Aircraft Dynamic Stability & Design of Stability Augmentation System
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Module 1
Lecture No 04
Laplace Transform

Good morning friends. In continuation to whatever we discussed in our last lecture, if I recall what are we trying? We are trying to understand a second order system which is typically mass spring damper system and we are doing all these things because we are preparing a Brussels to understand the dynamics of airplane when it is disturbed about this equilibrium. They are also very clear, we are talking about small disturbance about its equilibrium.

And we are trying to build our understanding of a second order system with the presumption that this knowledge will be useful in analysing the dynamics of an airplane under the influence of small disturbance. We also understand that we are more bothered about transient. And that is the part we need to understand if we want to characterise the dynamic stability of an airplane.
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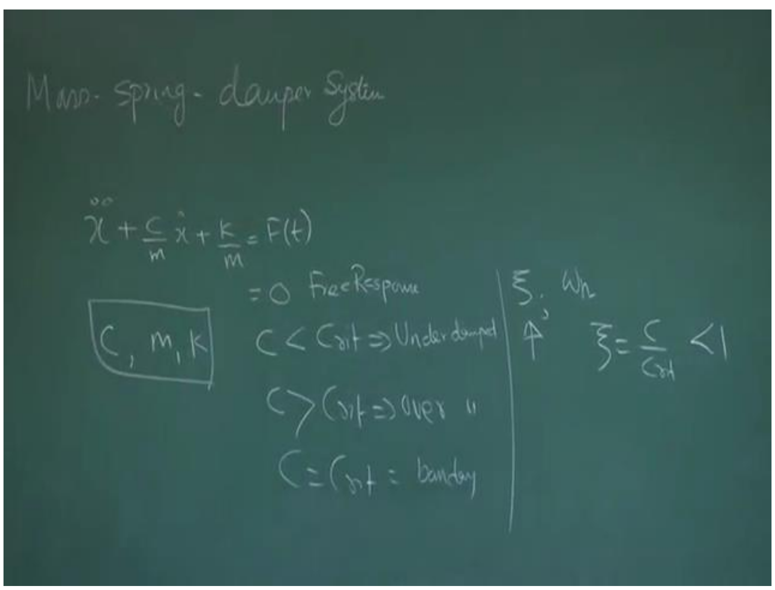

And if you recall, when we talk about Mass Spring Damper System, we have got equation of this form, X double dot + C by M X dot + K by M equal to FT but if it is a free response, it is equal to 0 for free response. And this is what we are looking for. We also understood when we tried to find its characteristic equation, we realised that depending upon the value of C, M and K, the combination, I could have situations where C, the damping coefficient is less than a particular value, say C less than C critical, then we say that this will be under damped case.

If C is greater than C critical, we say it is over damped. And C is equal to C critical, it is the boundary between the over damped and under damped case but we also know one thing that if the damping coefficient, C is chosen such that it is equal to C critical, then the time to return to the equilibrium will be not only non-oscillatory, it will be fastest. We also realised that if it is over damped case, there will not be any oscillations. But if it is an under damped case, then there will be oscillations and it will take some time to come back to equilibrium.

In understanding this, we try to define 2 things which are important, Zeta and Omega N. And Zeta is the damping ratio. It is actually the ratio of actual damping C divided by C critical. Recall, Zeta is C by C critical. So if Zeta is less than 1, that means the damping of the system is less than C critical. So it will have an under damped response. If Zeta is greater than 1, it will have an over damped response because at that time C is greater than C critical. And if C is equal to C critical, then there is a boundary. It will also be non-oscillatory and it will come back fastest, return fastest to equilibrium once the disturbance is withdrawn.

We are all talking about disturbances, we are assuming that there is a small disturbance about equilibrium. And, so we realise that if this equation, which is equation of motion for a mass spring damper system, that is how this X coordinate is changing, how this gentleman is moving in X direction. So this is equation of motion. And we realise that if we can transform this equation of motion using Zeta and Omega N, where Omega N is the natural frequency and we understand natural frequency means the frequency of oscillation if damping is 0.

