**Introduction to Finite Volume Methods-I**
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**Lecture – 10**

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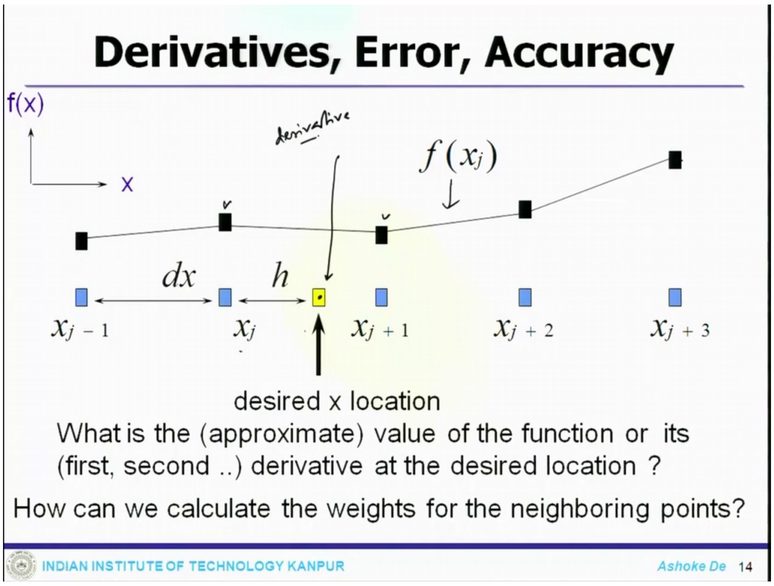


Welcome to the lectures of this Finite Volume Method. Now, if you move forward and try to find out the point, let us say I have these kind of systems which are not so, called looking like an uniform system. I have a point here, I have a point here, here and here. These are the function which actually represents point and distance between this is dx. And I try to find out a position which is exact location from this point to this edge distance and I try to find out the derivative at this location.

So, how do I calculate? Because this is does these particular point that yellow mark point it does not belong to neither of this nodal point. So, it is somewhere between this point and this point. So, I have to get some sort of a weighting function or weights which can actually represent properly to find out the derivative at that point ok. Now how do you do that?

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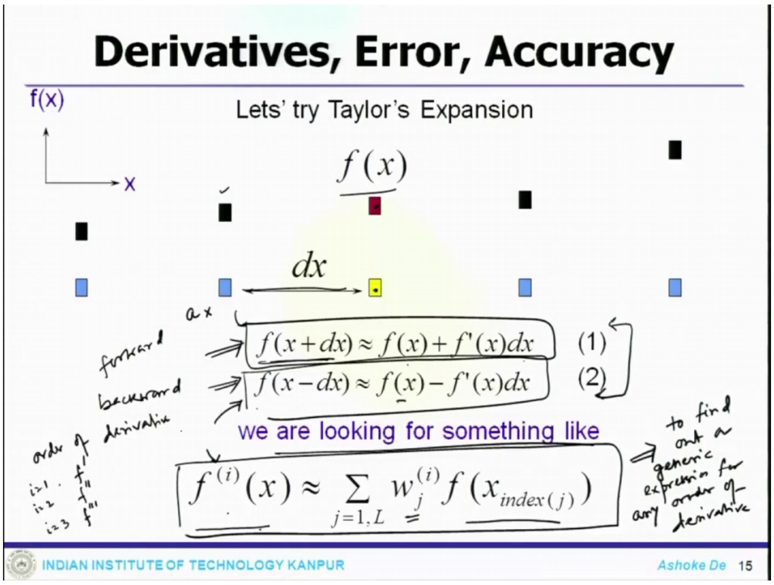


So, try the Taylor series expansion ok. So, we try to find out fx. Now I have a point here, and I have a point which I am interested here. So, this is a distance d x. So, I write x plus dx approximated as fx plus f prime x and d x ok. So, this is the point of my interest. And then I write so, this essentially write the forward formula ok. And the second one you go back to this is the point where fx is defined. So, this is actually the backward formula ok. And this is approximated as fx minus f prime x dx. So, you get back these 2 system.

