Aircraft Dynamic Stability & Design of Stability Augmentation System
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Module 8
Lecture No 43
Numericals: Mode Shapes

Hello friends. Throughout this course, we have seen about equation of motion for longitudinal motion, equation of motion for lateral directional motion, various modes of longitudinal motion. That is Phugoid and short period. Your lateral directional modes, spiral, Dutch roll and roll. Then we saw about transfer functions and how the transfer function will respond to step input and impulse signal.

We also saw stability augmentation system and how your required Zeta and Omega N will be changed using the stability augmentation system. Now today we will be focusing on mode shapes and what we will be learning? What is the importance of mode shapes and what information do we get from mode shapes?

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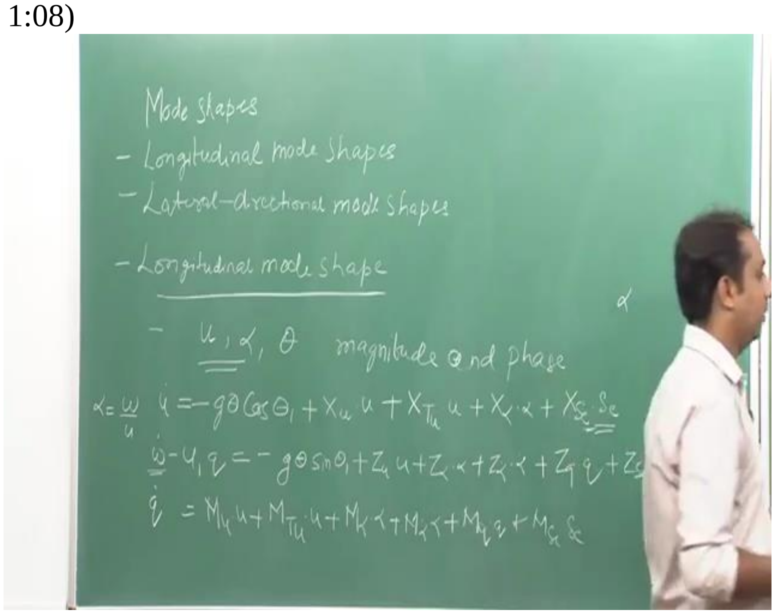

So mode shapes, we will be learning 1[st] longitudinal mode shapes and lateral directional mode shapes. So let us 1[st] focus on longitudinal mode shapes. No 1[st] question which comes into our mind is, what is a mode shape? Now mode shape is basically your relation between longitudinal derivatives, that is the relation between your U, your alpha and Zeta.

What will be the relationship or how they will affect each other in terms of magnitude and phase? What is the relation between these 3 variables in terms of magnitude and phase? So the best way of deriving the relationship between this derivative will be to use the longitudinal equation of motion or you can say longitudinal equation of motion free from elevator deflection and gust.

As we have already seen, while deriving transfer function and various modes of longitudinal motion, the derivatives, the differential equations for longitudinal motion was U dot - G Theta into Cos Theta 1 + X of U into U + XTU into U + X alpha into alpha + X Delta A into Delta A. W dot - U1Q equals to - G Theta sin Theta 1 + ZU into U + Z alpha into alpha + Z alpha dot into alpha dot + ZQ into Q + Z Delta E into Delta E.

And 3rd equation was Q dot equals to M U into U + MT U into U + M + M alpha into alpha + M alpha dot into alpha dot + M Q into Q + M Delta E into Delta E. This was already derived when we were driving transfer function and Phugoid and short period mode. Now as we told to derive the relationship between these longitudinal variables, we have to neglect the elevator inputs as well as any disturbance or gust.

So let us remove this or neglect this term due to Delta E and making suitable substitution in this equation. As we know that Alpha equals to W upon U, so using this equation or using this formula, you can substitute the value of W in this equation and your equation will be in terms of alpha.

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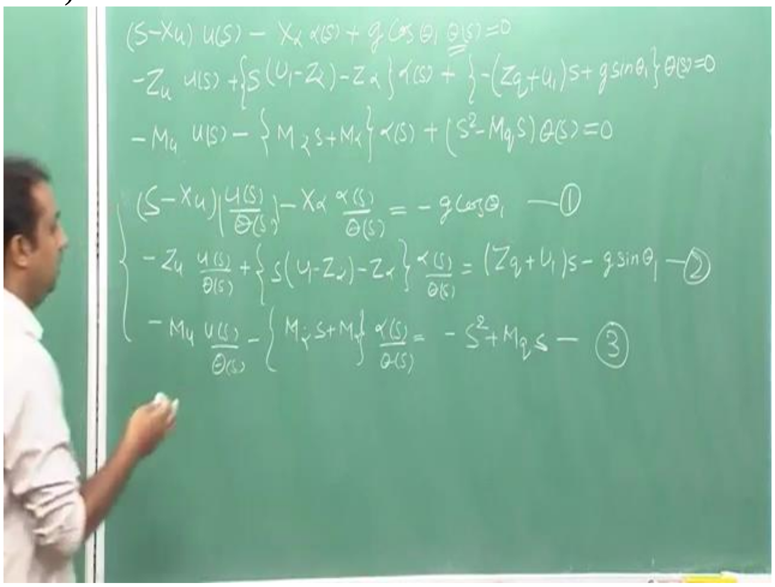

So making the required substitution and neglecting Delta E and taking Laplace transform of that, the equation we will be getting will be of this sort. S - X U into U of S - X alpha into alpha of S + G Cos Theta 1 in Theta of S equals to 0. - Z U into U of S + S into U1 - Z alpha dot - Z alpha into alpha of S + bracket - of ZQ + U1S + G sin Theta 1 into Theta of S equal to 0.