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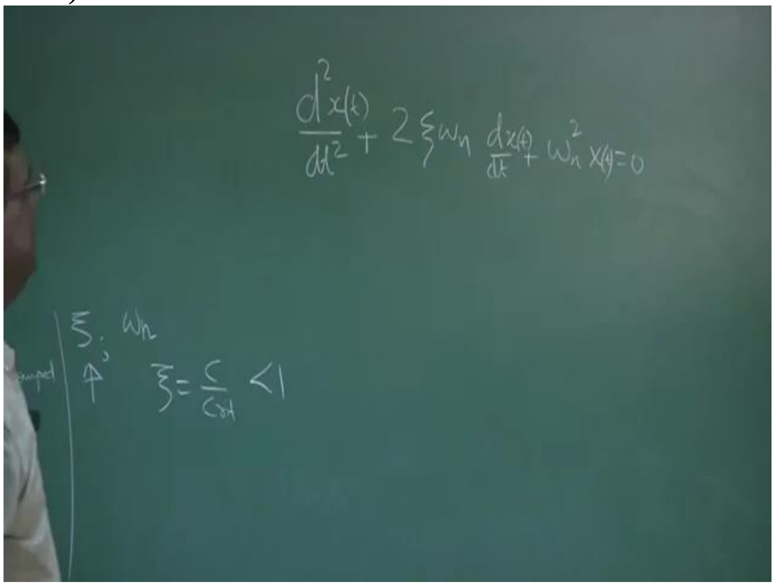

So in doing that, we came up to a point where it is D Square X by DT square + 2 Zeta Omega N into DX by DT + Omega N Square X equal to 0. And it goes without saying when I write X, X means X function of time. This is function of time: this is function of time, this is function of time.

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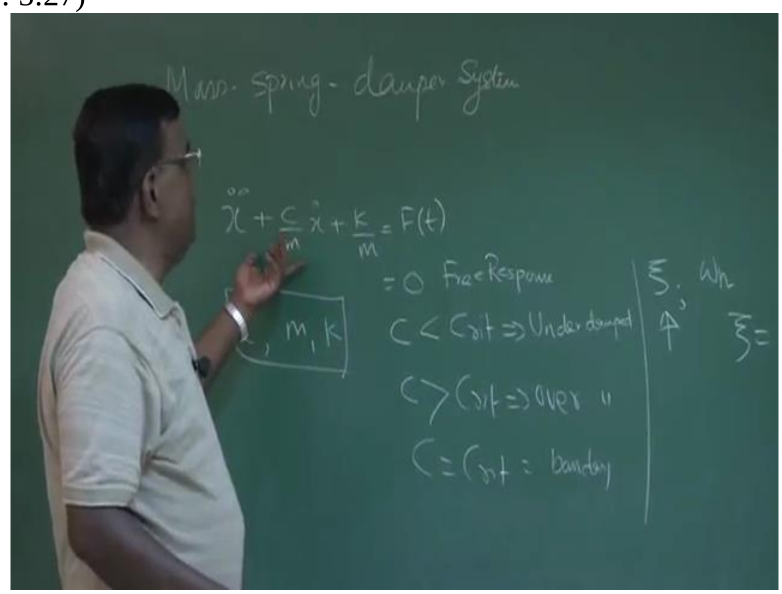

This is same equation as the equation given by this for a free response, written into using Zeta and Omega N, okay. Now we will have a little bit of halt and we will try to learn the last thing which is required from mathematical background and then as I promised you, after this lecture you are ready for utilising this understanding or analysing dynamic stability of airplane.

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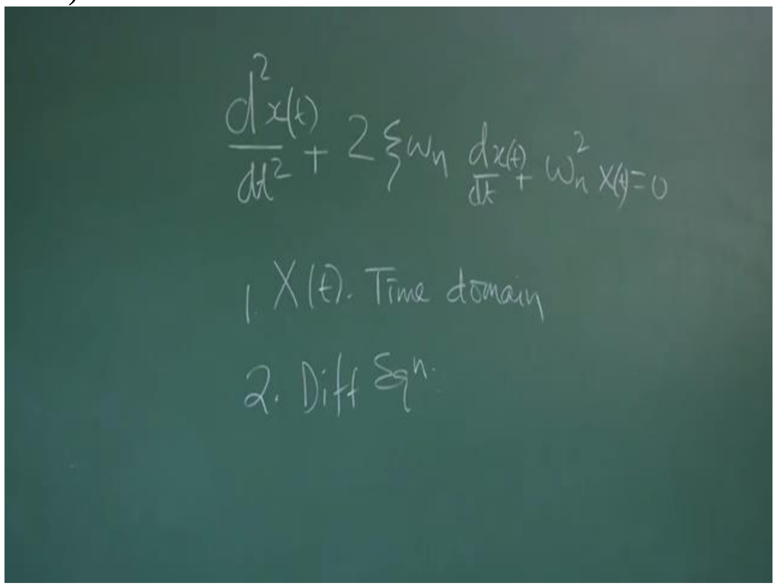