Now, what are we looking for? We are looking for something like, we want to this i stands for the derivative order of derivative; that means, if i is one; that means, I am trying f prime if i is 2, I am trying f double prime. If i is 3 I am trying for f triple prime. So, it actually tells you the particular derivative that you are interested in with some sort of a weight function. And then these index function, where the fx is defined.

So, you want to write something like that, but in a more generic form. Once you can come across with this then you can divides any derivative with any sort of point. So, essentially this is very useful to find out a generic expression for any order of derivative. So, whether I want first order derivative, second order derivative, and what is the order of accuracy I want everything can be kind of calculated for.

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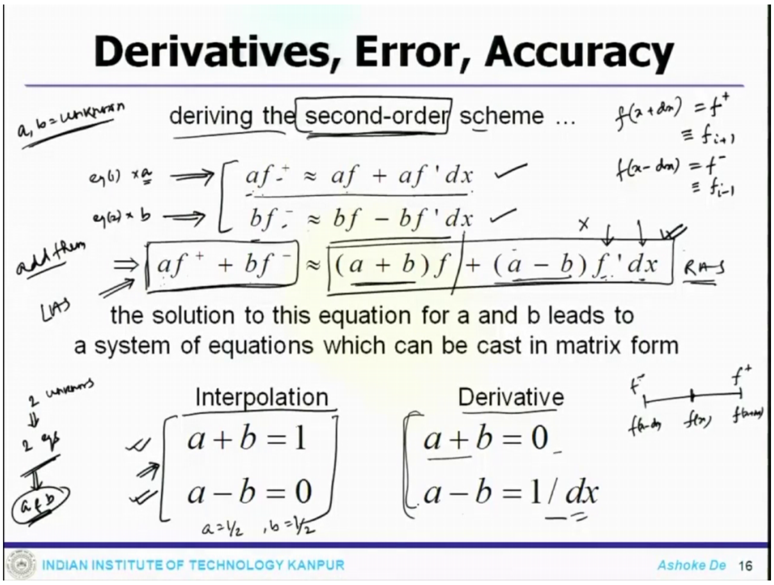


Now, you put some sort of an algebraic system. So, I want to derive a second order system ok. So, my numerical system should be second order activate. When we talk second order scheme. So, these are all sort of a synonym when you say that the second order scheme these are second order accurate system. So, when you say that second order accurate system. So, you get so, if you go back here and look at this expression in equation 1 and 2 and write down a f plus equals to af plus af prime dx. So, which is this expression f x plus dx. So, x plus dx is nothing but represented as f plus or f i plus 1. Similarly, if fx minus dx is f minus which is equivalent to f i minus 1.

So, first case this is equation 1 multiply to with weighting factor a. So, a is unknown here. So, a b these are unknown and what we are trying here we are trying to find out, a and b in a systematic way; so, that you can get an expression which is second order accurate ok. To in order to find that you multiplied the system with a first system; so, you get expression like this you multiplied the second system with b you get a system like this. So, if you compare my x plus dx is approximated like this x minus dx is approximated like this. So, this guy is multiplied with a this, guy is multiplied with b once you multiplied you write af plus approximated as af plus af prime dx number 1, number 2 bf minus approximated as bf bf prime dx. So, you get back the 2 more system.

So, essential equation one and 2 multiplied with a and b get back to this system. Once you get back these you add up. So, you add them together. If you have them left hand side you get this expression, af plus plus bf minus which is equivalent or approximated as a plus bf and a minus b a prime dx. So, this is what you get at the right hand side ok. And this is what you get at the left hand side. Now, if you see you have unknown 2 unknowns. And you got only ones equation. Now you have to solve a and b to find out the solution. So, how do you find out the solution? Because once I find out a and b that will lead to the particular derivative. So, how do you find out the solution? Because now, from these particular expression I can actually devise system of equations; where so, essentially what I need I have 2 unknowns. So, I need 2 equations. Then only I can solve for a and b.