And - MU U of S - M alpha dot S + M alpha into alpha of S + S square - MQS into Theta S equal to 0. Any thirst derivative which you encountered in this equation, either it is neglected or you can assume that it has already been incorporated in your derivatives here. Dividing this differential equation by Theta S they will get, because we want the relationship between these variables, we will get the equation as, S - XU U of S by Theta of S - X of alpha alpha of S by Theta of S equals to - G Cos Theta 1.

- ZU U of S by Theta of S + S into U1 - Z alpha dot - Z alpha alpha of S upon Theta of S equals to ZQ + U1S - G sin Theta 1. And your 3rd equation will be, - MU U of S upon Theta of S - of M alpha dot S + M alpha alpha of S by Theta of S equals to - S square + MQS.

You can write this differential equation in terms of matrix and we have already seen while deriving transfer function, you can use that Cramer’s rule to derive what will be the value of US upon Theta of S, alpha of S upon Theta of S. As you can see, this has 3 equations and 2 variables, you can choose any of the 2 equations and solve for U of S upon Theta of S, alpha of S upon Theta of S.

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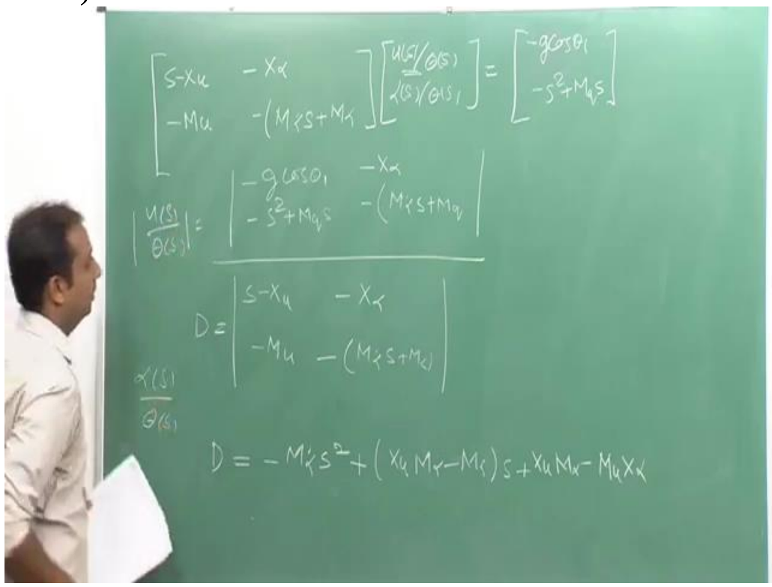

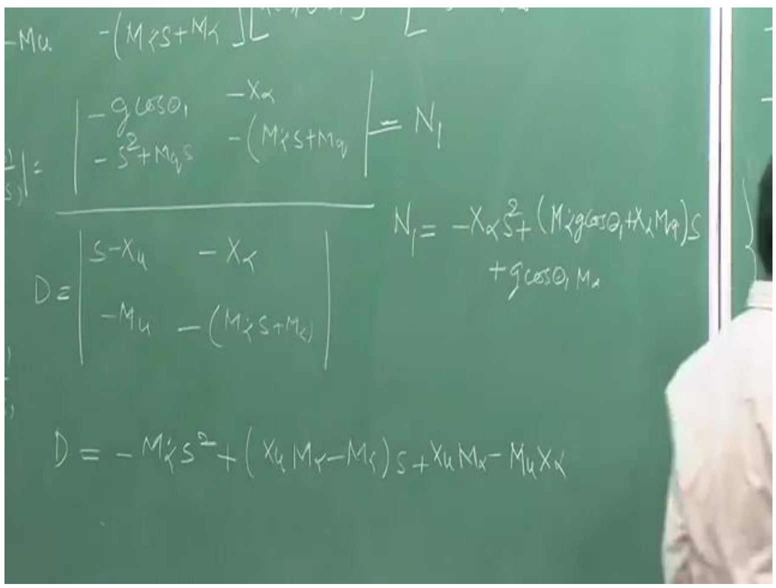

So representing it in a matrix form, we will get S - XU, - MU using equation 1 and 3. I am using equation 1 and 3. You can use any of the 2 equations. My matrix will be in the form, - X of alpha, - M alpha dot S + M alpha into U of S upon Theta of S, alpha of S divided by Theta of S equals to - gee Cos Theta 1, - S square + MQS. Now best way to derive the relationship between US and Theta of S is using Cramer’s rule.

So U of S upon Theta of S will be determinant of, representing this 1st column by the value of this matrix, that is - G Theta 1, - S square + MQS, - X alpha, - of M alpha dot S + MQ divided by determinant of this matrix. S - XU, - XU, - MU, - M alpha dot S + M alpha. Now the denominator in both the cases, that is U by S and Theta by S and alpha by S upon Theta by S will be same.

So determinant of this matrix will be, let us call it as D. D will be - M dot S square + XUM alpha dot - M alpha into S + XUM alpha - MU X alpha. Similarly the numerator of this ratio will be, let me call this as N1. N1 will be X alpha X square + M alpha dot G Cos of Theta 1 + X alpha MQS + G Cos Theta 1 M of alpha.