If you see here, we are talking in terms of time domain because X of T, right. This is in time domain. We also observing that this is a differential equation, right. Now historically, one of the approach is, why do not we change this equation from time domain to a frequency domain.

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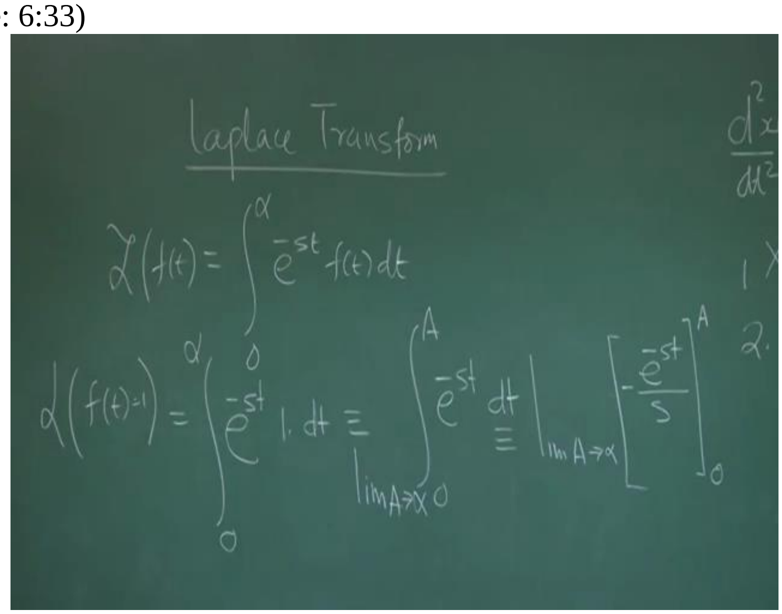

And that is essentially Laplace Transform. We will just use the gist of Laplace Transform and whatever I am telling you, if you recall this much, it is good enough for our future analysis. Laplace Transform of a function is defined as, that is F of T is the function, you multiply it with

E the power - ST where S is equal to J Mega. Omega is the frequency. And integral from 0 to infinity and you could see, it is an improper integral. Right? Now what is the advantage of this?

Once I take Laplace Transform for a function, let us say the function is F of T equal to 1. Then what will happen? Laplace of F of T which is equal to 1 will be 0 to infinity E to the power - ST, F of T is 1 into DT. However, you understand, this is improper integral. So we have to follow a procedure. We will write integral 0 to A, limit, do not get confused with limit and all such things. Limit A tends to infinity, E to the power - ST into DT. And then, this will be equal to limit A tends to infinity and this will be E to the power - ST by S 0 to A and A tends to infinity. Right?

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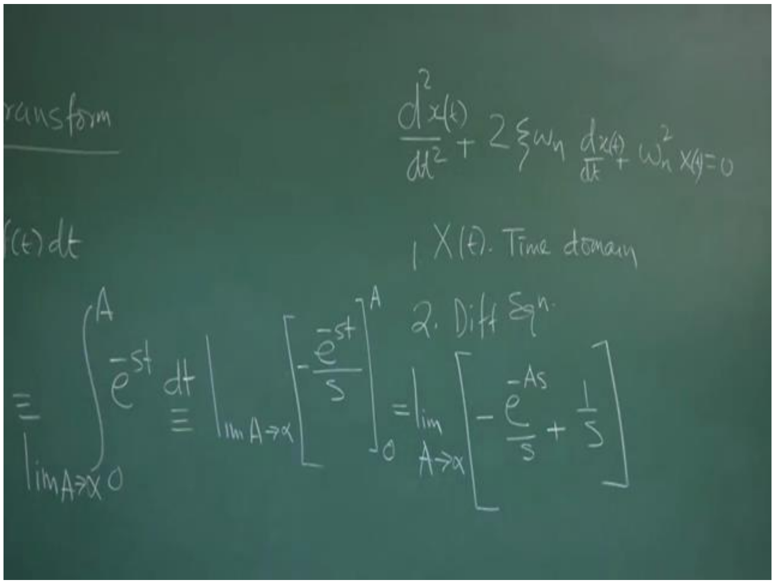

