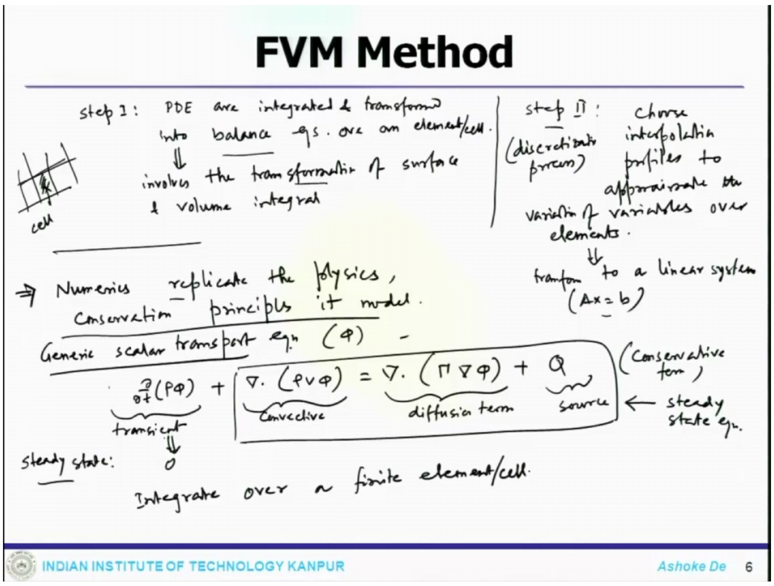

Now, moving ahead we will start with the finite volume discretizations for system and initially we will talk about how you system I mean discretize the system and then you look at the; so, there are couple of steps. One is step 1 what you do and then there will be step 2. So, there and two steps actually you achieve this process. So, first what you do is that all your PDE’s, all your PDE’s are essentially integrated and transformed into some sort of a balance equations over and element or you can call it cell whatever it is.

So, at the first step what you do? You have a set of governing equations or the partial differential equations which are essentially integrated and transform into the balance system. So, what does that involve? That essentially involves the transformation of surface and volume integral ok. So, what happens is that you have set of partial differential equation.

So, you have let us say if you look at a simple system like this; this is your one of the cell or finite element over this particular element. Let us look at it a 2D element like here. So, you integrate the governing equations, once you integrate the governing equations they finally come down to the balance equation over this particular cell and that involves the transformation or conversion of the surface and volume integral. And so, what do you do in the second step? Second step you choose some choose the interpolation profiles to approximate the variables ok. So, one step you first transform your governing equation to the balance equation over a particular cell.

So, essential I have a set of governing equations and I will transform that while doing that I transferred both my surface and volume integral and the second step some interpellation profiles are chosen to approximate the variable ok; approximate the you can put the approximate the variation of variables over elements ok.

So, and this will allow you to transform to a linear system. So, essentially here you get back the linear system on A x equals to b. So, this is what so, second step whole idea is that this is what is your discretization process. So, with this particular process or the discretization process you convert your all these system to the linear system and finally, you solve for it ok.

But so, the one of the very important property one of the important property of this process is that once you transform this system, so your numerics exactly replicate the physics that is one of the very important properly property and also the conservation principle it model.

So, two important things that come along with it is the numerics actually replicate the physics along with the conservation property. So, your governing equations variable that what you actually transform they are sort of conserved ok.

Now, if I now go back to a system of equation or some generic scalar transport equation, so, we will start with that the generic scalar transport equation on phi. So, if you write down that in the conservative form, so, the conservative form would be the term del t rho phi plus delta dot rho v phi equals to delta dot gamma del phi plus some source term, let us say q ok. So, this is in a conservative form ok. So, this term is your essentially the transient term ok, this is your convective term, this is your diffusion term and this is my source term.

Now, if you have a steady state system from here; that means, the transient term actually goes off. So, you get back the system like this much, this is will be the system for my steady state case for steady state system ok. Now, if you integrate the equation now to get the system you have to integrate over a finite element or cell ok.

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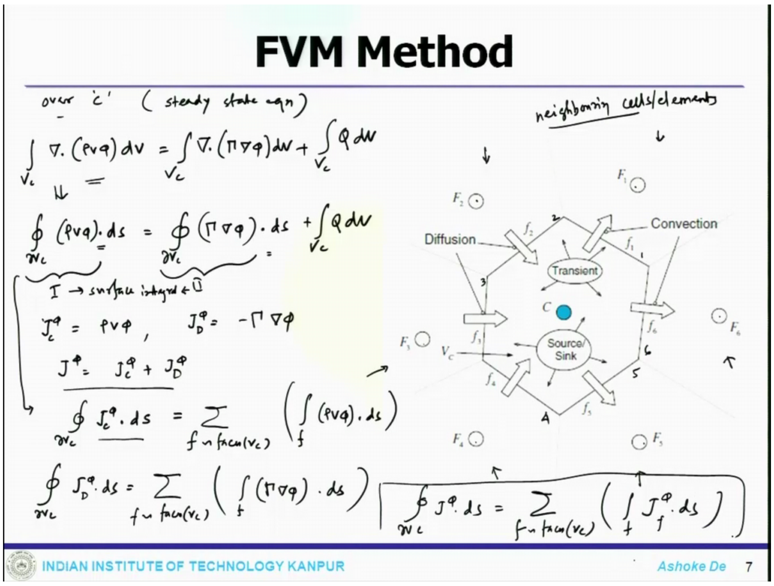