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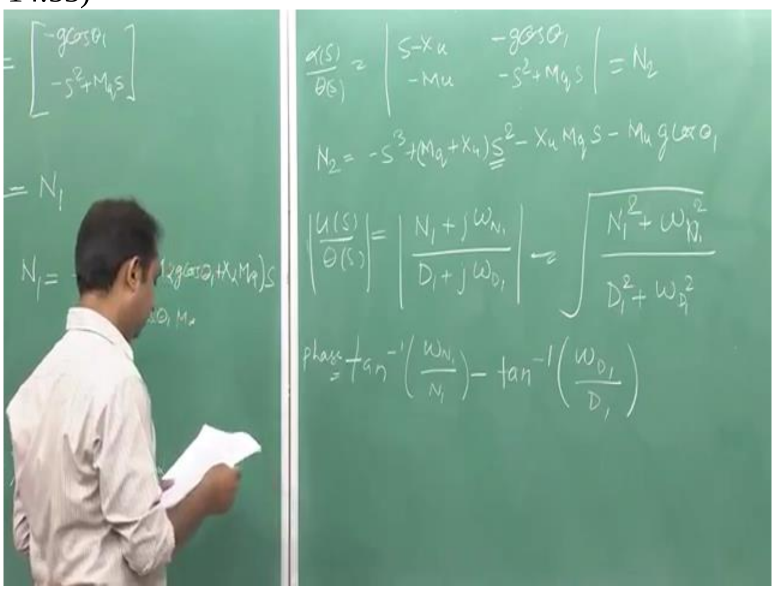

Similarly you can derive numerator for alpha of S upon Theta of S which will be determinant of S - XU, - MU, and replacing the 2nd column by your matrix, - G Cos Theta 1 and - S square + MQS. Let me call this as N2, the numerator 2. The value of N2 will be - SQ + MQ + XU into S square - XUMQS - MU G Cos Theta 1.

Now I have got the relationship between U of S upon Theta of S in terms of numerator and denominator as well as alpha of S upon Theta of S in terms of numerator and denominator. Now the ratio of this will be in complex form since the value of S are complex. We will be getting this ratio in terms of complex number.

Let U of S upon Theta of S be in some form of complex number. Say numerator, 1 + J of Omega N1 and B1 J of Omega D. So the magnitude of this will be equal to route under N1 square + Omega N square divided by D square + Omega D1 square. And your phase will be Tan inverse of Omega N1 upon N1 - Tan inverse Omega-D1 upon D1.

Now using the values of longitudinal derivatives for a business jet aero plane, I have already shared the derivatives on the forum. You can go and check the values of these derivatives. These derivatives were used for deriving the transfer function as well as when we will be doing the numerical for this section, the same derivatives are used in both the sections, transfer function as well as for mode shapes. You can go and check the forum.

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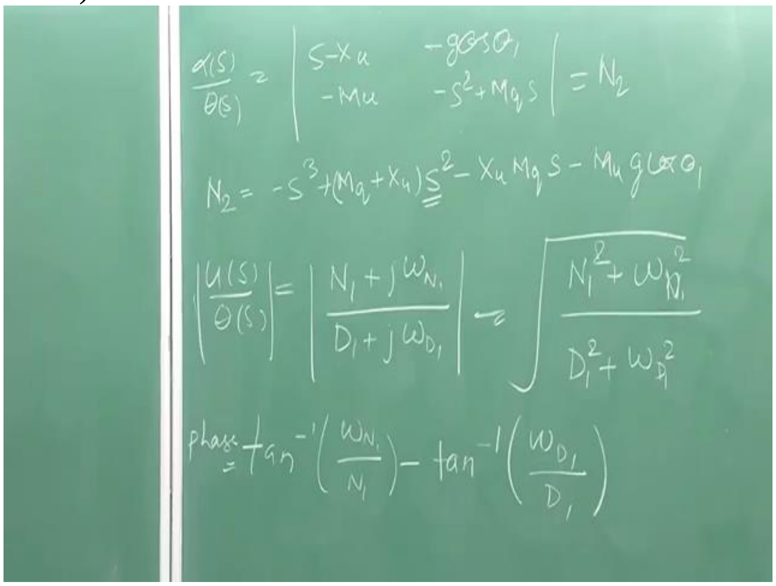

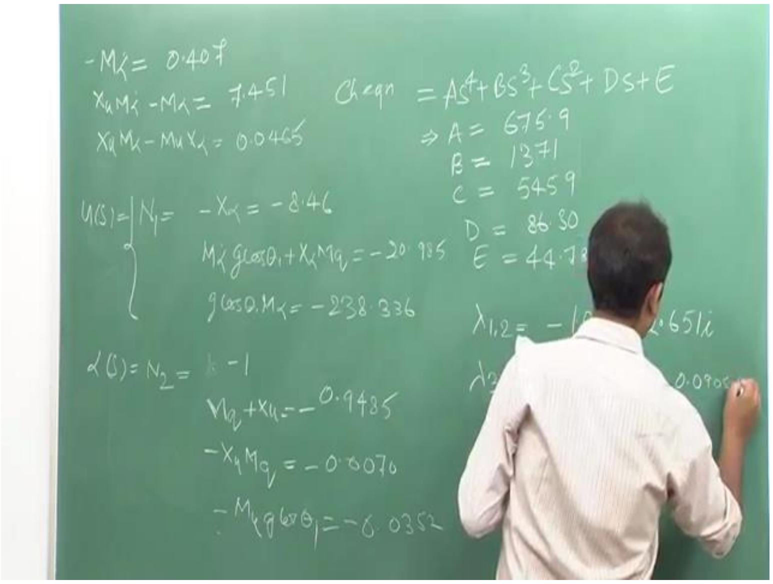

So for derivatives, longitudinal derivatives for a business jet aircraft, my values of these derivatives, MQ, XU are my M alpha dot equals to 0.407, XU M alpha dot - M alpha equals to 7.451. XU M alpha - MU X of Alpha equals to 0.0465. For U of S numerator N1, the coefficients which we calculated will be - X of alpha equal to - 8.46 N alpha dot G Cos Theta 1 + X of alpha MQ equals to - 20.985.