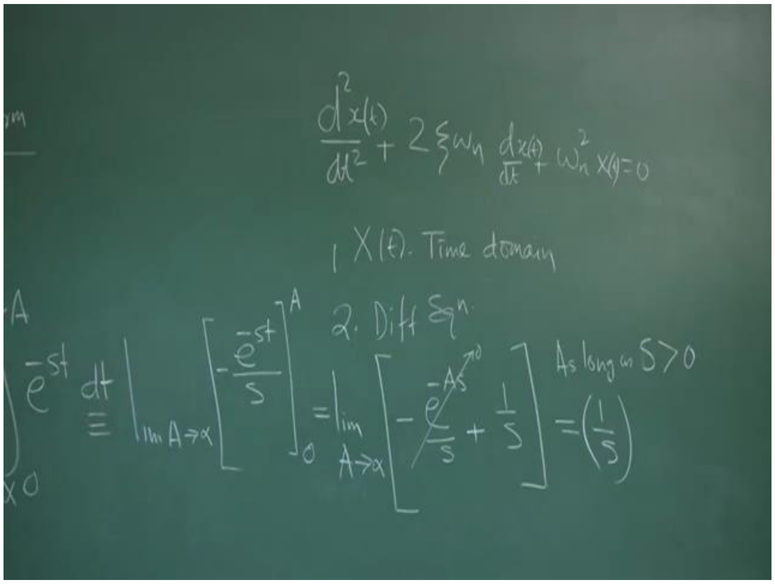

This again, once I put the limits, I say limit A tends to infinity, this will be - E to the power - AS by S + 1 by S. Right? I put A and 0 to be the limits. So what is happening if I do that? See, understand, this is very important. As long as S is positive, as long as S is greater than 0, if I take this limit, this gentleman will go to 0 and the result will be 1 by S. This is extremely important that S has to be greater than 0. Otherwise, it will not converge.

So this is a simple trick for finding Laplace Transform. For any other function, you can put, replace FT by that function, do the integral. Sometimes you have to do by parts and you should be able to get the desired expression. In fact, Laplace Transform for all the standard functions are available. You need not do all those functions. You should try 1 or 2. You can refer book by Creswick, Engineering Mathematics by Creswick and in half an hour you can sap through and get the hang of it. So this is to introduce you in Laplace Transform, what is happening you see.

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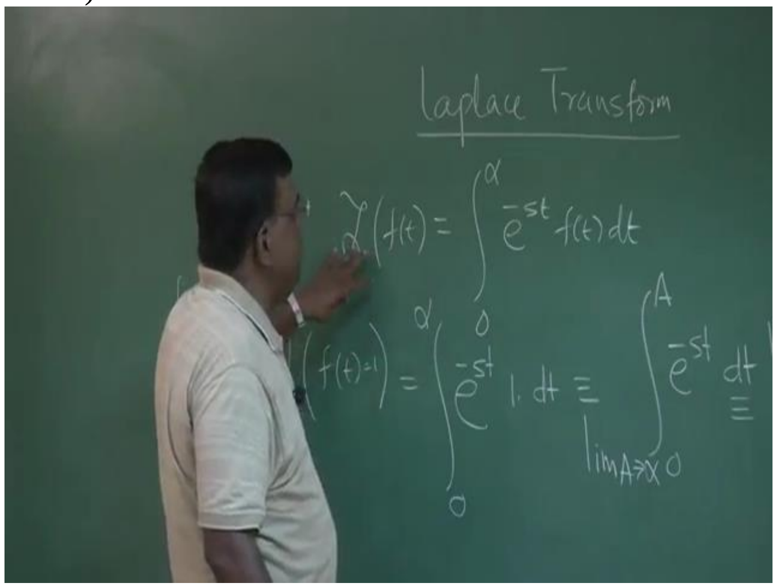