So, from this particular expression you can devise that. When I said you say that I only want the interpolation, I do not want any derivative to be present; that means, these functions f plus and f minus if I want to only calculate the interpolation then I need to get interpolation between f plus and f minus. So, that will have some values of a and b. Now some values of a and b only get me interpolations. So, that will retain. So, if you look at here is my function fx here is my function x plus d x and here is my function x minus d x. So, this is my f plus this is my f minus.

So, what I want I want that the function to be interpolated at this point. And I have expression like this or I have an expression in hand then f plus f minus can be retained f can be retained. We do not need to have this derivative term to be present. So, when I only interested in interpellation, I do not need this term. So, theoretically f prime cannot be 0 because this is having a finite value dx does have some final value. So, the term which can be taken to 0 is a minus b. So, that leads to the one equation. Secondly, I am trying to get an interpolation. So, this should be 1; that means, in the interpolation of these functions I will retain the left inside which would be also retuning the this portion no derivative. So, that leads to 2 equation.

So, if I solve these 2 equations, I will get an value for a see one can quickly find out the value for a would be have and b would be also have. So, if I put half and half and half and half. So, I get an interpolation value I do not have any derivative exists. Now if I want the derivative to be present; that means, I want let say f prime. So, the derivative to be present here I, what I need? I do not need these interpolated values. So, a plus b goes to 0 and a minus b that becomes 1 by dx. Why? Because I want only f prime to be present f prime to be present here I have to get rid of completely this term along with this dx. So, no matter what value a minus b has it should be divided by dx. So, these dx gets cancelled, out you get an system of equation.

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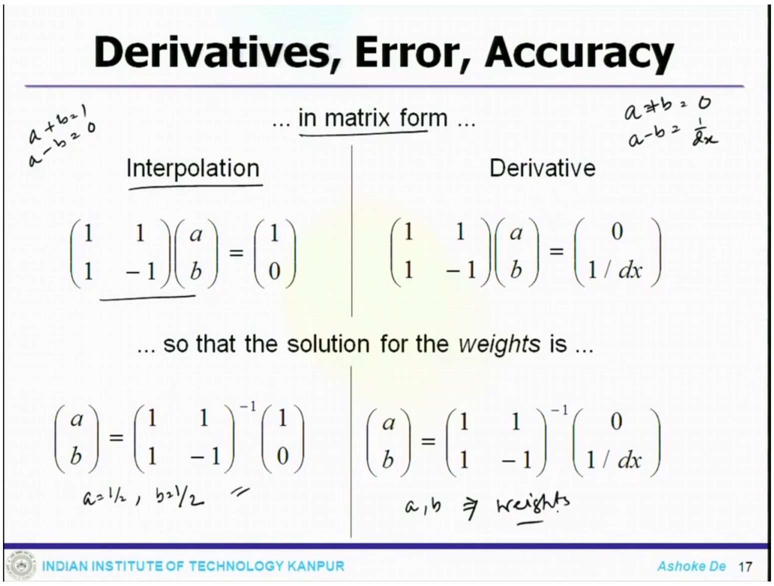


From here if you solve you get the weights. Now if you write in a matrix form for the interpolation where only I am interested show equation that I had in the interpolation a plus b equals to 1 and a minus b equals to 0. So, I write 1 1 1 minus 1 ab 1 0. So, this is the matrix if you solve you get the system. So, theoretically a should be half and b should be half. One can solve either these matrix or one can solve and other case a plus b equals to 0 a minus b is 1 by dx. So, if we put them together in the matrix 1 1 0 1 minus 1 1 by dx if you solve that. So, here if you look at this a b these are nothing but the weights that is used to find out the derivative.