G Cos Theta 1 M alpha equals to - 238.336. These were for U of S. Alpha of S similarly, that is your numerator N2, the coefficients will be - 1. MQ + X of U equals to - 0.9485. - XUMQ equals to - 0.0070. - MU G Cos Theta 1 equals to - 0.0352. Using the derivatives, now we can calculate what will be the transfer function of longitudinal perturbed equation of motion which we already derived while we were studying transfer function which came in the form of AS4 + BS cube + CS square + DS + E. This was the character city question of my longitudinally question of motion.

Now substituting the derivatives we have already shared on the forum, the value of A to E will be, A equals to 675.9, B equals to 1371, C equals to 5459, D equals to 86.30, E equals 44.78. So the roots of my characteristic equation will be Lambda 1 and 2 equal to - 1.008 + - 2.651I and Lambda 34 equals to - 0.0069 + - 0.0905I.

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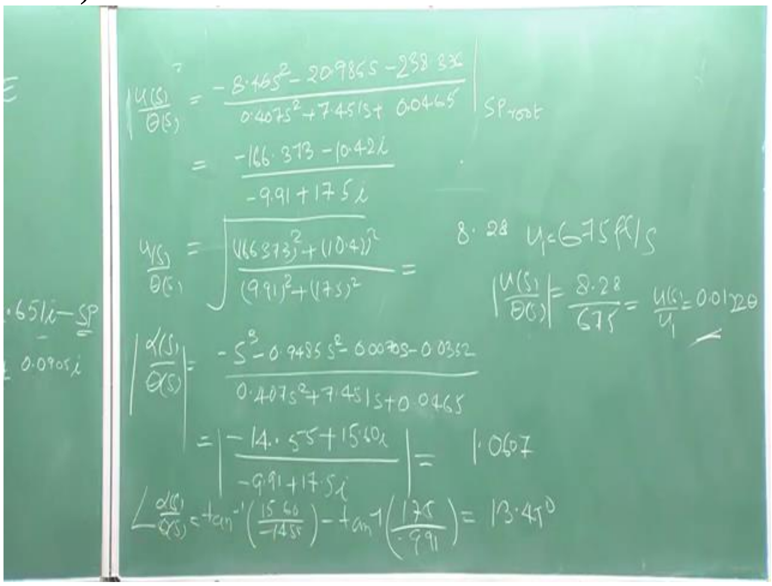

The ratio for U of S upon Theta of S, substituting the value of derivatives, we will get hit .4 6S square - 20.985S - 238.336 divided by 0.407 S square + 7.451S + 0.0465. Now substituting the value of S, that is roots of this characteristic equation, now as you know this will be corresponding to your short period mode roots and this will be corresponding to your Phugoid mode.

So let us substitute the value of short period mode in my equation for deriving, what will be the magnitude and phase of U of S upon Theta of S. Substituting the value of short period mode roots, we will get final value in terms of complex number will be - 166.373 - 10.42I divided by - 9.91 + 17.5I. This value at short period mode. Now magnitude of this complex number will be root under 166.373 square + 10.42 square divided by 9.91 square + 17.8 square which will come about 8.28.

Similarly, alpha of S upon Theta of S was - S cube - 0.9485S square - 0.0070S - 0.0352 divided by, since the denominator was same, so this will be same., 0.407 S square + 7.451S + 0.0465. The magnitude of alpha of S upon Theta of S at your short period mode root will give me - 14.55 + 15.60I divided by - 9.91 + 17.5I. The magnitude of this will be 1.0607 and phase of this will be, phase of alpha of S upon Theta of S is equal to Tan inverse 15.60 divided by - 14.55 - of Tan inverse 17.5 divided by - 9.91 which will be around 13.49 degrees.

Now as you can see, my initial velocity which was 675 ft./s, this was my U1. U of S upon Theta of S, magnitude of this is 8.28. When divided by your initial velocity U1, that is 675, you will get U of S upon U1 in terms of Theta of S, 0.122 Theta which is a very small quantity.

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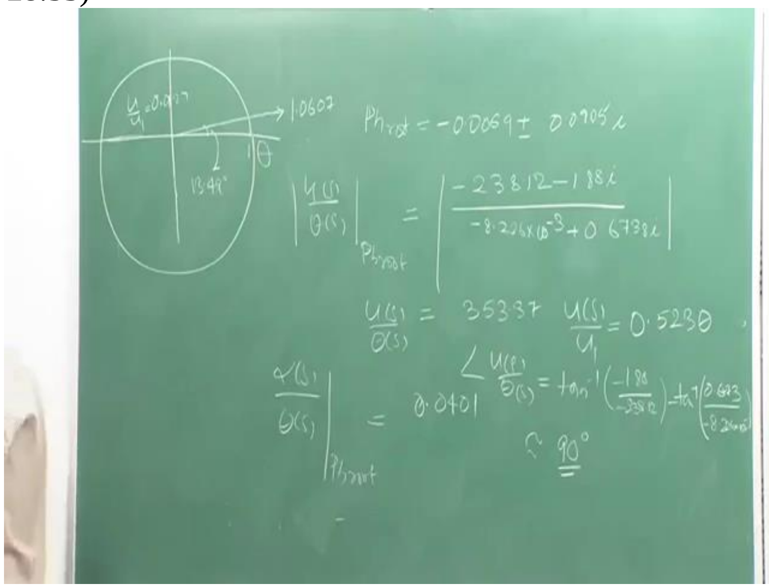

So now drawing the mode shape of this analysis for short period mode, this is a unit circle of Theta. As I saw during the analysis, your alpha upon Theta, the magnitude is what? 1.067. So the vector, this is of unit, this is a circle of unit radius. So your magnitude of alpha will be 1.06 and at an angle of 13.49 degrees. So at 13.49 degrees, your magnitude will be. This is 13.49 and the magnitude of your vector is 1.067.