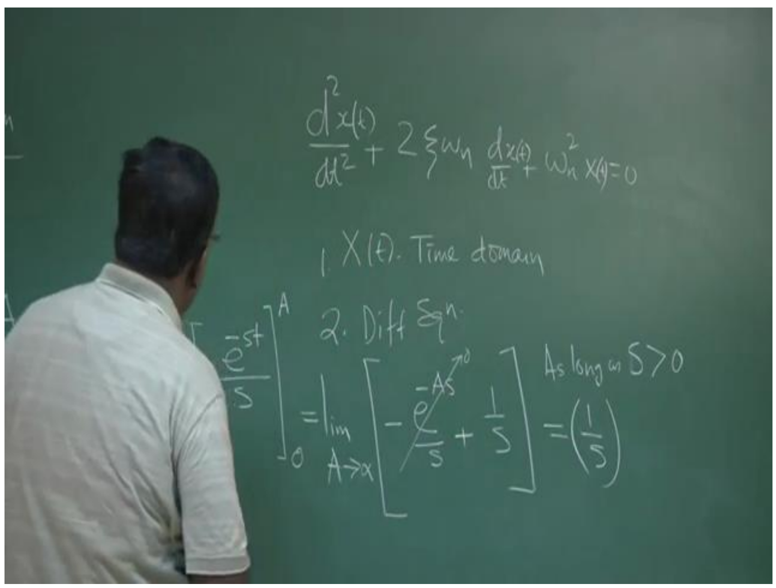

The time domain function is being transformed into a frequency domain. That is the important thing. If I stretch this understanding for function which are derivatives, then you can see that, I am just writing the result.

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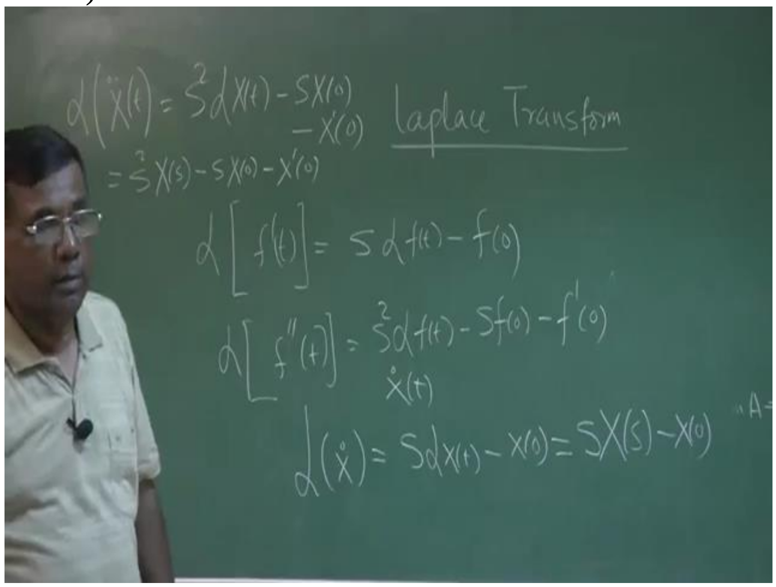

Laplace of F prime T is S of Laplace of F of T - F of 0. Similarly, Laplace of F double prime T is S Square Laplace of F of T - SF of 0 - F prime 0. I again strongly advise you, you can do this by using the definition of Laplace integral, you would then use integration by parts, you will get these results. Just to make you familiar, suppose I am trying to do for X dot. So as per this, Laplace of X dot will be equal to S Laplace of X of T - X of 0 which I write as SX of S - S of 0. Is this clear?

Laplace of X of T is X of S but my notation. Similarly, if I try to find out Laplace of X double dot T by this, I will get S Square into Laplace of X of T - SX of 0 - X prime evaluated at 0. So this I can again write as S Square XS - SX of 0 - X prime 0. Since we are dealing with a linear system, and for linear system, you can always put these initial conditions to be 0 because this will not affect the stability characteristics. If it is non-linear, you can alter that.