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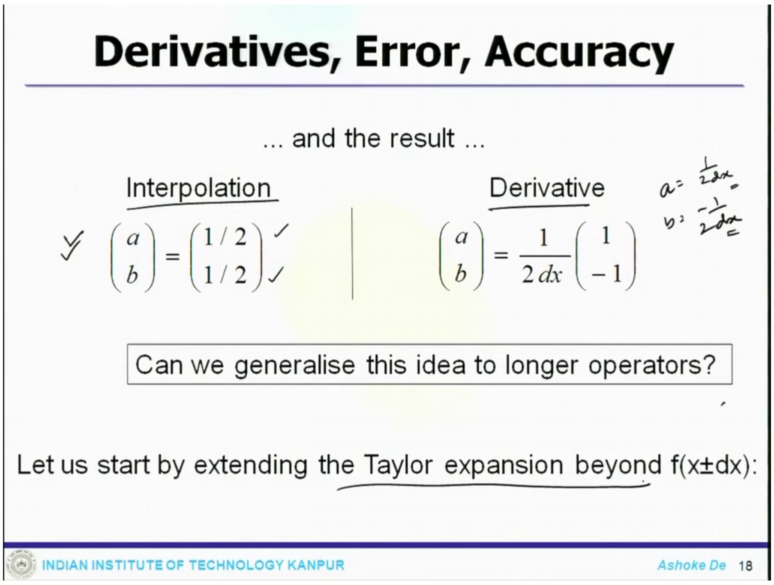


Now if you calculate that and the result as I said for interpolation, a and b would be half and half and this case a and b would be. So, here a is 1 by 2 dx b is minus 1 by 2 dx. So, essentially if you look at these 2, the derivative and the interpolation there is a nice similarity one case you get half half other case the half half is setting there for 1 by dx is also setting there.

So now, the question is that can you generalize this idea this weighting idea, and use it for the other operators or maybe finding trying to find out the higher order derivative. So, certainly we can how do you do that again the underline story, or the theory which remain same is the Taylor series expression, but we have to now expand the Taylor series expansion for other term.

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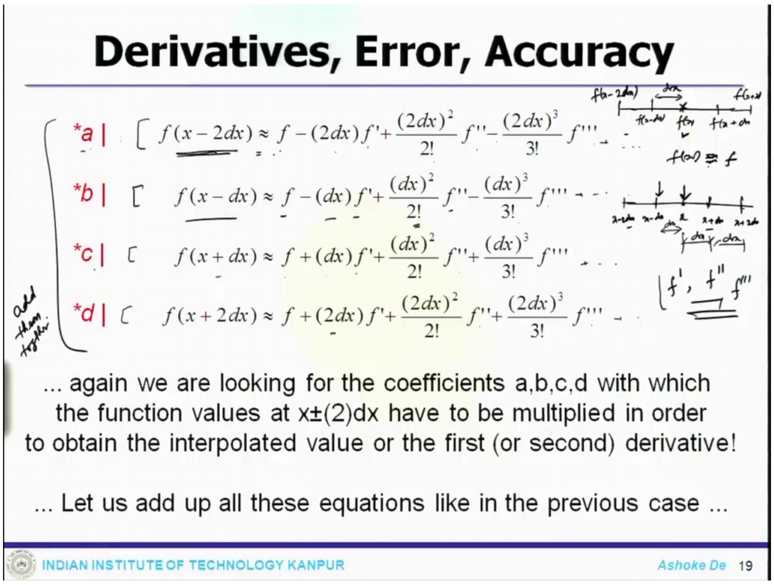


And then finding that how one can do that? Let us look at it one after another. So, again we will concentrate on a system like this. Simple system where fx is defined, you have x plus dx you have x minus dx. And this distance are given as dx. You can also have one point ahead it x plus 2 dx ok. So, first thing that we want to find out x minus 2 dx; that means, the point which is this is my point of interest.

The point which is staying here x minus 2 d x and these distance between each point they are uniform and dx. So, x minus 2 d x again I am approximating using the function at fx ok. So, fx which is written as f fx minus 2 dx f prime plus 2 d x square factorial to f double prime minus 2 d x cube triple prime and so on. Ok this particular equation you multiplied with a that is my first equation.