And U of S upon U1 in terms of Theta is very small, 0.0122. So we can neglect that part. Now doing the same analysis for another root, that is Phugoid mode root which was - 0.0069 + - 0.0905I. Now substituting this value in your equation of U of S upon Theta of S and alpha of S upon Theta of S, magnitude of U of S upon Theta of S while substituting at, substituting roots of Phugoid mode you will get - 238.12 - 1.88I by - 8.226 into 10 to the power of - 3 + 0.6738I.

Now magnitude of this number will be, the magnitude of this complex number ratio which will come about U of S upon Theta of S. Now dividing this by your initial velocity, that is U1, so U of S upon U1 in terms of Theta will be 0.523 Theta. Similarly magnitude of alpha of S upon Theta of S at Phugoid mode root equals to 0.0401 and phase of U of S upon Theta of S equals to

Tan inverse of - 1.88 by - of 238.12 - Tan inverse 0.673 - 8.226 into 10 to the power - 3 which will come about 90 degree.

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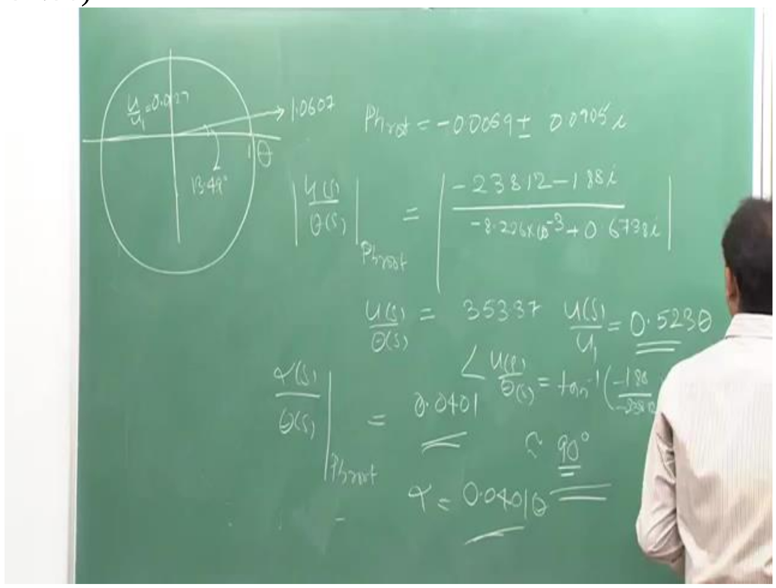

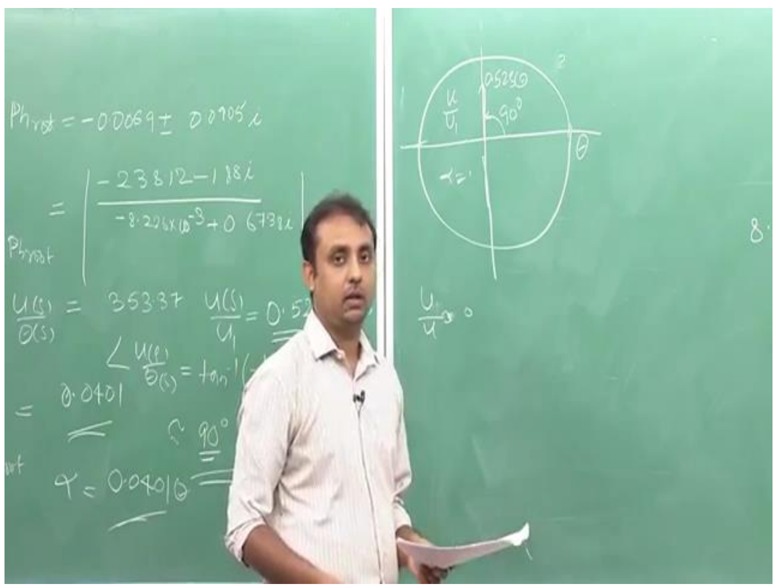

Now drawing the mode shift diagram for Phugoid mode, this is a unity circle with the Theta as parameter, your U upon U1 was 0.523 Theta and angle of degree which will be represented by this vector. This is 0.523 Theta and this was 90 degree. And as you can see, the value of alpha in this case, alpha upon Theta S in this case was very small. That is alpha in terms of Theta will be 0.0412 Theta which is dismal. So you can neglect that.

Now as I was telling, the information which we get for mode shapes are, in case of Phugoid mode, as you can see, since alpha is very small, so we can say for Phugoid mode approximation, alpha remains constant. Whereas for short period mode, as you saw, U upon U1 was very small. So you can say, while approximating short period mode, your velocity, speed remains constant.

From the analysis, we can see that the short period mode takes place at constant speed. While Phugoid mode takes place at constant angle of attack. Similar to analysis which we did for longitudinal equation of motion, they will be doing analysis for lateral directional equation of motion. So we will be seeing the lateral directional mode shapes.