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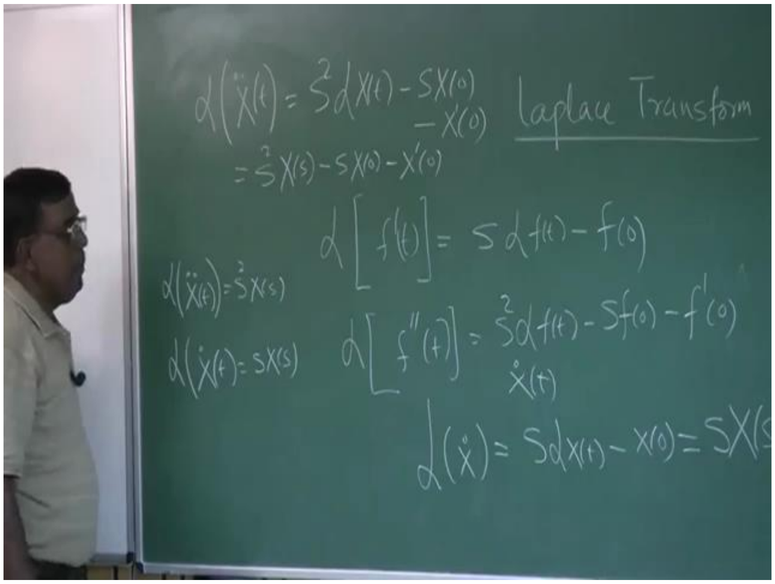

If you are talking about lean year system, small perturbations, I can always approximately write Laplace of X double dot T equal to S Square X of S Laplace of X dot T is equal to SX of S. Right? Please understand, I have taken these initial conditions to be 0 because as far as stability is concerned for linear system, it does not matter what is the initial condition.

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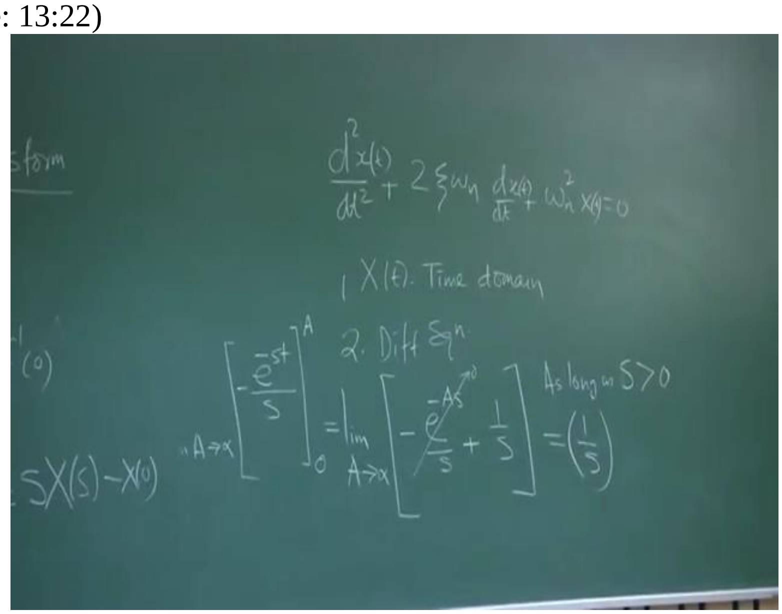

If I now apply this understanding to that equation, then what happens? Let us try to see that. Now let us revise, Laplace of X double dot will be S square X of S because I am putting all the initial conditions 0. Laplace of X dot T will be SX of S. Clear?