So, here what we are trying to write here is that we are trying to generalize this idea for higher derivative. And to do that let us consider these 5 points. One is x x plus dx x plus 2 d x x minus dx x minus 2 dx ok. Now, first find f minus 2 dx then we will find out x minus dx; that means, using the value at x I will try to find out the function value at x minus dx again f minus dx f prime dx square factorial 2 f double prime minus dx whole cube factorial 3 f triple prime and so on.

Now, these are the points now this guy is multiplied with b. Now you look at the point which is ahead of it; that means, x plus dx. So, plus dx would be f plus dx f prime dx square by factorial 2 f double prime dx cube by factorial 3 f triple prime and so on. And the last one this guy is multiplied with c and the last term which you want to evaluate at x plus 2 x. So, x plus 2 x is f plus 2 dx f prime 2 dx square factorial 2 f double prime 2 dx whole cube factorial 3 f triple prime and so on and this one is multiplied with d. So, essentially I want to find out the derivative and I have adopted points 2 points behind that particular point and 2 points ahead of that. And all of them are default by a distance dx from the preceding one. So, essentially the distance between the points, are uniform and they are given as dx.

And now, each of this equation first one multiplied with a second one b third one c, last one d. And what we are trying to find out? We try to find out any derivative it could be first derivative it could be second derivative and you can also find out third derivative. So, you can find out, but let us see what happens to first derivative or the second derivative. And as we have done earlier you add of this thing together. So, all these equations multiplied with a second one b third one c and d after multiplication you add them together ok.

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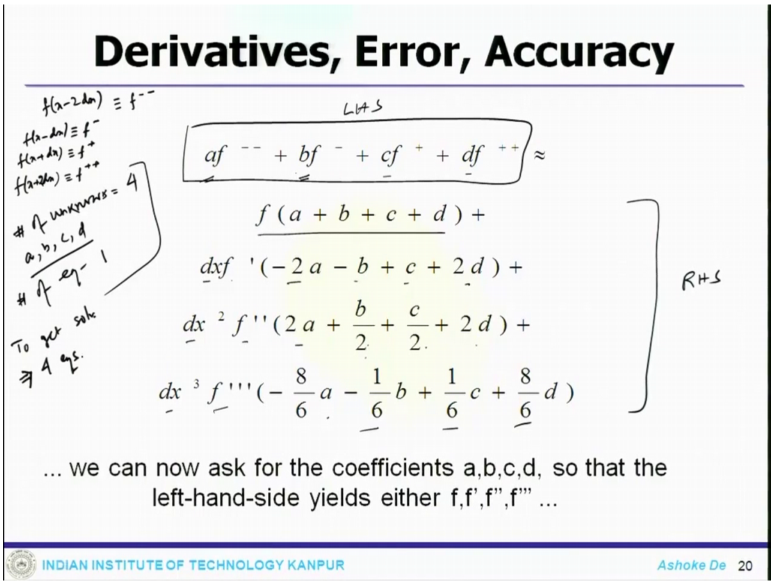


So, what it gives you back? It is a very nice expression. So, here x minus 2 dx which is equivalent to f minus minus that is what you see here a multiplied with f minus minus, x minus dx equivalent to f minus second equation multiplied with bf minus x plus dx equivalent to a plus and f x plus dx 2 dx equivalent to f plus plus. So, that is here. So, if you just have this essentially f minus minus this is f minus, this is f plus this is f plus plus f minus minus multiplied with a first term and second term equation multiplied with b that is the this equation third equation multiplied with c that is ca plus 4th equation is multiplied with d that is f double plus.