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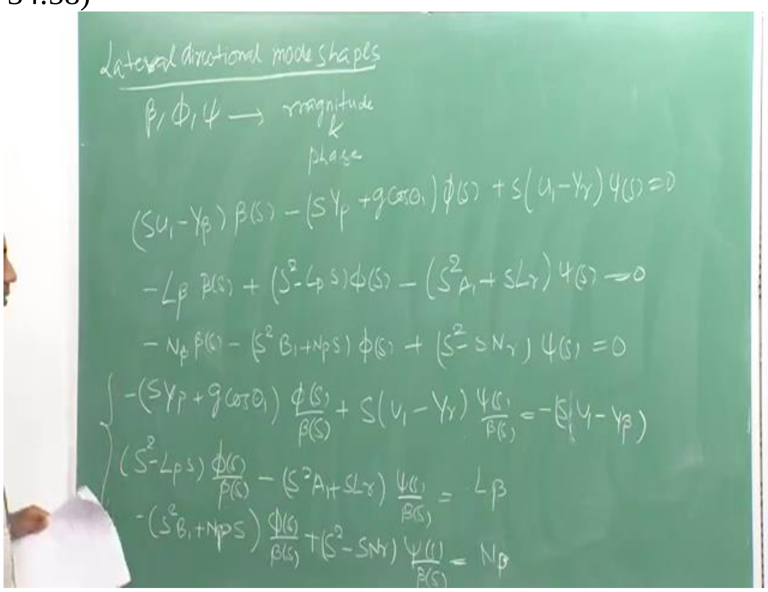

Lateral directional mode shapes. We will be finding the relationship between your beta, Phi and Psi in terms of magnitude and phase. So the best way of deriving that will be writing your lateral directional equation and taking laplace transform of that provided that you have neglected aileron deflection, rudder deflection and any gust, your differential equation in Laplace transform will be SU - Y of beta into bit of S - SYP + G of Theta1 into Phi of S + SU1 - YR into Psi of S equals to 0.

- L beta into beta of S + S square - LPS into Phi of S - S square A1 + SLR into Psi of S equals to 0. - N beta into beta of S - S square V1 + NPS into Phi of S + S square - SNR into Psi of S equals to 0. Now you can rearrange this equation by dividing the whole differential, 3 differential equations by beta of S, you will get.

- of SYP + G Cos Theta 1 into Phi of S upon beta of S + SU1 - Y of R into Psi of S upon beta of S equals to - SU1 - Y of beta. S square - LPS into Phi of S upon beta of S - S square A1 + SLR into Psi of S upon beta of S equals to L beta. And 3rd, - S square beta 1 + NPS Phi of S upon beta of S + S square - SNR into Psi of S upon beta of S equals to N beta. You can use any 2 equation.

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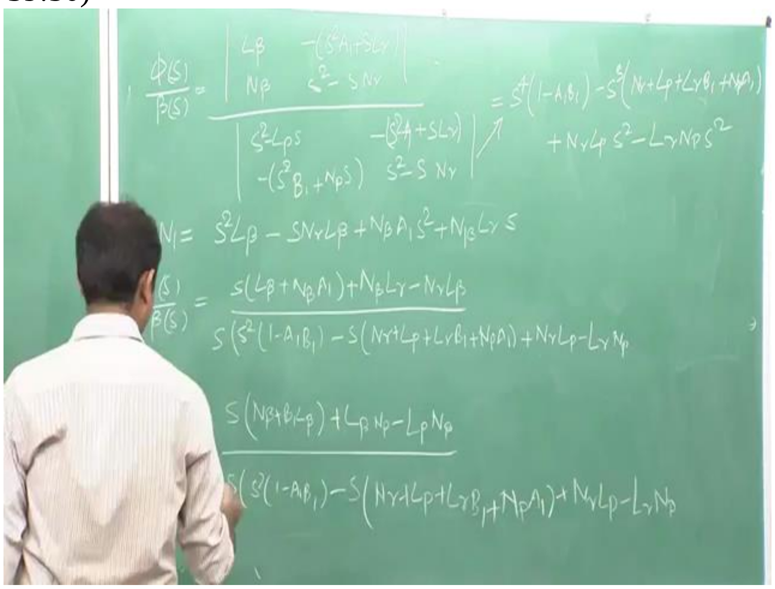

Since it is a 2 variable equation, and you have 3 equations, so you can take any 2 of these and you can write that in matrix form. And we have already seen, for longitudinal case as well while doing transfer functions, you can derive 1st variable by using Kramer's role. So I am directly writing your transfer function which will be Phis of S upon beta of S, taking variable 2 and 3.

So it will be determinant of L beta, N beta, - S square A1 + SLR, S square - SNR divided by determinate S square LPS, - S square B1 + NPS, - S square A1 + SLR, and S square - SNR which will give S to the power of 4 1 - A1B1 - S cube NR + LP + LR B1 + NPA1 + NR LP S square - NR NP S square. This is your denominator, the determinant of denominator.

The numerator will be N1 S square L beta - SNR L beta + N beta A1S square + N beta LRS. So Phi of S upon beta of S will be S into L beta + N beta A1 + N beta LR - NR L beta divided by S square 1 - A1 B1 S into - SNR + LP SLR B1 + NP A1 + NR LP - LR NP. Similarly, Psi of S upon beta of S will be S into N beta + B1 L beta + L beta NP - LP N beta divided by S into S square 1 - A1 B1 - SNR + LP + LR B1 + NP A1 + NR LP - LR NP.

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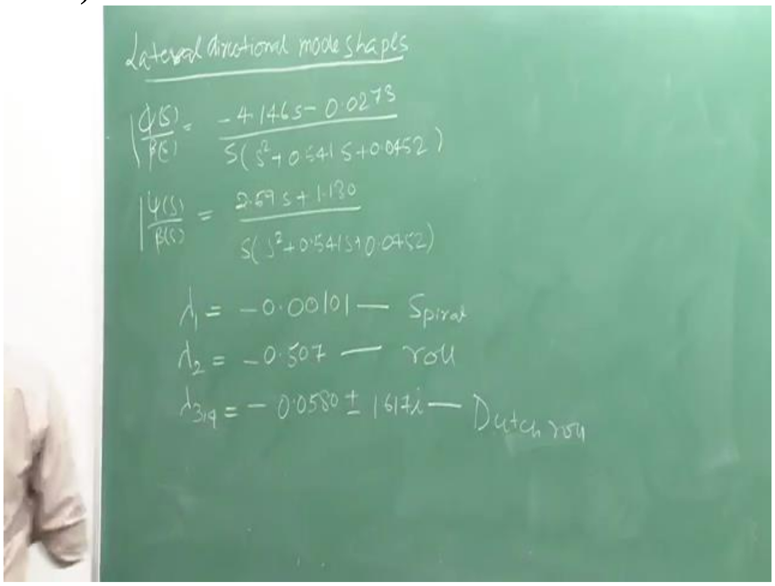

Using the value of lateral directional derivatives, you will get the transfer function of the ratio between Psi of S upon beta of S and Phi of S and beta of S and Psi of S and beta of S as Phi of S upon beta of S equals to - 4.146S - 0.0273 by S into S square + 0.541S + 0.0452. And Psi of S upon beta of S will be 2.59S + 1.130 divided by S into S square + 0.541S + 0.0452.