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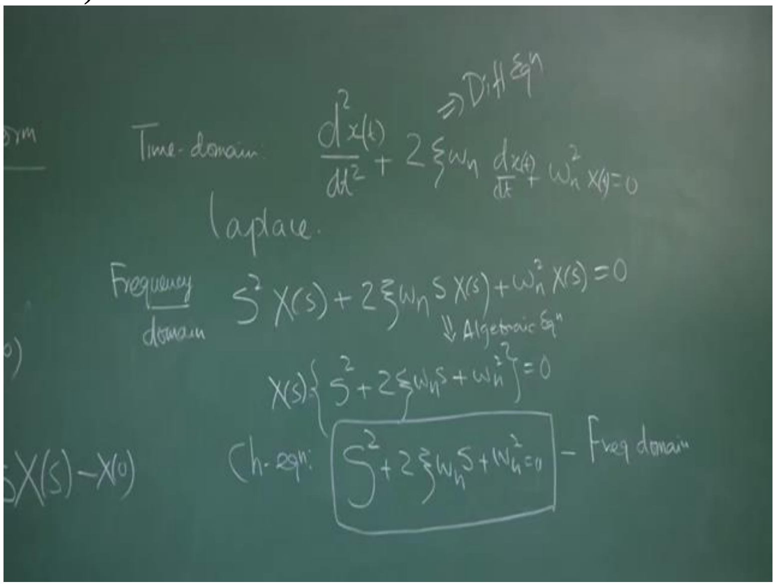

So now come here. If I take the Laplace on both sides, this is X double dot, so this will be S square X of S + 2 Zeta Omega N SX of S because this is X dot. + Omega N square X of S equal to 0. Please understand, this was the characteristic equation assuming free response because I am interested in the transient. This is in time domain and this is now in frequency domain. S is basically J Mega. So this is in frequency domain. We are all taking advantage of linear system and this is in time domain.

Now what are the differences we are finding here? Please see here. The moment I take a Laplace transform of this equation, this resulting characteristic equation is algebraic equation. Is not it? This was differential equation and this is algebraic equation. This is the first advantage we got and that will be our prime advantage for working in the frequency domain or using the Laplace transform.

So now what is the characteristic equation? The equation is X of S, if I take common, then it will have S Square + 2 Zeta Omega N S + Omega N square equal to 0. So my characteristic equation now becomes S Square + 2 Zeta Omega N S + Omega N S Square equal to 0. Right? Clear? So this is the characteristic equation in frequency domain. Now see how we can use it. Why we have come down to this sort of equation? Because our aim was to find out Zeta and Omega N of the system, what is the damping ratio so that I know whether the free response is oscillatory or it is critical damping, over damping, or under damping and also need to know what is the natural frequency?

That is frequency of oscillation when damping is 0. And if I know this, I can easily find out what is the pseudo-damped frequency which derivation we have already made. This was our purpose. Now, how to use this equation that is the question.

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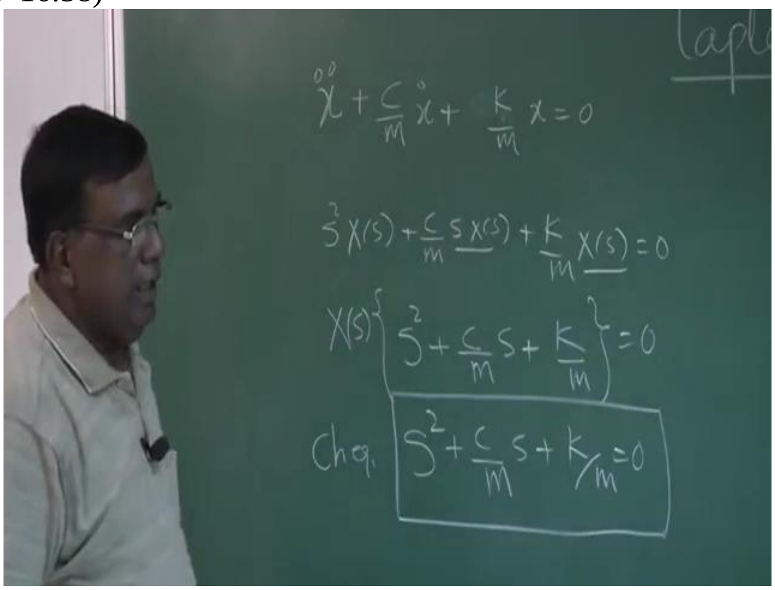

Let us take mass spring damper system and which is X double dot + C by MX dot + K by M is equal to 0. This is in a Mass Spring Damper System. Now I want to find out what is the value of Zeta and what is the value of natural frequency. What I do? I take Laplace transform. So I get S square X of S + C by M SX of M + K by M X of S equal to 0. I have just taken Laplace transform here. I know, for X double dot, it is S square X of S and for X dot, it is S X of S and for X, it is X of S. If I take X of S common I have S Square + C by MS + K by M equal to 0.