Now, once you collect the term together now you see if you multiplied with a you get a af here, and we are adding them together after multiplication you are adding them together. So, you get here af a into 2 dx f prime into like that. So, af bf cf df. So, you collect them together. So, this is my left hand side and this is completely my right hand side and right hand side. What do you get if you collect them together; this is what you get a b c d. Now you collect all the term corresponding to f prime. So, f prime first equation is multiplied with a so much 2 d x second is b minus 2 dx third is c plus d x last one d 2 dx. So, you collect dx f prime outside minus 2 x minus b plus c plus 2 d.

Now, similarly you look at the second third term second derivative. So, that is having 2 dx per by factorial 2. All of them this one multiplied with a this one b c d. You collect them together dx square f double prime, 2 a the reason is that you get 4 here divided by 2. So, 2 a plus b by 2 here you get dx square which is outside 1 by factorial 2. So, b by 2 third case also c by 2, 4th case 4 by 2 2; so, you get 2 a b by 2 c by 2, 2 d.

Now, remains the third derivative here. 2 cube is 8 divided by factorial 3. So, there will be a factorial dx cube, here dx cube multiplied with b here dx cube multiplied with c here 2 dx cube d. You collect dx cube f triple prime minus 8 by 6 a that comes from this particular term with minus sign second term b by 6 that comes from this term, third term is c by 6 comes from this term last term is 8 by 6 d. Now you get a system with all a b c d now if you look at this particular system you have a left hand side right hand side. And your number of unknowns number of unknowns here 4 a b c d, but equation you have.

So, number of equation is one, just like previous case previous case. We are 2 unknowns one equation, but we have devised 2 different system and solve for a and b similarly here we have 4 unknowns. So, at least to get solution we need 4 equations. So, how do we derive those 4 equation? As we have done earlier we can find out these coefficients by mapping the coefficients from one side to another.

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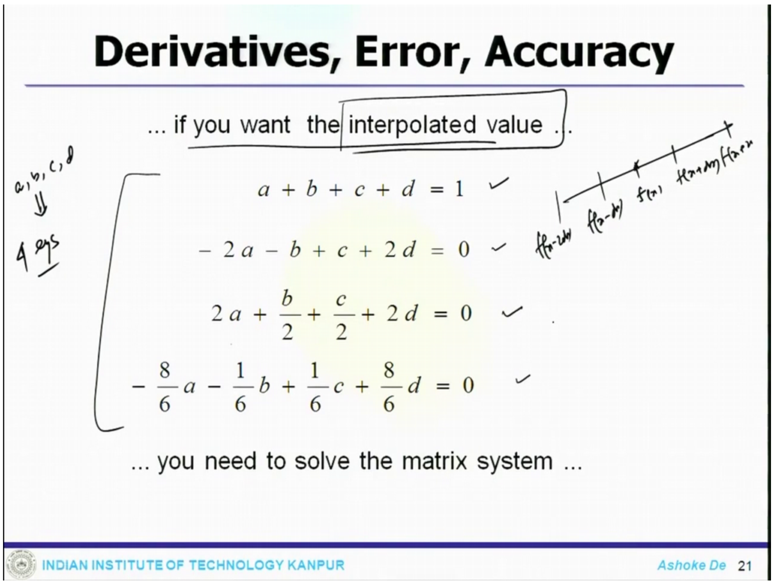


Now, as we have done first we are interested in the interpolated value; that means, all these points, fx x plus dx x plus 2 dx, x minus dx x minus 2 dx. I want everything to be interpolated. If that is the case left hand side will be there and right hand side this guy a plus b plus c plus d should be 1 because, I want only interpolated value. Any other derivatives which belongs to this right hand side they must be 0; that means, a plus b plus c plus d should be 1 minus 2 a which is connected with the first derivative minus 2 a minus b plus c plus 2 d they must be 0 that is second.

second derivative is connected with 2 a plus b by 2 c by 2 plus 2 d that is 0, third derivative connected with minus 8 by 6 1 by 6 1 by 6 8 by 6. So, that is also 0. So, you get a system of equations. Now, here if you look at it you got a b c d you got 4 equations. Now, one can solve algebraically one can convert them to a matrix and find out the solution.