Now using this lateral directional derivatives, you will get the characteristic root of lateral directional equation of motion which will be 4^{th} order equations. So it will be having 4 roots. So the roots of that characteristic equations are, the values are already given in the sheet which I have shared on the forum. You can go and check the values.

Lambda 1 will be - 0.00101. Lambda 2 equals to - 0.507 and Lambda 3, 4, that will be a complex conjugate, this is - 0.0580 + - 1.617I. Now we have got the ratios between Phi of S and beta of S and Psi of S and beta of S. And evaluating each values at a different roots, this will be corresponding to your spiral mode.

This will be corresponding to roll mode. And this will be corresponding to Dutch roll mode. So evaluating these ratios at each of these roots, you will get mode shapes for different modes, corresponding to spiral, corresponding to roll and corresponding to Dutch roll.

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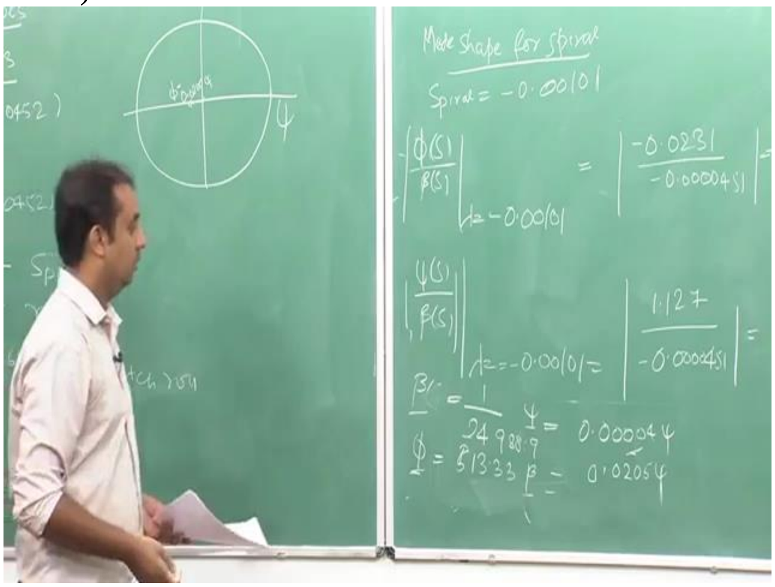

Let us 1st see the mode shape regarding spiral. Mode shape for spiral. The spiral root was - 0.00101. Substituting this value in your ratio of Phi of S upon beta of S and Psi of S upon beta of S, we will get Phi of S upon beta of S at Lambda equals to - 0.00101.

Magnitude of this will be - 0.0231 divided by - 0.0000451 which will be 513.33. And phase of this will be 0 degree. Both Phi and beta are in phase. Similarly, Psi of S upon beta of S at Lambda, magnitude of this at Lambda equal to - 0.00101 will give me 1.127 divided by - 0.0000451 which is equal to 24988.9. But the phase here are apart by 180 degrees or they are out of phase by 180 degree. Phase of Psi of S upon beta of S equals to 180 degrees.

So while drawing the mode shapes for spiral mode, this will be unit circle with Psi S parameter. From this ratio, your beta of S beta will be 1 by 24988.9 into Psi which will be somewhere around 0.0004 Psi and your Phi will be 513.33 into beta and since beta is 0.0004 Psi, so Phi and Psi, relationship between them will be, Phi will be 0.0205 Psi.

Now we saw than the phase difference between beta and Psi was 180 degree. Whereas the phase for Psi and beta was 0. So when plotting this magnitude and phase in my mode shape, we will be getting time attitude of Phi with respect to Psi as 0.025 but phase will be 180 degree. Say, somewhere here. This is 0.05 Psi.

Phi will be equal and phase will be 180 degree. And this is very small. You can neglect it. It will be very close to origin. So you can neglect the. Now from this analysis, what we came to know that in case of spiral mode, your beta is very small since the ratios are very large. So beta or slight slip angle plays a very insignificant role in spiral mode. And as you can see, the relationship between Phi and Psi is 0.025, phase difference of 180 degree.

So it can be plotted with, as seen in spiral mode. So from this analysis, we can infer that your beta is very small. Such ratios are very large. So you can neglect that. And spiral mode is majorly concerned with your side slip angle since the ratio of Psi of S upon beta of S is very large.