So, let us consider that and if you look at this particular picture let us say we consider one element C and these are the faces marked as 1 2 3 4 5 6 and the neighboring cells are F 1 F 2 F 3 F 4 F 5 F 6. So, the neighboring cells and these are the F 1 F 2 F 2 this marked points are essentially the cell centre of the neighboring cells.

So, these are the neighboring cell or elements so the element C which is actually surrounded by all these 6 elements. Now if you integrate over C the equation. So, what you write the lets say look at the study equation first, steady state equation. So, if you integrate over C what do you get? You get the volume integral over C del dot rho v phi dv over V c. So, del dot gamma del phi dv plus V c q dv. So, this is my, if you look at the steady state equation you have convective term, diffusion term, source term. So, you integrate over this equation over the cell C.

So, once you do that this particular case now you use the divergence theorem and convert them to surface integral ok. So, now, rho v phi dot V s equals to gamma phi dot ds plus this remains as a volume integral. So, now, these 2 terms 1 and 2 these are now essentially surface integral. So, what you have done by doing that integration? You actually converted the using the divergence theorem. This is purely the application of I mean algebra, so, vector algebra you just convert them to surface integral and here is a dot product which is sitting there once you transform them.

So, now you would comes to the calculation of the fluxes. So, the convective fluxes if you calculate this would be rho v phi and the diffusion fluxes would be gamma delta phi ok. So, if you now integrate these fluxes; the total flux would be if I write that total flux would be convective fluxes plus diffusive fluxes. So, that is the total component of the fluxes.

Now, about this particular cell C, if I actually integrate this fluxes from this equation my first term would become del v C j. So, that C dot d s would now become summation f faces of that volume; that means, here it will go from 1 2 3 4 5 6 to the integration of that rho v phi dot ds ok. So, the term 1 which is the surface integral of convective fluxes that will become like a; if you look at this here you had integral here you get back an algebraic systems.

Similarly, if you convert the diffusion fluxes so, this will also become f faces f gamma del phi dot ds. So, that also get transform and the source term would be the volume integration of the source term that show, if I look at that essentially f. So, this is the expression for my total flux over that element surfaces ok.

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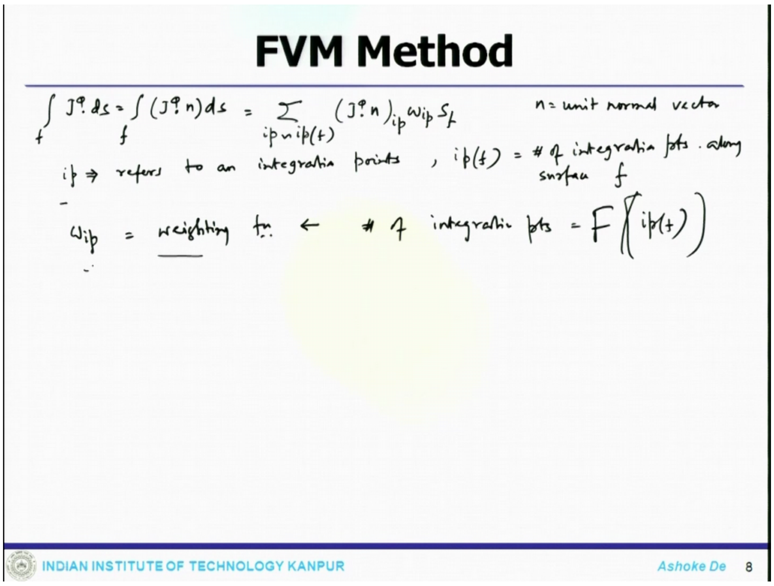

Now, if I write that that J phi dot ds over faces, so, this would J phi dot nds; n is the unit normal vector then this will become my i p; i p of faces J phi dot n i p w i p s f ok. Now i p here refers to an integration points and i p f; i p f is the number of integration points along surface f ok. So, you got integration points and the number of integration point ok. So what you can do? So, and depending on the situation how many number of points are there? So, you can always find out the complete integral and also you need this w i p. So, w i p is the waiting function and it primarily depends on number of integration points; that means, essentially this becomes a function of IPL.

So, as I have different kind of number of integration points the waiting function or the factor is going to be different ok. So, we will stop here today and will take from here in the follow up lectures.

Thank you.