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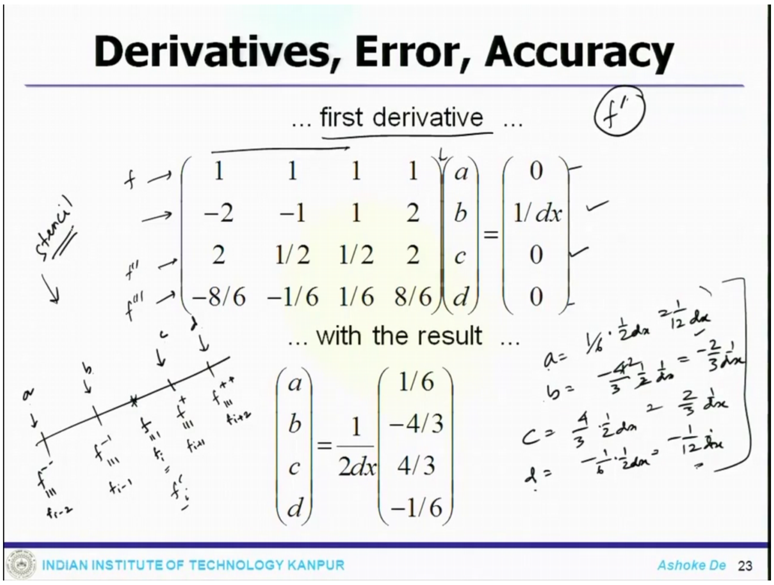


So, again for interpolations if you put them together a b c d 1 so, the first equation corresponds to the first system. This if you put the coefficients a b c d 0 0 0; you solve this matrix you get a equals to minus 1 by 6 b equals to 2 by 3 c equals to 2 by 3 and d equals to minus 1 by 6. So, all this weights if you use here, a b c d that you have got if you use all this derivative terms will vanish. They will disappear only thing you get some interpolated value.

Now, similarly we can get first derivative if you want to find out the first derivative; that means, you are looking for f prime. So, if you look at that here, a plus b plus c plus d. So, this and f prime is connected with this. So, this will be there any term which is connected with f double prime. They would be 0; that means, this term would be 0 this term would be 0 only first derivative sum would be there and this guy would be 0. So, the first derivative a plus b plus c, which is connected with the f that function that is 0.

Term which is connected with f prime that is giving 1 by dx term which is connected with the f double prime that is also 0, term which is connected with f triple prime that is also 0. So, you got a matrix once you solve it you get a equals to 1 by 6 into 1 by 2 dx. Essentially 1 by 12 dx b equals to minus 4 by 3 into 1 by 2 dx; that means, 2 by 3 minus 2 by 3 dx c, equals to 4 by 3 into 2 dx, again 2 by 3 1 by dx d equals to minus 1 by 6; so, 1 by 12 dx.

Now, if you see interestingly the points are like this. This is my f this is f plus f double plus f minus f double minus. Which is equivalent to f i f i plus 1 fi plus 2 f i minus 1 f i minus 2, and if you look at it that the weights, which are multiplied they are uniformly distributed one is a and d they are connected with this point and this point of the system. Because, if you relook back the equation a is multiplied with f double minus d multiplied with f double plus. So, a is multiplied with this.

So, the weight is 1 by 12 dx. And d is 1 by minus 12 dx b and c which is multiplied with this is b this is a this is d this is c. So, if you see the weights a and b are having equal values with different sign b and c they do have same value, but with different sign and you are trying to find out. So, you are trying to find out the derivative of f i and taking 2 points ahead of it and 2 points behind that. Essentially they becomes this is centre derivative. That is why you have equal weights of a and d, but opposite in sign you have equal weights of b and c that is also opposite in sign and then you evaluate the f prime at that location.

So, this is essentially centre derivative involving 5 different points. When you involve so many points this actually leads to the stencil or this is called the numerical stencil that you have to find out the derivative. And we will look at it in the next lecture.

Thank you.