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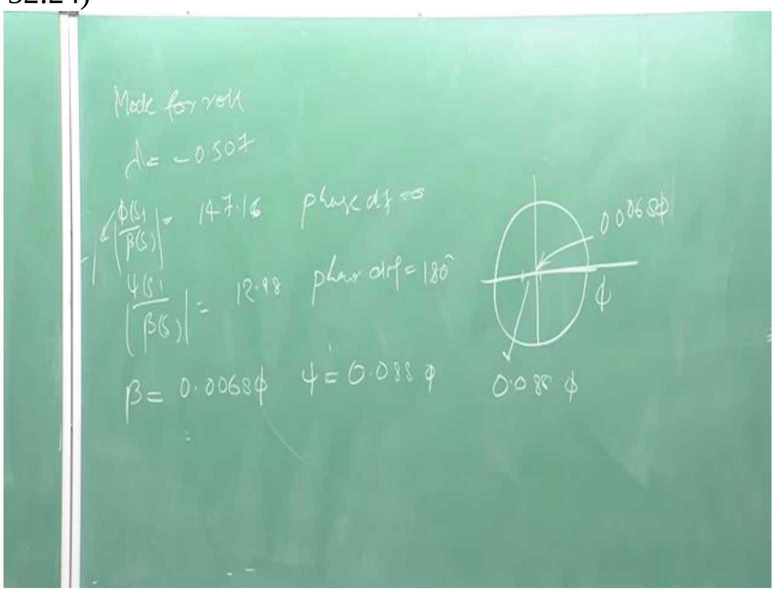

Now 2nd analysis will be for roll mode. So mode shape for roll, the roots was given by - - 507. Now when substituting this value in the ratios of Phi upon beta S and Psi upon beta S, you will get Phi S upon beta S as 147.16 with phase difference 0 and Psi of S upon beta of S, magnitude of this will be 12.98 with phase difference 180 degree.

Similarly we can write that beta in terms of Phi will be 0.00685 and your Psi will be 0.885. When plotting this in mode shape for roll mode, a unit circle in terms of Phi, the magnitude will be a small value for beta and Phi relation, this will be 0.00685 and your Psi and beta relation, Psi and Phi relation, 0.0885. Now from this analysis, you can see for rolling mode, Phi is a dominant, the bank angle is dominant for roll mode.

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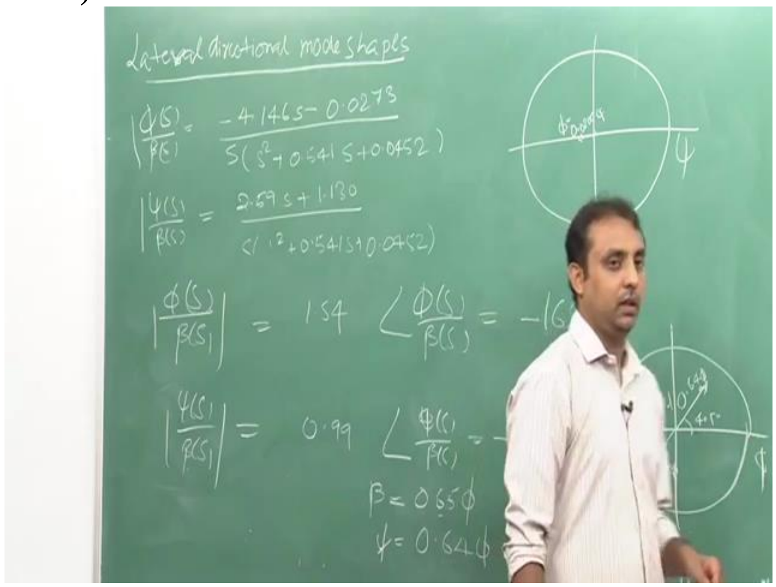

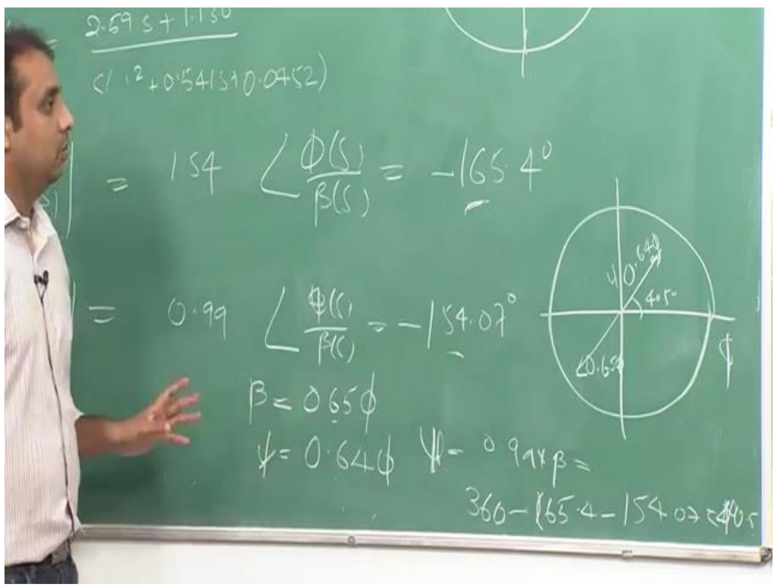

And the last analysis will be for your Dutch roll. Substituting the values of Dutch roll in these ratios, we will be getting, magnitude of Phi S upon beta S as 1.54 angle Phi of S upon beta of S as - 165.4 degree. There will be 2, magnitude and phase angle because these are complex roots. Similarly Psi of S upon beta of S magnitude will be 0.99 and angle of Psi of S upon beta of S equals to - 164.07.

When growing the mode shape for this Dutch roll, a unit circle in terms of Phi, you will get beta equals to 0.65 Phi and Psi equals to 0.64 Phi. Since this direct relation, so this will be at an angle of - 165 degree. This is - 165 degree and magnitude will be 0.65 Phi. And this will be, since Psi is obtained by 0.99 into beta and beta is already 0.65 into Phi, the angle which you will be getting is 360 - of this angle, 165.4 - 154.07 which will be around 40.40 degree. So your magnitude will be, this will be 40.40 degree. And magnitude will be 0.645. Service was all regarding to mode shapes. And I hope you enjoyed this lecture. Thank you.