What is my characteristic equation? It is S Square + C by M S + K by M equal to 0. Now I want to find out what is the expression for Zeta and Omega N for the second order system. We are very comfortable now.

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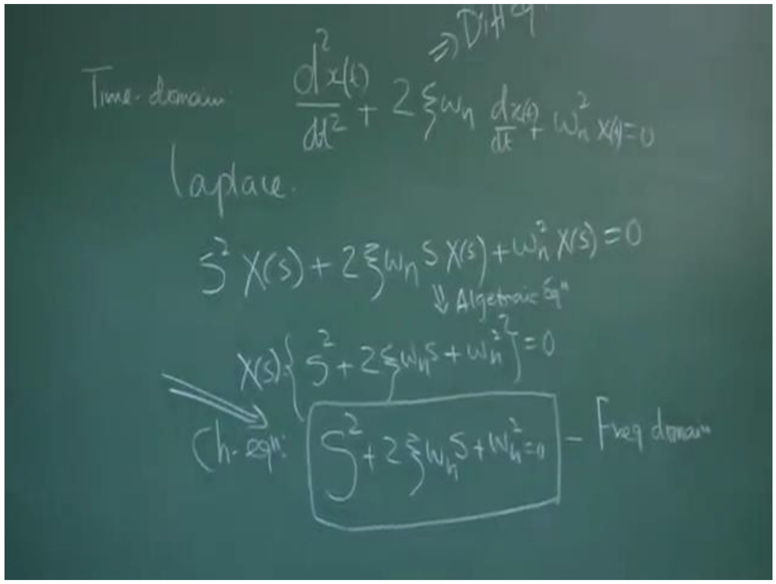

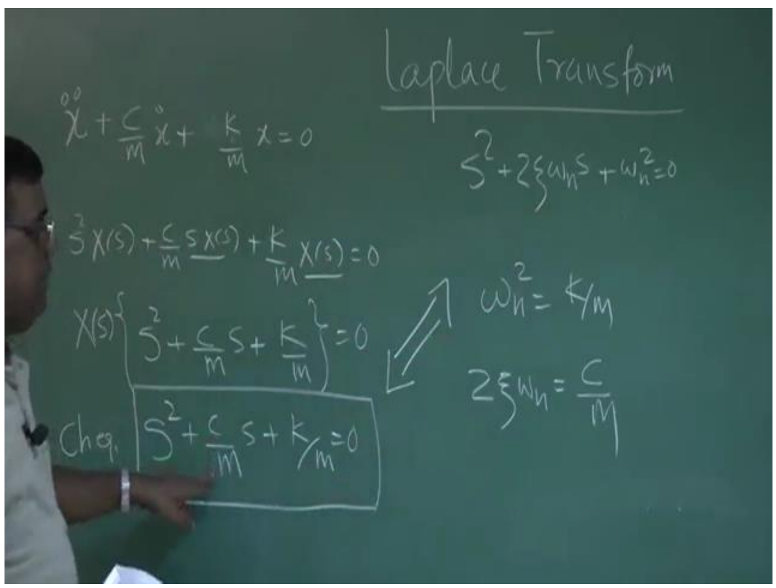

We know that for second order system, the characteristic equation can be also written in this form, that is S Square + 2 Zeta Omega N S + Omega N square equal to 0. If I can map this and this and find out what is the value of Omega N and Zeta, if I compare this, I get Omega N square equal to K by M and 2 Zeta Omega N equal to C by M. I am just comparing these 2 equations. S Square and S Square. 2 Zeta Omega N is nothing but C by M and K by M is nothing but Omega N square. So what does this tell you?

The natural frequency is under root K by M and Zeta will be equal to C by 2M into one by under root K by M. See, Zeta will be equal to C by 2M and Omega N is here. That is, if I know the value of spring constant K for the spring, if I know what is the mass of the spring, I will be able to get the value of Zeta and natural frequency. Is this clear? So that is how it is very handy. Okay?

We will be solving a few examples and you will realise, it is also important to know whether the system will be oscillatory or it will be just without any oscillation come back to equilibrium. Although finite details, we will see as we progress.